Analysis of cooperation and competition relationship among two and three firms

MIHAELA NEAMŢU NICOLETA SÎRGHI
West University of Timişoara
Dept. of Economics and Modelling
Pestalozzi Street 16, 300115 Timisoara
ROMANIA
mihaela.neamtu@e-uvt.ro, nicoleta.sirghi@e-uvt.ro

DORIANA RAMESCU
West University of Timişoara
Faculty of Economics and Business Administration
Pestalozzi Street 16, 300115 Timisoara
ROMANIA
doriana.ramescu@e-uvt.ro

Abstract: This paper studies the cooperation or competition relationship from the perspective of dynamic games, where two and three firms are considered, respectively. The associated mathematical models are described by systems with nonlinear delayed differential equations. The equilibrium points are determined and the existence of the Hopf bifurcation is investigated. The conclusions and future works are provided.

Key–Words: equilibrium points, Hopf bifurcation, replicator dynamics, stability

1 Introduction

The economical evolutionary game comes from the biological evolutionary theory which relies on the idea of the survival of the fittest entity that it is based on the interaction of behavior strategy and the iteration process. It can use differential equations for modeling the choice of the population among strategies. Each player gains a fitness associated to a certain strategy [12]. The strategy of choice is the one that provides a payoff greater than or equal to the average payoff. A standard approach is one of the replicator dynamics, where the relative adjustment of the strategy dynamics is proportional to the range of payoffs that is more than the average payoff. In the existing literature, due to the fact that the outcome of a chosen strategy involves delays, there has been several works that studied the delayed replicator dynamics as shown in [1], [3], [5], [9], [11], [15].

The evolutionary game theory takes into account the bounded rationality, repeatedly performs game activities through constant imitating and learning [10]. The replicator dynamics describes the strategy evolution and is given by an ordinary differential equation [14]. In [2], the authors analyze the replicator dynamics with distributed delays. In [11], a single fixed delay is considered in the fitness function and the critical delay is determined when the stability of the equilibrium point is lost. In [15], the existence of the Hopf bifurcation in the two-strategy replicator equations is examined, where the time delay is included in the fitness of each strategy.

Based on the previous considerations, in Section 2 we present the mathematical model that describes two firms as bounded rational players in a game which have to choose between two strategies, one being cooperation and the other one competition. Section 3 analyses the existence of the Hopf bifurcation. In Section 4 we study the mathematical model with three firms and two strategies. Section 5 provides the conclusions and future works.

2 The mathematical model for two firms and two strategies

First of all, we consider two firms as bounded rational players in a game which have to choose between two strategies, one being cooperation and the other one competition. The difference between the two strategies is related to the cost to be paid by each player when they choose cooperation while the cost is 0 when the competition is preferred.

When both players go for cooperation, the resources and benefits are shared. Let a_1 and b_1 be the net income of the players, respectively.

In case of competition being chosen by both players, there are neither shared resources nor benefits, each player pays the transaction cost alone and therefore the income is not shared. The net incomes are a_4 and b_4, respectively.

When the players go for different strategies, for the one which chooses cooperation it implies that his/her resources are shared, the net income is uncertain and the costs are greater than or equal to the ones associated to the cooperation of two sides. For the one which chooses competition it implies: neither his/her resources nor the net income are shared and the cost is 0.

If the first firm chooses cooperation and the second one chooses competition, the net income for the first one is a_2 and for the second one is b_2. If the first firm chooses competition and the second one chooses cooperation, the net income for the first one is a_3, and for the second one is b_3.

Based on the above considerations the payoffs matrixes of the firms are given by:

Nr	Portfolio Strategies	Firm 1	Firm 2
1	x y	a_1	b_1
2	x 1-y	a_2	b_2
3	1-x y	a_3	b_3
4	1-x 1-y	a_4	b_4

The payoff of the first player in case of cooperative strategy is:

$$E(U_1)_x = ya_1 + (1-y)a_2 \tag{1}$$

and

$$E(U_1)_{1-x} = ya_3 + (1-y)a_4. \tag{2}$$

for a competitive strategy. Thus, the average payoff of the mixed strategy is:

$$E(\bar{U}_1) = xE(U_1)_x + (1-x)E(U_1)_{1-x}. \tag{3}$$

The preferred strategy is the one that delivers a payoff greater than or equal to the average payoff. The change rate of the probability for the first player to choose cooperation is proportional to the range of payoffs that is more than the average payoff [13]:

$$\dot{x}(t) = x(t)(E(U_1)_x - E(\bar{U}_1)). \tag{4}$$

The payoff of the second player having a cooperative strategy is:

$$E(U_2)_y = xb_1 + (1-x)b_3 \tag{5}$$

and

$$E(U_2)_{1-y} = xb_2 + (1-x)b_4. \tag{6}$$

for a competitive strategy. Thus, the average payoff of the mixed strategy for the second player is:

$$E(\bar{U}_2) = yE(U_2)_y + (1-y)E(U_2)_{1-y}. \tag{7}$$

Therefore, the change rate of the probability for the second player to choose cooperation is proportional to the range of payoffs that is more than the average payoff:

$$\dot{y}(t) = y(t)(E(U_2)_y - E(\bar{U}_2)). \tag{8}$$

Replacing (1) with (2) in (3) and (5) with (6) in (7) we obtain the dynamic replicator is given by:

$$\begin{aligned} \dot{x}(t) &= x(t)(1-x(t))(\alpha_0 + \alpha_2 y(t)) \\ \dot{y}(t) &= y(t)(1-y(t))(\beta_0 + \beta_1 x(t)) \end{aligned} \tag{9}$$

where

$$\alpha_0 = a_2 - a_4, \alpha_2 = a_1 - a_2 - a_3 + a_4 \tag{10}$$

$$\beta_0 = b_3 - b_4, \beta_1 = b_1 - b_2 - b_3 + b_4. \tag{11}$$

Due to the fact that a firm chooses a strategy at time t, the payoff is gained after a delay, in what following we analyze the dynamic replicator with time delay given by:

$$\begin{aligned} \dot{x}(t) &= x(t)(1-x(t-\tau_1))(\alpha_0 + \alpha_2 y(t)) \\ \dot{y}(t) &= y(t)(1-y(t-\tau_2))(\beta_0 + \beta_1 x(t)) \end{aligned} \tag{12}$$

with $\tau_1 \geq 0$, $\tau_2 \geq 0$.

3 Hopf bifurcation analysis

The equilibrium points of (12) are: $O(0,0)$, $A(0,1)$, $B(1,0)$, $C(1,1)$.

If $d_1 = -\frac{\alpha_0}{\alpha_2}$ and $d_2 = -\frac{\beta_0}{\beta_1}$ with $|\beta_0| < |\beta_1|$, $\beta_0\beta_1 < 0$, $|\alpha_0| < |\alpha_2|$, $\alpha_0\alpha_2 < 0$, then $D(d_1, d_2)$ is an equilibrium point as well.

Let $E(x_0, y_0)$ be an equilibrium point. The characteristic equation of (12) is given by:

$$\begin{aligned} &\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} - \\ &-b_{22}(\lambda - a_{11})e^{-\lambda\tau_2} - b_{11}(\lambda - a_{22})e^{-\lambda\tau_1} + \\ &b_{11}b_{22}e^{-\lambda(\tau_1+\tau_2)} = 0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} &a_{11} = (1-x_0)(\alpha_0 + \alpha_2 y_0), \\ &b_{11} = -x_0(\alpha_0 + \alpha_2 y_0), \\ &a_{22} = (1-y_0)(\beta_0 + \beta_1 x_0), \\ &b_{22} = -y_0(\beta_0 + \beta_1 x_0), \\ &a_{12} = x_0(1-x_0)\alpha_2, a_{21} = y_0(1-y_0)\beta_1. \end{aligned} \tag{14}$$

Using (13) and (14) we have:

Proposition 1 *If $a_2 - a_4 < 0$, $b_3 - b_4 < 0$, the equilibrium point $O(0,0)$ is locally asymptotically stable, for all $\tau_1 \geq 0$, $\tau_2 \geq 0$.*

Proposition 2 *If $a_1 - a_3 < 0$, $b_3 - b_4 > 0$, $\tau_2 = 0$, the equilibrium point $A(0,1)$ is locally asymptotically stable, for all $\tau_1 \geq 0$.*

If $\tau_2 > 0$, then the characteristic equation (13) is:

$$(\lambda - (a_1 - a_3))(\lambda + (b_3 - b_4)e^{-\lambda\tau_2}) = 0. \quad (15)$$

Let $a_1 - a_3 < 0$, $b_3 - b_4 > 0$, $\tau_2 > 0$. Let $\lambda = -i\omega_0$ be a root of the equation:

$$\lambda + (b_3 - b_4)e^{-\lambda\tau_2} = 0. \quad (16)$$

From (16) we obtain:

$$\omega_0 = b_3 - b_4, \tau_{20} = \frac{\pi}{2(b_3 - b_4)}.$$

Then:

Proposition 3 *If $a_1 - a_3 < 0$, $b_3 - b_4 > 0$, the equilibrium point $A(0,1)$ is locally asymptotically stable, for $\tau_1 = 0$, $0 < \tau_2 < \tau_{20}$. When $\tau_2 = \tau_{20}$ a Hopf bifurcation occurs.*

Proposition 4 *If there are no delays, $b_1 - b_2 < 0$, $a_2 - a_4 > 0$, the equilibrium point $B(1,0)$ is locally asymptotically stable. If $b_1 - b_2 < 0$, $a_2 - a_4 > 0$, $B(1,0)$ is locally asymptotically stable, for any $0 < \tau_1 < \tau_{10}$. When $\tau_{10} = \dfrac{\pi}{2(a_2 - a_4)}$ a Hopf bifurcation occurs.*

Proposition 5 *If there are no delays and $a_1 - a_3 > 0$, $b_1 - b_2 > 0$, the equilibrium point $C(1,1)$ is locally asymptotically stable. If $a_1 - a_3 > 0$, $b_1 - b_2 > 0$, $C(1,1)$ is locally asymptotically stable, for any $0 < \tau_1, \tau_2 < \min\{\tau_{10}, \tau_{20}\} = \tau_{12}$, where $\tau_{10} = \dfrac{\pi}{2(a_1 - a_3)}$, $\tau_{20} = \dfrac{\pi}{2(b_1 - b_2)}$. A Hopf bifurcation occurs when $\tau_1 = \tau_2 = \tau_{12}$.*

Proposition 6 *If $\tau_1 = 0$, $\tau_2 = 0$ the equilibrium point $D(d_1, d_2)$ is a saddle point.*

4 The mathematical model with three firms and two strategies

Let F_i, $i = 1,2,3$ be three firms. Firm F_1 chooses cooperation with the probability $x(0 \le x \le 1)$ and then the probability $1 - x$ is the probability for competition. Firm F_2 chooses cooperation with the probability $y(0 \le y \le 1)$ and competition with $1 - y$. For the third firm the associated probabilities are z and $1 - z$.

In the tripartite game there are eight portfolio strategies and the corresponding payoffs are:

Nr	Strategies	Firm 1	Firm 2	Firm 3
1	x y z	a_1	b_1	c_1
2	x y 1-z	a_2	b_2	c_2
3	x 1-y z	a_3	b_3	c_3
4	x 1-y 1-z	a_4	b_4	c_4
5	1-x y z	a_5	b_5	c_5
6	1-x y 1-z	a_6	b_6	c_6
7	1-x 1-y z	a_7	b_7	c_7
8	1-x 1-y 1-z	a_8	b_8	c_8

In a similar case as in the previous case the delayed dynamics replicator is given by:

$$\begin{array}{l}
\dot{x}(t) = x(t)(1 - x(t-\tau_1))(\alpha_0 + \alpha_2 y(t) + \alpha_3 z(t) + \\
+\alpha_{23} y(t) z(y)) \\
\dot{y}(t) = y(t)(1 - y(t-\tau_2))(\beta_0 + \beta_1 x(t) + \beta_3 z(t) + \\
+\beta_{13} x(t) z(t)) \\
\dot{z}(t) = z(t)(1 - z(t-\tau_3))(\gamma_0 + \gamma_1 x(t) + \gamma_2 y(t) + \\
+\gamma_{12} x(t) y(t))
\end{array} \quad (17)$$

with $\tau_1 \ge 0$, $\tau_2 \ge 0$, $\tau_3 \ge 0$ and

$$\begin{array}{l}
\alpha_0 = a_4 - a_8, \alpha_2 = a_2 - a_4 - a_6 + a_8, \\
\alpha_3 = a_3 - a_4 - a_7 + a_8, \\
\alpha_{23} = a_1 - a_2 - a_3 + a_4 - a_5 + a_6 + a_7 - a_8, \\
\beta_0 = b_6 - b_8, \beta_1 = b_2 - b_4 - b_6 + b_8, \\
\beta_3 = b_5 - b_6 - b_7 + b_8, \\
\beta_{13} = b_1 - b_2 - b_3 + b_4 - b_5 + b_6 + b_7 - b_8, \\
\gamma_0 = c_7 - c_8, \gamma_1 = c_3 - c_4 - c_7 + c_8, \\
\gamma_2 = c_5 - c_6 - c_7 + c_8, \\
\gamma_{12} = c_1 - c_2 - c_3 + c_4 - c_6 + c_7 - c_8.
\end{array} \quad (18)$$

The equilibrium point that do not depend on the coefficients of the matrix are: $A_0(0,0,0)$, $A_1(1,0,0)$, $A_2(0,1,0)$, $A_3(0,0,1)$, $A_{12}(1,1,0)$, $A_{13}(1,0,1)$, $A_{23}(0,1,1)$, $A_{123}(1,1,1)$.

Let $S(x_0, y_0, z_0)$ be an equilibrium point. The characteristic equation of (19) is given by:

$$\begin{array}{l}
(\lambda - a_{11} - b_{11}e^{-\lambda\tau_1})(\lambda - a_{22} - b_{22}e^{-\lambda\tau_2}) \cdot \\
\cdot(\lambda - a_{33} - b_{33}e^{-\lambda\tau_3}) - (a_{13}a_{31} + a_{12}a_{21} + a_{23}a_{32})\lambda - \\
-a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13} + a_{13}a_{31}a_{21} + a_{12}a_{21}a_{33} + \\
+a_{23}a_{32}a_{11} + a_{13}a_{31}b_{22}e^{-\lambda\tau_2} + a_{12}a_{21}b_{33}e^{-\lambda\tau_3} + \\
+a_{23}a_{32}b_{11}e^{-\lambda\tau_1} = 0,
\end{array} \quad (19)$$

where

$$
\begin{aligned}
a_{11} &= (1-x_0)(\alpha_0+\alpha_2 y_0+\alpha_3 z_0+\alpha_{23}y_0z_0),\\
a_{22} &= (1-y_0)(\beta_0+\beta_1 x_0+\beta_3 z_0+\beta_{13}x_0z_0),\\
a_{33} &= (1-z_0)(\gamma_0+\gamma_1 x_0+\gamma_2 y_0+\gamma_{12}x_0y_0),\\
a_{12} &= x_0(1-x_0)(\alpha_2+\alpha_0 z_0),\\
a_{21} &= y_0(1-y_0)(\beta_1+\beta_{13}z_0),\\
a_{13} &= x_0(1-x_0)(\alpha_3+\alpha_{23}y_0),\\
a_{31} &= z_0(1-z_0)(\alpha_1+\alpha_{12}y_0),\\
a_{23} &= y_0(1-y_0)(\alpha_3+\alpha_{23}y_0),\\
a_{32} &= z_0(1-z_0)(\gamma_2+\gamma_{12}x_0),\\
b_{11} &= -x_0(\alpha_0+\alpha_2 y_0+\alpha_3 z_0+\alpha_{23}y_0z_0),\\
b_{22} &= -y_0(\beta_0+\beta_1 x_0+\beta_3 z_0+\beta_{13}x_0z_0),\\
b_{23} &= y_0(1-y_0)(\beta_3+\beta_{13}x_0),\\
b_{33} &= -z_0(\gamma_0+\gamma_1 x_0+\gamma_2 y_0+\gamma_{12}x_0y_0).
\end{aligned} \tag{20}
$$

Using (19) and (20) we have the following statements:

Proposition 7 *If* $a_4-a_8<0$, $b_6-b_8<0$, $c_7-c_8<0$ *the equilibrium point* $A_0(0,0,0)$ *is locally asymptotically stable, for all* $\tau_1\geq 0$, $\tau_2\geq 0$, $\tau_3\geq 0$.

Proposition 8 *If* $a_4-a_8>0$, $b4-b_2<0$, $c_4-c_3<0$, $\tau_i=0$, $i=1,2,3$, *the equilibrium point* $A_1(1,0,0)$ *is locally asymptotically stable. If* $a_4-a_8>0$, $b_4-b_2<0$, $c_4-c_3<0$ *and* $\tau_1\in[0,\tau_{10})$, *where* $\tau_{10}=\dfrac{\pi}{2(a_4-a_8)}$, *then* $A_1(1,0,0)$ *is locally asymptotically stable for any* $\tau_2\geq 0$, $\tau_3\geq 0$. *A Hopf bifurcation occurs when* $\tau_1=\tau_{10}$.

Proposition 9 *If* $a_2-a_6<0$, $b_6-b_8>0$, $c_5-c_6<0$, $\tau_i=0$, $i=1,2,3$, *the equilibrium point* $A_2(0,1,0)$ *is locally asymptotically stable. If* $a_2-a_6<0$, $b_6-b_8>0$, $c_5-c_6<0$ *and* $\tau_2\in[0,\tau_{20})$, *where* $\tau_{20}=\dfrac{\pi}{2(b_6-b_8)}$, *then* $A_2(0,1,0)$ *is locally asymptotically stable for any* $\tau_1\geq 0$, $\tau_3\geq 0$. *A Hopf bifurcation occurs when* $\tau_2=\tau_{20}$.

Proposition 10 *If* $a_3-a_7<0$, $b_5-b_7<0$, $c_7-c_8>0$, $\tau_i=0$, $i=1,2,3$, *the equilibrium point* $A_3(0,0,1)$ *is locally asymptotically stable. If* $a_3-a_7<0$, $b_5-b_7<0$, $c_7-c_8>0$ *and* $\tau_3\in[0,\tau_{30})$, *where* $\tau_{30}=\dfrac{\pi}{2(c_7-c_8)}$, *then* $A_3(0,0,1)$ *is locally asymptotically stable for any* $\tau_1\geq 0$, $\tau_2\geq 0$. *A Hopf bifurcation occurs when* $\tau_3=\tau_{30}$.

Proposition 11 *If* $a_2-a_6>0$, $b_2-b_4>0$, $c_3-c_4+c_5-c_6-c_7+c_8>0$, $\tau_i=0$, $i=1,2,3$ *the equilibrium point* $A_{12}(1,1,0)$ *is locally asymptotically stable. If* $a_2-a_6>0$, $b_2-b_4>0$, $c_3-c_4+c_5-c_6-c_7+c_8>0$ *and* $\tau_1=\tau_2\in[0,\tau_{12})$, *where* $\tau_{12}=\min\{\tau_{10},\tau_{20}\}$, $\tau_{10}=\dfrac{\pi}{2(a_2-a_6)}$, $\tau_{20}=\dfrac{\pi}{2(b_2-b_4)}$, *then* $A_{12}(1,1,0)$ *is locally asymptotically stable for any* $\tau_3\geq 0$. *A Hopf bifurcation occurs when* $\tau_1=\tau_2=\tau_{12}$.

Proposition 12 *If* $a_3-a_7>0$, $b_2-b_4+b_5-b_6-b_7+b_8<0$, $c_3-c_4>0$, $\tau_1=\tau_2=\tau_3=0$, *the equilibrium point* $A_{13}(1,0,1)$ *is locally asymptotically stable. If* $a_3-a_7>0$, $b_2-b_4+b_5-b_6-b_7+b_8<0$, $c_3-c_4>0$ *and* $\tau_1=\tau_3\in[0,\tau_{13})$, *where* $\tau_{13}=\min\{\tau_{10},\tau_{30}\}$, $\tau_{10}=\dfrac{\pi}{2(a_3-a_7)}$, $\tau_{30}=\dfrac{\pi}{2(c_6-c_4)}$, *then* $A_{13}(1,0,1)$ *is locally asymptotically stable for any* $\tau_2\geq 0$. *A Hopf bifurcation occurs when* $\tau_1=\tau_3=\tau_{13}$.

Proposition 13 *If* $a_1-a_5<0$, $b_7-b_5>0$, $c_6-c_5>0$, $\tau_1=\tau_2=\tau_3=0$, *the equilibrium point* $A_{23}(0,1,1)$ *is locally asymptotically stable. If* $a_1-a_5<0$, $b_7-b_5>0$, $c_6-c_5>0$ *and* $\tau_2=\tau_3\in[0,\tau_{23})$, *where* $\tau_{23}=\min\{\tau_{20},\tau_{30}\}$, $\tau_{20}=\dfrac{\pi}{2(b_7-b_5)}$, $\tau_{30}=\dfrac{\pi}{2(c_6-c_5)}$, *then* $A_{23}(0,1,1)$ *is locally asymptotically stable for any* $\tau_1\geq 0$. *A Hopf bifurcation occurs when* $\tau_2=\tau_3=\tau_{23}$.

Proposition 14 *If* $a_1-a_5>0$, $b_1-b_3>0$, $c_2-c_1>0$, $\tau_1=\tau_2=\tau_3=0$, *the equilibrium point* $A_{123}(1,1,1)$ *is locally asymptotically stable. If* $a_1-a_5>0$, $b_1-b_3>0$, $c_2-c_1>0$ *and* $\tau_1=\tau_2=\tau_3\in[0,\tau_{123})$, *where* $\tau_{23}=\min\{\tau_{10},\tau_{20},\tau_{30}\}$, $\tau_{10}=\dfrac{\pi}{2(a_1-a_5)}$, $\tau_{20}=\dfrac{\pi}{2(b_1-b_3)}$, $\tau_{30}=\dfrac{\pi}{2(c_2-c_1)}$, *then* $A_{123}(1,1,1)$ *is locally asymptotically stable. A Hopf bifurcation occurs when* $\tau_1=\tau_2=\tau_3=\tau_{123}$.

The equilibrium points that depend on the model's parameters are unstable.

5 Numerical simulations

For the numerical simulations, with respect to the mathematical model with two firms and two strategies, we consider the following values for the parameters: $a_1=0.3$, $a_2=0.3$, $a_3=0.4$, $a_4=0.5$, $b_1=0.2$, $b_2=0.5$, $b_3=0.4$, $b_4=0.8$. From Proposition 1, the equilibrium point is $(0,0)$ is locally asymptotically stable for any $\tau_1\geq 0$, $\tau_2\geq 0$ and the orbits of $(t,x(t))$, $(t,y(t))$ can be visualized in Fig. 1 and Fig. 2:

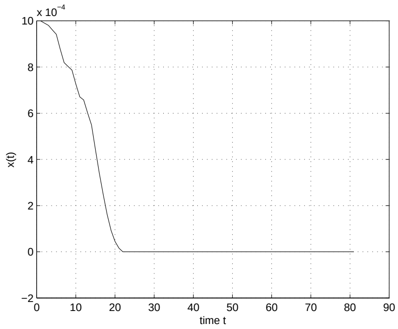

Fig. 1. (t, x(t))

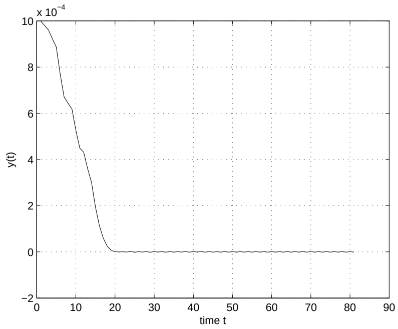

Fig. 2. (t, y(t))

For $a_1 = 0.4$, $a_2 = 0.5$, $a_3 = 0.1$, $a_4 = 0.1$, $b_1 = 0.4$, $b_2 = 0.2$, $b_3 = 0.8$, $b_4 = 0.4$, from Proposition 5 in the neighbourhood of the equilibrium point $C(1, 1)$ there is a Hopf bifurcation for $\tau_1 = 5.058$ and $\tau_2 = 1.93$. We can visualize the orbits of $(t, x(t))$, $(t, y(t))$ in Fig. 3 and Fig. 4:

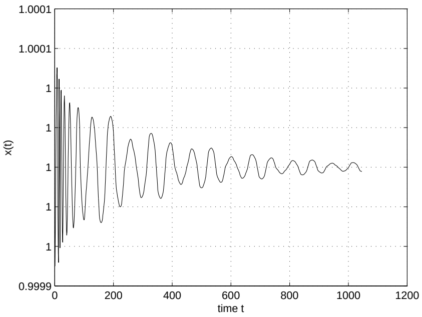

Fig. 3. (t, x(t))

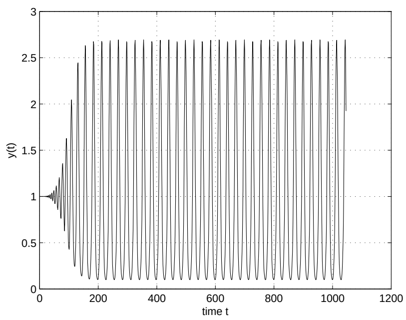

Fig. 4. (t, y(t))

For the mathematical model with three firms and two strategies, we use the following values for the parameters: $a_1 = 0.2$, $a_2 = 0.3$, $a_3 = 0.4$, $a_4 = 0.8$, $a_5 = 0.1$, $a_6 = 0.6$, $a_7 = 0.1$, $a_8 = 0.1$, $b_1 = 0.6$, $b_2 = 0.2$, $b_3 = 0.1$, $b_4 = 0.6$, $b_5 = 0.3$, $b_6 = 0.8$, $b_7 = 0.2$, $b_8 = 0.2$, $c_1 = 0.4$, $c_2 = 0.8$, $c_3 = 0.2$, $c_4 = 0.1$, $c_5 = 0.3$, $c_6 = 0.6$, $c_7 = 0.3$, $c_8 = 0.7$. From Proposition 9 in the neighbourhood of the equilibrium point $A_2(0, 1, 0)$ there is a Hopf bifurcation for $\tau_{20} = 2.85$. Fig. 5, Fig. 6 and Fig. 7 show the orbits of $(t, x(t))$, $(t, y(t))$, $(t, z(t))$:

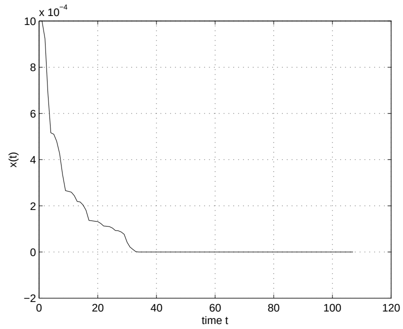

Fig. 5. (t, x(t))

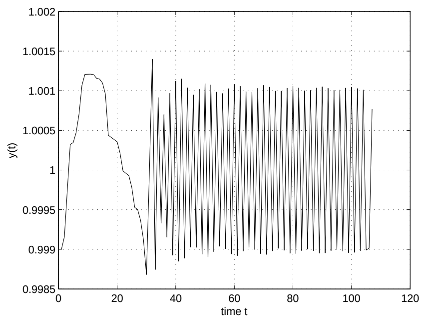

Fig. 6. (t, y(t))

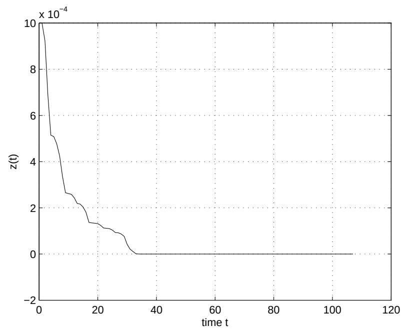

Fig. 7. (t, z(t))

6 Conclusions

This paper formulates the mathematical models of cooperation and competition relationship among two and three firms from the dynamical games point of view. For both cases the models are described by nonlinear systems with delayed differential equations. We have introduced time delays, because the choice of a strategy at time t leads to the payoff after a time delay. The equilibrium points have been determined and their local asymptotic stability has been analyzed. The existence of the Hopf bifurcation has been studied. We have found periodic solutions due to the introduction of lags. As in [4] the study will be continued by considering distributed time delays and numerical simulations will be carry out. Moreover, the stochastic approach can be taken into consideration as in [8].

References:

[1] J. Alboszta, M. Jacek, Stability of evolutionarily stable strategies in discrete replicator dynamics with time delay, *Journal of Theoretical Biology*, 231(2), 2004, pp. 175-179.

[2] N. Ben Khalifa, R. El-Azouzi, Y. Hayel, Random time delays in evolutionary game dynamics, *Proceedings of IEEE Control and Decision Conference* (CDC), Osaka, Japan, 2015, pp. 3840-3845.

[3] N. B. Khalifa, El A. Rachid, H. Yezekael, Delayed evolutionary game dynamics with nonuniform interactions in two communities, *Proceedings of IEEE Control and Decision Conference* (CDC), Los Angeles, California, Dec. 2014.

[4] N. B. Khalifa, El A. Rachid, H. Yezekael, Hopf Bifurcations in Replicator Dynamics with Distributed Delays, *preprint arXiv*:1703.06721, 2017.

[5] R. Iijima, On delayed discrete evolutionary dynamics, *Journal of Theoretical Biology*, 300, 2012, pp. 1-6.

[6] L. Ma, Y. Wang, The Tripartite Evolutionary Game Analysis of Collusive Behavior Regulation in EPR System, *International Journal of Humanities and Management Sciences*, 4(4), 2016.

[7] M. Neamtu, Deterministic and stochastic Cournot duopoly games with tax evasion, *WSEAS Transactions on Mathematics*, ISSN: 1109-2769, 8(9), 2010, pp. 618-627.

[8] N. Sirghi, M. Neamtu, Deterministic and stochastic advertising diffusion model with delay, *WSEAS Transactions on systems and control*, E-ISSN: 2224-2856, 4(8), 2013, pp. 141-150.

[9] N. Sirghi, M. Neamtu, D. Opris, Dynamical evolutionary games with delay, *Proceedings of the 13th Int. Conf. on Mathematics and Computers in Business and Economics* (MCBE), 2012.

[10] Q. Yi, Y. Liu, F. Luo, Study on mechanism of behavioral ecology of innovation cluster. *J Huxiang Forum*, vol. 2, 2007, pp.9-73.

[11] T. Yi, Z. Wang, Effect of time delay and evolutionarily stable strategy, *Journal of Theoretical Biology*, 187(1), 1997, pp. 111-116.

[12] P. Zhao, L. Yuxin, Evolutionary Game of Cooperation between Cultural and Creative Enterprises from Both Sides of the Taiwan Strait: From the Perspective of Anthropology, *Anthropologist*, 2(1-2), 2015, pp. 61-70.

[13] Y. Zhao, D. Li, L. Pan, Cooperation or Competition: An Evolutionary Game Study between Commercial Banks and Big Data-Based E-Commerce Financial Institutions in China, Hindawi Publishing Corporation, *Discrete Dynamics in Nature and Society*, 2015, Article ID 890972, http://dx.doi.org/10.1155/2015/890972, 8 pages.

[14] J. W. Weibull, Evolutionary game theory, MIT press, 1997.

[15] E. Wesson, R. Richard, Hopf bifurcations in two-player delayed replicator dynamics, *International Journal of Bifurcation and Chaos*, 26(1), 2016.