Probability Aspects of the Multiple Choice Question Tests

JINDRICH KLUFA
Department of Mathematics
University of Economics
W. Churchill Sq. 4, Prague 3
CZECH REPUBLIC
klufa@vse.cz

Abstract: - In this paper we shall analyse entrance examinations at Prague University of Economics from probability point of view. Entrance examinations in mathematics and entrance examinations in English are used for admission process at the university. These tests in mathematics and English are the multiple choice question tests. We shall find the probability distribution of random variables "number of points in the test in mathematics" and "number of points in the test in English" (in this case model of binomial distribution can be used). On the base of these probability distributions we can answer e.g. the following questions (under assumption of random choice of answers): what is the probability, that number of right answers exceeds given number, what is expected number of right answers, etc.

Key-Words: - Entrance exams, probability distribution, test in mathematics, test in English, binomial distribution, examples.

1 Introduction

In many situations the following type of test is used for checking students' knowledge. The test has n questions, each question has m answers (the multiple choice question test). Multiple choice question tests are also applied for entrance examinations in mathematics and entrance examinations in English at Prague University of Economics – see Klůfa (2011). Multiple choice question tests are suitable for entrance examinations. These tests are objective, results can be evaluated easily for large number of students. On the other hand, a student can obtain certain number of points in the test purely by guessing the right answers. This problem we shall solve in present paper for admission process at Prague University of Economics. Similar problem is addressed in other education research – see e.g. Zhao (2005), Premadasa (1993), Zhao (2006), Klůfa (2015), Hrubý (2016).

From probability point of view multiple choice test means: Let us consider n independent random trials having two possible outcomes, say "success" (right answer) and "failure" (wrong answer) with probabilities p and $(1-p)$ respectively. Under assumption that each answer has a same probability is $p=1/m$.

Let us denote X number of successes (right answers) that occur in n independent random trials. Random variable X is distributed according the binomial law with parameters n and p. Probability, that number of successes is k (see e.g. Rao, 1973) is

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \qquad (1)$$

for $k=0,1,\dots,n$. The expected value and the variance (dispersion) of random variable X distributed according the binomial law with parameters n and p is

$$E(X) = n\,p, \quad D(X) = n\,p\,(1-p) \qquad (2)$$

2 Entrance Exams in Mathematics

Entrance examination in mathematics has 10 questions for 5 points and 5 questions for 10 points (100 points total). Each question has 5 answers. Under assumption that each answer has a same probability, probability of right answer is *p=1/5.*

Example 1. Under assumption of random choice of answers we shall find probability that number of points in the test in mathematics is 15.

Let us denote

Y = number of points in the test in mathematics

X_1 = number of right answers in the first 10 issues

X_2 = number of right answers in 10-points tasks

Holds

$P(Y=15) = P[\ (X_1=1 \cap X_2=1)\ U\ (X_1=3 \cap X_2=0)\ ] =$

$=P[\ (X_1=1 \cap X_2=1)\ ] + P[\ (X_1=3 \cap X_2=0)\ ]$

Random variables X_1, X_2 are independent, therefore we have - see e.g. Rényi (1972)

$P(Y=15) = P\ (X_1=1)\ P(X_2=1) + P\ (X_1=3)\ P(X_2=0)$

Random variable X_1 has binomial distribution with parameters $n=10$ and $p=0{,}2$. Random variable X_2 has binomial distribution with parameters $n=5$ and $p=0{,}2$. According to (1) we obtain

$$P(Y=15) =$$
$$= \binom{10}{1} 0{,}2\ 0{,}8^9 \binom{5}{1} 0{,}2\ 0{,}8^4 + \binom{10}{3} 0{,}2^3\ 0{,}8^7 \binom{5}{0}\ 0{,}8^5 = = 0{,}175922$$

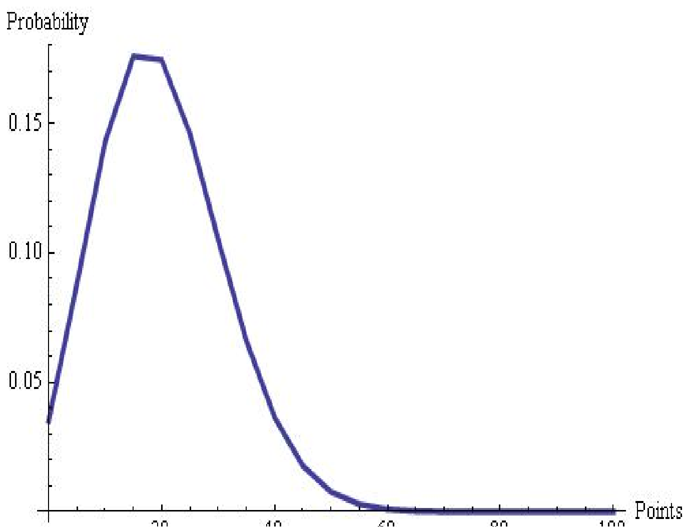
Analogously, we can calculate the probabilities *P(Y=k)* for other *k=0, 5, 10, 15, ... , 95, 100* (see Table 1 and Figure 1). For this calculation we used software Mathematica (Statistics 'Discrete Distributions') – see Wolfram (1991).

Table 1 Distribution of number of points in the test (mathematics)

Points from the test	Probability	Points from the test	Probability
0	0,035184	55	0,002890
5	0,087961	60	0,000957
10	0,142937	65	0,000275
15	0,175922	70	0,000067
20	0,174547	75	0,000014
25	0,146098	80	0,000002
30	0,105227	85	3×10^{-7}
35	0,066057	90	2×10^{-8}
40	0,036467	95	1×10^{-9}
45	0,017761	100	3×10^{-11}
50	0,007634	Sum	1,000000

Source: Own construction

Fig. 1 Distribution of number of points in the test (polygon) – mathematics



Source: Own construction

Example 2. Under assumption of random choice of answers we shall find probability that number of points from the test in mathematics is

(a) 30 and more,
(b) 40 and more,
(c) 50 and more.

(a) Using notation from example 1 we have

$$P(Y \geq 30) = 1 - P(Y<30) = 1 - P[(Y=0) \cup (Y=5) \cup (Y=10) \cup (Y=15) \cup (Y=20) \cup (Y=25)] =$$

$$= 1 - [\, P(Y=0)+ P(Y=5)+ P(Y=10)+ P(Y=15)+ P(Y=20)+ P(Y=25)\,]$$

Finally from Tab.1 we obtain

$$P(Y \geq 30) = 1 - 0{,}762649 = 0{,}237351.$$

Under assumption of random choice of answers almost a quarter of students get the test score 30 or more points.

(b) Analogously, we obtain

$$P(Y \geq 40) = 1 - P(Y<40) =$$

$$= 1 - [\, P(Y=0)+ P(Y=5)+ P(Y=10)+ P(Y=15)+ P(Y=20)+ P(Y=25) + P(Y=30) + P(Y=35)\,]$$

Finally from Tab.1

$$P(Y \geq 40) = 1 - 0{,}933933 = 0{,}066067.$$

Under assumption of random choice of answers approximately 6,6% of students get the test score 40 or more points.

(c) Finally

$$P(Y \geq 50) = 1 - 0{,}988161 = 0{,}011839.$$

Under assumption of random choice of answers approximately 1,2% of students get the test score 50 or more points.

Example 3. Under assumption of random choice of answers we shall find expected number of points in the test in mathematics and mode.

Using notation from example 1 we have

$$Y = 5\,X_1 + 10\,X_2$$

Therefore - see e.g. Feller (1970)

$$E(Y) = E(5X_1 + 10X_2) = 5\,E(X_1) + 10\,E(X_2)$$

According to (2) we obtain (e.g. $E(X_1) = 10 \,.\, 0{,}2 = 2$)

$$E(Y) = 5 \,.\, 2 + 10 \,.\, 1 = 20.$$

Expected number of points in the test in mathematics is 20. The mode is the most probable number of points. From Tab.1 is

$$\hat{y} = 15.$$

.

3 Entrance Exams in English

Entrance examinations in English has 50 questions for 2 points (100 points total). Each question has 4 answers. Under assumption that each answer has a same probability, probability of right answer is *p=1/4.*

Example 4. Under assumption of random choice of answers we shall find probability that number of points from the test in English is 20.

Let us denote

Z = number of points in the test in English

X = number of right answers in the 50 issues

Random variables X has binomial distribution with parameters *n=50* and *p=0,25.* According to (1) we obtain

$$P(Z = 20) = P(X = 10) = \binom{50}{10} 0{,}25^{10}\, 0{,}75^{40} = 0{,}098518$$

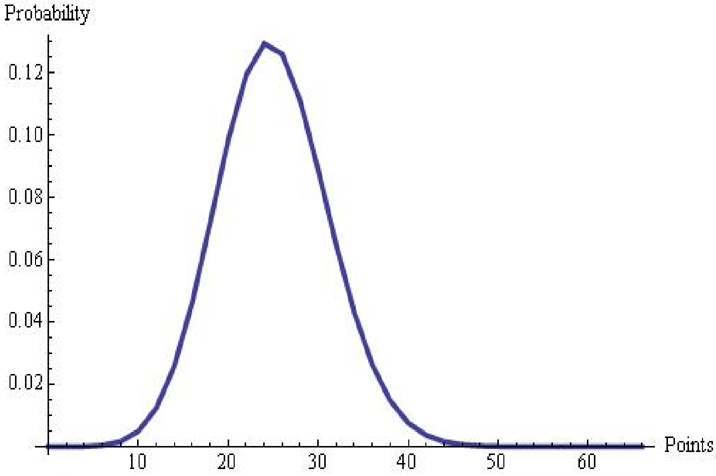
Analogously, we can calculate the probability *P(Z=k)* for other *k=0, 2, 4, 6, 8, ... , 98, 100* - see Table 2 (only for *k=0, ... , 66*, other probabilities are less than 10^{-9}) and Figure 2. For this calculation we used software Mathematica (Statistics ‘DiscreteDistributions’) – see Wolfram (1991).

Table 2 Distribution of number of points in the test (English)

Points from the test	Probability	Points from the test	Probability
0	0,000001	36	0,026390
2	0,000009	38	0,014816
4	0,000077	40	0,007655
6	0,000411	42	0,003645
8	0,001610	44	0,001602
10	0,004938	46	0,000650
12	0,012345	48	0,000244
14	0,025865	50	0,000084
16	0,046341	52	0,000027
18	0,072087	54	0,000008
20	0,098518	56	0,000002
22	0,119416	58	0,000001
24	0,129368	60	1×10^{-7}
26	0,126050	62	3×10^{-8}
28	0,111044	64	6×10^{-9}
30	0,088836	66	1×10^{-9}
32	0,064776	***	***
34	0,043184	Sum	1,000000

Source: Own construction

Fig. 2 Distribution of number of points in the test (polygon) – English



Source: Own construction

Example 5. Under assumption of random choice of answers we shall find probability that number of points from the test in English is

(a) 30 and more,
(b) 40 and more,
(c) 50 and more.

(a) Using notation from example 4 we have

$$P(Z \geq 30) = 1 - P(Z<30) = 1 - [\, P(Z=0)+ P(Z=2)+ P(Z=4)+ \ldots + P(Z=26)+ P(Z=28)\,]$$

Finally from Tab.2 we obtain

$$P(Z \geq 30) = 1 - 0{,}748080 = 0{,}251920.$$

Under assumption of random choice of answers approximately quarter of students get the test score 30 or more points.

(b) Analogously, from Tab.2 we obtain

$$P(Z \geq 40) = 1 - P(Z<40) = 1 - 0{,}986082 = 0{,}013918.$$

Under assumption of random choice of answers approximately 1,4% of students get the test score 40 or more points.

(c) Finally

$$P(Z \geq 50) = 1 - 0{,}999878 = 0{,}000122.$$

Under assumption of random choice of answers approximately 0,01% of students get the test score 50 or more points.

Example 6. Under assumption of random choice of answers we shall find expected number of points in the test in English and mode.

Using notation from example 4 we have $Z = 2X$. According to (2) we obtain

$$E(Z) = E(2X) = 2\,E(X) = 2 \,.\, 12{,}5 = 25.$$

Expected number of points in the test in English is 25. The mode is the most probable number of points. From Tab.2 is

$$\hat{z} = 24.$$

4 Conclusion

Probability that number of points from both tests (mathematics and English) is 50 and more is (random variables Y and Z in example 1 and 4 are independent)

$$P(Y \geq 50) \times P(Z \geq 50) = 0{,}011839 \times 0{,}000122 = 0{,}000001.$$

Approximately one student from million (under assumption of random choice of answers) successfully makes the entrance examinations at University of Economics. Multiple choice question tests are optimal for entrance examinations at University of Economics. These tests are objective (there is clearly no impact of any subjective factor in evaluation). Moreover, results can be evaluated quite easily for large number of students. From results of this paper follows that risk of acceptance students with lower performance levels is negligible.

Acknowledgment

This paper was processed with contribution of long term institutional support of research activities by Faculty of Informatics and Statistics, University of Economics, Prague (IP400040).

References:

[1] Feller, W. *An Introduction to Probability Theory and its Application.* John Wiley, 1970.

[2] Hrubý, M. (2016) 'Feedback improvement of question objects ', *International Journal of Continuing Engineering Education and Lifelong Learning*, vol. 26, no 2, pp. 183–195.

[3] Klůfa, J. *Mathematics for entrance examinations at University of Economics.* Ekopress, 2011. ISBN 978-80-86929-77-4

[4] Klůfa, J. (2015) 'Analysis of entrance examinations', Efficiency and Responsibility in Education, *Proceedings of the 12th International Conference*, Prague, pp. 250–256.

[5] Premadasa, I. (1993) 'A reappraisal of the use of multiple-choice questions', *Medical Teacher*, vol. 15, no. 2-3, pp. 237-242.

[6] Rao, C.R. *Linear Statistical Inference and Its Applications.* John Wiley, 1973.

[7] Rényi, A. *Teorie pravděpodobnosti.* Academia, 1972.

[8] Wolfram, S. *Mathematica.* Addison-Wesley, 1991.

[9] Zhao, Y. (2005) 'Algorithms for converting raw scores of multiple-choice question tests to conventional percentage marks', *International Journal of Engineering Education*, vol. 21, , no. 6, pp. 1189-1194.

[10] Zhao, Y. (2006) 'How to design and interpret a multiple-choice-question test: A probabilistic approach', *International Journal of Engineering Education*, vol. 22, no. 6, pp. 1281-1286.