On Buffon needle problem for an irregular lattice

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Abstract: In the previous papers [1] and [6] the authors introduced in the Buffon-Laplace type problems so-called obstacles. They considered two lattices and considering a classic Buffon type problem introducing in the first moment the maximum value of probability, i.e. reducing the probability interval and in the second considering an irregular lattice. In [5] Caristi and Ferrara considered also a Buffon type problem considering the possibles deformations of the lattice and in [2] Caristi, Puglisi and Stoka considered another particular regular lattices with eight sides. Fengfan and Deyi [4] study similar problem using two concepts, the generalized support function and restricted chord function, both referring to the convex set, which were introduced by Delin in [3]. In this paper, we consider another particular irregular lattice (see fig. 1) and considering the formula of the kinematic measure of Poincaré [7] and the result of Stoka [9] we study a Buffon problem for this irregular lattice. We determine the probability of intersection of a body test needle of length l, $l < \frac{a}{3}$.

Key–Words: Geometric probability, integral geometry, Buffon problem, lattice of regions, kinematic measure
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1 Preliminaires

In this section we present some results and considerations that will be needed in the rest of the paper.

Consider the irreguar lattice $\Re$ with a fundamental region C_0 composed of the union by four trinagles and an exagon (fig. 1) with $a \leq b$:

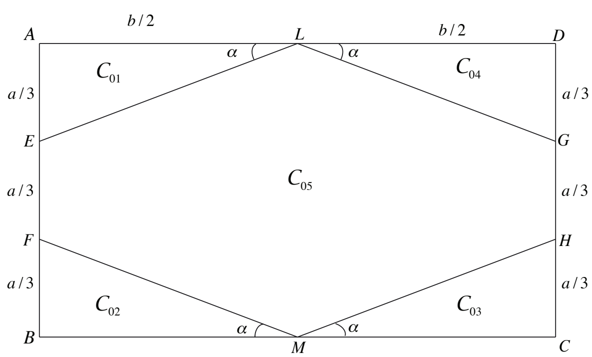

fig.1

We know that, any congruent polygon can be inlaid in a plane. In this way we obtain a lattice that covers the plane. A set of points in the plane is called a domain if it is open and connected. A set of points is called a region if it is the union of a domain with some, or all of its boundary points. From the lattice of fundamental regions in the plane, we understand a sequence of congruent regions that represent the Santalò conditions [8]:

With the notations of this figure we have

$$b = \frac{2a}{3} ctg\alpha, \qquad |GL| = |HM| = |LE| =$$

$$|MF| = \frac{a}{3\sin\alpha},$$

$$areaC_0 = \frac{2a^2}{3}, \qquad Arctg\frac{2}{3} \leq \alpha \leq \frac{\pi}{4}.$$

We want to compute the probability that a segment s with random position and of constant length l, $l < \frac{a}{3}$ intersects a side of lattice $\Re$, i.e. the probability P_{int} that a segment s intersects a side of the fundamental cell C_0.

The position of the segment s is determinated by its middle point and by the angle φ that s formed with the line AD o BC.

To compute the probability P_{int} we consider the limiting positions of segment s, for a specified value of φ, in the cells $C_{0i}, (i = 1, 2, 3)$(fig.2).

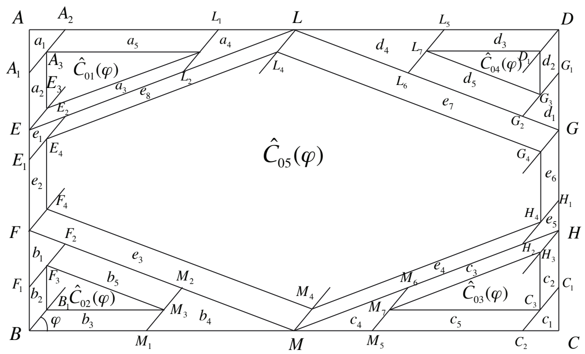

fig.2

By denoting $M_i\,(i=1,...,5)$ as the set of segments s which have their center in C_{0i} and N_i the set of segments s all contained in the cell C_{0i} we have [9]:

$$P_{int}=1-\frac{\sum_{i=1}^{5}\mu\left(N_i\right)}{\sum_{i=1}^{5}\mu\left(M_i\right)}, \qquad (1)$$

where μ is the Lebesgue measure in the Euclidean plane.

To compute the above measure $\mu\left(M_i\right)$ and $\mu\left(N_i\right)$ we use the Poincaré kinematic measure [7] $dk=dx\wedge dy\wedge d\varphi$, where x, y are the coordinates of the middle point of s and φ is the fixed angle.

2 Main results

Considering that $l<\frac{a}{3}$ we can prove

Theorem. The probability that a random segment s of constant length $l<\frac{a}{3}$ intersects a side of lattice $\Re$ is:

$$P_{int}=\frac{3tg\alpha}{\left(\pi-2\alpha\right)a^2}\left\{\frac{al}{3}\left(4-4\sin\alpha+ctg\alpha+5ctg\alpha\cos\alpha\right)+\frac{l^2}{4}\left[3+2\sin2\alpha-5\cos2\alpha+\left(1-tg\alpha+ctg\alpha\right)\left(\pi-2\alpha\right)\right]\right\}. \qquad (2)$$

Proof. Taking into account the symmetries of the lattice and the different values of φ we have:

$$area\widehat{C}_{01}\left(\varphi\right)=areaC_{01}-\sum_{i=1}^{5}areaa_i\left(\varphi\right),$$

$$area\widehat{C}_{02}\left(\varphi\right)=areaC_{02}-\sum_{i=1}^{5}areab_i\left(\varphi\right)$$

$$area\widehat{C}_{03}\left(\varphi\right)=areaC_{03}-\sum_{i=1}^{5}areac_i\left(\varphi\right).$$

$$area\widehat{C}_{04}\left(\varphi\right)=areaC_{04}-\sum_{i=1}^{5}aread_i\left(\varphi\right)$$

$$area\widehat{C}_{05}\left(\varphi\right)=areaC_{05}-\sum_{i=1}^{5}areae_i\left(\varphi\right)$$

We obtain that:

$$\mu\left(M_i\right)=\int_{\alpha}^{\frac{\pi}{2}}d\varphi\int\int_{\{(x,y)\epsilon C_{0i}\}}dxdy=\int_{\alpha}^{\frac{\pi}{2}}\left(areaC_{0i}\right)d\varphi=\left(\frac{\pi}{2}-\alpha\right)areaC_{0i},$$

$$\left(i=1,...,5\right).$$

then

$$\sum_{i=1}^{5}\mu\left(M_i\right)=\left(\frac{\pi}{2}-\alpha\right)\sum_{i=1}^{5}areaC_{0i}=\left(\frac{\pi}{2}-\alpha\right)areaC_0=\frac{\left(\pi-2\alpha\right)ctg\alpha}{3}a^2. \qquad (3)$$

In same way to compute $\mu\left(N_i\right)$ we have that:

$$A_1\left(\varphi\right)=A_3\left(\varphi\right)=\sum_{i=1}^{5}areaa_i\left(\varphi\right)=\frac{al}{6}\left[ctg\alpha\cos\varphi+\left(ctg\varphi+1\right)\sin\varphi\right]-\frac{l^2}{4}\left[\left(1+ctg\alpha\right)\sin2\varphi+1-\cos2\varphi\right],$$

$$A_2\left(\varphi\right)=A_4\left(\varphi\right)=\sum_{i=1}^{5}areab_i\left(\varphi\right)=\frac{al}{3}\left(\cos\varphi+ctg\alpha\sin\varphi\right)-\frac{l^2}{4}\left[2\sin2\varphi+\left(tg\alpha-ctg\alpha\right)\cos2\varphi+tg\alpha+ctg\alpha\right],$$

and

$$A_5\left(\varphi\right)=\sum_{i=1}^{8}areae_i\left(\varphi\right)=\frac{al}{3}\left(\cos\varphi+ctg\alpha\sin\varphi\right)-\frac{l^2}{4}\left[\sin2\varphi-tg\alpha\cos2\varphi-tg\alpha\right].$$

Then we obtain that:

$$\mu(N_i) = \int_{\alpha}^{\frac{\pi}{2}} d\varphi \int\int_{\left\{(x,y)\epsilon \widehat{C}_{0i}(\varphi)\right\}} dxdy =$$

$$\int_{\alpha}^{\frac{\pi}{2}} \left[area\widehat{C}_{0i}(\varphi)\right] d\varphi = \int_{\alpha}^{\frac{\pi}{2}} \left[areaC_{0i} - A_i(\varphi)\right] d\varphi =$$

$$\left(\frac{\pi}{2} - \alpha\right) areaC_{0i} - \int_{\alpha}^{\frac{\pi}{2}} \left[A_i(\varphi)\right] d\varphi.$$

and

$$\sum_{i=1}^{3} \mu(N_i) = \frac{(\pi - 2\alpha)\, ctg\alpha}{3} a^2 - \int_{\alpha}^{\frac{\pi}{2}} \left[\sum_{i=1}^{3} A_1(\varphi)\right] d\varphi. \tag{4}$$

In the end, from (1), (3) and (4) we obtain the probability (2).

References:

[1] D. Barilla, G. Caristi, A. Puglisi and M. Stoka, A Buffon-Laplace type problems for an irregular lattice with maximum probability, Applied Mathematical Sciences, vol. 8 (2014),no. 165, pp. 8287-8293.

[2] G. Caristi, A. Puglisi and M. Stoka, A Laplace type problem for regular lattices with octagonal cell, Far East Journal of Mathematical Sciences, vol. 48, issue 1, January 2011, pp. 103-118.

[3] R. Delin, Topics in Integral Geometry, Singapore, New Jersey, London, and Hongkong: World Scientic, 1994.

[4] X. Fengfan and L. Deyi, On generalized Buffon Needle problem for lattices, Acta Mathematica Scientia 2011, 31B(1), pp. 303-308.

[5] G. Caristi and M. Ferrara, On Buffon's problem for a lattice and its deformations. Beitrage zur Algebra und Geometrie, 2004, 45(1), pp. 13-20.

[6] G. Caristi and M. Stoka, A Buffon-Laplace type problem for an irregular lattice with "body test" rectangle, Applied Mathematical Sciences, vol. 8 (2014), pp. 8395–8401.

[7] H. Poincarè, Calcul des probabilitès, 2nd ed., Gauthier-Villard, Paris, 1912.

[8] L. A. Santalò, Integral Geometry and Geometric Probability, London: Addison-Wesley Publishing Company, 1976.

[9] M. Stoka, Probabilités géométriques de type "Buffon" dans le plan euclidien, Atti Accd. Sci. Torino, T, 110.