On Buffon needle problem for an irregular lattice

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Abstract: In the previous papers [1] and [6] the authors introduced in the Buffon-Laplace type problems so-called obstacles. They considered two lattices and considering a classic Buffon type problem introducing in the first moment the maximum value of probability, i.e. reducing the probability interval and in the second considering an irregular lattice. In [5] Caristi and Ferrara considered also a Buffon type problem considering the possibles deformations of the lattice and in [2] Caristi, Puglisi and Stoka considered another particular regular lattices with eight sides. Fengfan and Deyi [4] study similar problem using two concepts, the generalized support function and restricted chord function, both referring to the convex set, which were introduced by Delin in [3]. In this paper, we consider another particular irregular lattice (see fig. 1) and considering the formula of the kinematic measure of Poincaré [7] and the result of Stoka [9] we study a Buffon problem for this irregular lattice. We determine the probability of intersection of a body test needle of length $l$, $l < \frac{a}{3}$.

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1 Preliminaires

In this section we present some results and considerations that will be needed in the rest of the paper.

Consider the irregular lattice $\mathcal{R}$ with a fundamental region $C_0$ composed of the union by four triangles and an exagon (fig. 1) with $a \leq b$:

![fig.1](image)

We know that, any congruent polygon can be inlaid in a plane. In this way we obtain a lattice that covers the plane. A set of points in the plane is called a domain if it is open and connected. A set of points is called a region if it is the union of a domain with some, or all of its boundary points. From the lattice of fundamental regions in the plane, we understand a sequence of congruent regions that represent the Santalò conditions [8]:

With the notations of this figure we have

$$b = \frac{2a}{3} \cot \alpha, \quad |GL| = |HM| = |LE| = |MF| = \frac{a}{3 \sin \alpha},$$

$$\text{area} C_0 = \frac{2a^2}{3}, \quad \arctg \frac{2}{3} \leq \alpha \leq \frac{\pi}{4}.$$

We want to compute the probability that a segment $s$ with random position and of constant length $l, l < \frac{a}{3}$ intersects a side of lattice $\mathcal{R}$, i.e. the probability $P_{int}$ that a segment $s$ intersects a side of the fundamental cell $C_0$.

The position of the segment $s$ is determined by its middle point and by the angle $\varphi$ that $s$ formed with the line $AD \circ BC$.

To compute the probability $P_{int}$ we consider the limiting positions of segment $s$, for a specified value of $\varphi$, in the cells $C_{0i}, (i = 1, 2, 3)$(fig.2).
By denoting $M_i (i = 1, ..., 5)$ as the set of segments $s$ which have their center in $C_{0i}$ and $N_i$ the set of segments $s$ all contained in the cell $C_{0i}$ we have [9]:

$$P_{int} = 1 - \frac{\sum_{i=1}^{5} \mu (N_i)}{\sum_{i=1}^{5} \mu (M_i)},$$

(1)

where $\mu$ is the Lebesgue measure in the Euclidean plane.

To compute the above measure $\mu (M_i)$ and $\mu (N_i)$ we use the Poincaré kinematic measure [7] $\text{dk} = dx \wedge dy \wedge d\phi$, where $x, y$ are the coordinates of the middle point of $s$ and $\phi$ is the fixed angle.

## 2 Main results

Considering that $l < \frac{a}{3}$ we can prove

**Theorem.** The probability that a random segment $s$ of constant length $l < \frac{a}{3}$ intersects a side of lattice $R$ is:

$$P_{int} = \frac{3tga}{(\pi - 2\alpha) a^2} \left\{ \frac{al}{3} \left( 4 - 4 \sin \alpha + \right. \right.$$  

$$\left. \left. ctga + 5cga \cos \alpha + \right. \right.$$  

$$\frac{l^2}{4} \left[ 3 + 2 \sin 2\alpha - 5 \cos 2\alpha + \right.$$  

$$(1 - tga + cga) (\pi - 2\alpha)) \right\} .$$

(2)

**Proof.** Taking into account the symmetries of the lattice and the different values of $\phi$ we have:

$$\text{area} \hat{C}_{01}(\phi) = \text{area} C_{01} - \sum_{i=1}^{5} \text{area} a_i(\phi),$$

$$\text{area} \hat{C}_{02}(\phi) = \text{area} C_{02} - \sum_{i=1}^{5} \text{area} b_i(\phi)$$

and

$$\text{area} \hat{C}_{03}(\phi) = \text{area} C_{03} - \sum_{i=1}^{5} \text{area} c_i(\phi).$$

We obtain that:

$$\mu (M_i) = \int_{\alpha}^{\frac{\pi}{2}} d\phi \int_{\{(x,y) \in C_{0i}\}} dx dy =$$

$$\int_{\alpha}^{\frac{\pi}{2}} (\text{area} C_{0i}) d\phi = \left( \frac{\pi}{2} - \alpha \right) \text{area} C_{0i},$$

$$(i = 1, ..., 5).$$

Then

$$\sum_{i=1}^{5} \mu (M_i) = \left( \frac{\pi}{2} - \alpha \right) \sum_{i=1}^{5} \text{area} C_{0i} =$$

$$\left( \frac{\pi}{2} - \alpha \right) \text{area} C_0 = \frac{(\pi - 2\alpha) \text{ctga} \alpha}{3} a^2 .$$

(3)

In same way to compute $\mu (N_i)$ we have that:

$$A_1(\phi) = A_3(\phi) = \sum_{i=1}^{5} \text{area} a_i(\phi) =$$

$$\frac{al}{6} [\text{ctga} \cos \phi + (\text{ctga} + 1) \sin \phi] -$$

$$\frac{l^2}{4} \left[ \left( 1 + \text{ctga} \right) \sin 2\phi + 1 - \cos 2\phi \right],$$

$$A_2(\phi) = A_4(\phi) = \sum_{i=1}^{5} \text{area} b_i(\phi) =$$

$$\frac{al}{3} \left( \cos \phi + \text{ctga} \sin \phi \right) -$$

$$\frac{l^2}{4} \left[ 2 \sin 2\phi + (\text{ctga} - \text{ctga}) \cos 2\phi + \text{ctga} + \text{ctga} \right],$$

and

$$A_5(\phi) = \sum_{i=1}^{8} \text{area} c_i(\phi) = \frac{al}{3} \left( \cos \phi + \text{ctga} \sin \phi \right) -$$

$$\frac{l^2}{4} \left[ \sin 2\phi - \text{ctga} \cos 2\phi - \text{ctga} \right].$$
Then we obtain that:

\[ \mu(N_i) = \int_{\alpha}^{\pi/2} d\phi \int \int \{ (x,y) \in C_0i(\phi) \} \, dx \, dy = \]

\[ \int_{\alpha}^{\pi/2} \left[ \text{area} \tilde{C}_0i(\phi) \right] \, d\phi = \int_{\alpha}^{\pi/2} \left[ \text{area} C_0i \, - \, A_i(\phi) \right] \, d\phi = \]

\[ \left( \frac{\pi}{2} - \alpha \right) \text{area} C_0i \, - \, \int_{\alpha}^{\pi/2} \left[ A_i(\phi) \right] \, d\phi. \]

and

\[ \sum_{i=1}^{3} \mu(N_i) = \frac{\left( \pi - 2\alpha \right) \text{ctg} \alpha \, a^2}{3} - \int_{\alpha}^{\pi/2} \left[ \sum_{i=1}^{3} A_i(\phi) \right] \, d\phi. \quad (4) \]

In the end, from (1), (3) and (4) we obtain the probability (2).

References:


