International Journal of Economics and Management Systems
http://iaras.org/iaras/journals/ijems

Must an Optimal Buy and Hold Portfolio Contain any Derivative?

ALEJANDRO BALBÁS
University Carlos III of Madrid
C/ Madrid, 126. 28903 Getafe (Madrid)
SPAIN
alejandro.balbas@uc3m.es

BEATRIZ BALBÁS
University of Alcalá de Henares
Pl. de la Victoria, 2. 28802 Alcalá (Madrid)
SPAIN
beatriz.balbas@uah.es

RAQUEL BALBÁS
University Complutense of Madrid
Somosaguas-Campus. 28223 Pozuelo de Alarcón (Madrid)
SPAIN
raquel.balbas@ccee.ucm.es

Abstract: Consider a portfolio choice problem maximizing the expected return and simultaneously minimizing a general (and frequently coherent) risk measure. This paper shows that every stock (or stock index) is often outperformed by a buy and hold strategy containing some of its derivatives and the underlying stock itself. As a consequence, every investment only containing international benchmarks will not be efficient, and the investors must properly add some derivatives. Though there is still a controversy, this finding had been pointed out in dynamic frameworks, but the novelty is that one does not need to rebalance the portfolio of derivatives before their expiration date. This is very important in practice because transaction costs are sometimes significant when trading derivatives.

Key–Words: Optimal Buy and Hold Strategy, Derivative Security, General Risk Measure.

1 Introduction

There are many papers whose main purpose is to study whether it is or it is not interesting to incorporate derivatives in order to compose efficient portfolios (Ahn *et al.*, 1999, Haugh and Lo, 2001, Constantinides *et al.*, 2011, etc.). Actually, though there still exits some controversy, it is becoming accepted that derivatives are frequently useful. This usefulness has been empirically pointed out by Balbás *et al.* (2016a), among others. They showed that the most important international stock indices may be outperformed (according to the Sharpe ratio) by combinations of their derivatives. Nevertheless, these authors dealt with a dynamic framework, and the investor had to rebalance her/his position frequently, provoking frictions and other transaction costs.

In this paper we will prove that derivatives also allow us to improve the portfolio $(risk, return)$ if one is looking for a buy and hold strategy, $i.e.$, if the investment is not going to be rebalanced within a significant time interval. Since the return variance (or the return standard deviation) is not a good risk measure when dealing with derivatives and other asymmetric securities because it is not compatible with the second order stochastic dominance (Ogryczak and Ruszczynski, 1999), we have selected alternative risk measures such as the conditional value at risk ($CVaR$, Rockafellar and Uryasev, 2000) the weighted $CVaR$ ($WCVaR$, Rockafellar *et al.*, 2006), and other recently introduced risk measures (Goovaerts and Laeven, 2008, Aumann and Serrano, 2008, Artzner *et al.*, 1999, Rockafellar *et al.*, 2006, etc.). Furthermore, in order to get a model-free approach, we will also deal with worst case risk measures such as the robust $CVaR$ ($RCVaR$, Balbás *et al.*, 2016b).[1]

The paper outline is as follows. Section 2 will be devoted to fixing notations and assumptions. The buy and hold portfolio choice problem will be presented and studied in Section 3. The most important results will be the necessary and sufficient optimality conditions of Theorem 1 and Corollary 2, as well as the characterization of Theorem 5, where we will show that the absence of derivatives in the optimal strategy

[1] It is worth to pointing out that many classical actuarial and/or financial problems have been revisited with the new risk measurement methodologies (Kalichenko *et al.*, 2012, Guan and Liang, 2014, Peng and Wang, 2016, Zhuang *et al.*, 2016, etc.).

only holds under very restrictive assumptions. Besides, Remark 6 will show that the conditions of Theorem 5 will barely hold if prices are given by a theoretical asset pricing model (binomial model, Black and Scholes, stochastic volatility, etc.).

We will present a simple numerical example in Section 4. This example will be very illustrative since it will show that the approach may be really model-free, and only the information provided by the market ($i.e.$, the market quotations) may matter. This is important because several reasons may provoke significant discrepancies between the theoretical prices provided by the available pricing models and the real quotations reflected by a real derivative market (Davis, *et al.*, 1993, Bondarenko, 2014, etc.). Moreover, the experiment of Section 4 will clearly illustrate how derivatives will really improve the portfolio performance, and therefore some derivatives will often belong to the optimal portfolio.

The last section presents the main conclusions of the paper.

2 Preliminaries and Notations

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of states of nature Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. As usual, denote by L^2 the Hilbert space of real valued random variables y on Ω such that $\mathbb{E}\left(y^2\right) < \infty$, endowed with the inner product $(x, y) \to \mathbb{E}(xy)$ and norm $\|y\|_2 = \left(\mathbb{E}\left(y^2\right)\right)^{1/2}$, $\mathbb{E}(.)$ representing mathematical expectation. We will be dealing with a finite collection of available securities $\{S_0, S_1, ..., S_m\} \subset L^2$, where $S_0 = 1$ will be the riskless asset, and S_1 will represent the underlying asset of the rest of securities $\{S_2, S_3, ..., S_m\}$, which will be European style derivatives with the same expiration date T. Assume that $\{S_0, S_1, ..., S_m\}$ are linearly independent,[2] and suppose that their current prices $p_0 = 1, p_1, ..., p_m$ are observable in the market. Since $p_0 = 1$ we are considering a null interest rate. Obviously, this assumption is not at all restrictive, and its fulfillment can be easily achieved by the usual the normalization method. In order to prevent some mathematical problems, impose Assumption 1 below;

Assumption 1 $\mathbb{P}(S_j \geq 0) = 1$, $j = 1, 2, ..., m$. *Consequently, the absence of arbitrage implies that* $p_j > 0$, $j = 1, 2, ..., m$. $\square$

[2] $i.e.$, there are no non-trivial linear combinations leading to the null asset, or, equivalently, the range of the covariance matrix of $\{S_1, S_2, ..., S_m\}$ equals m.

If $\rho : L^2 \longrightarrow \mathbb{R}$ is a risk measure then $\rho(y)$ may be understood as the "risk" associated with the wealth y, for every $y \in L^2$. Let us assume that ρ satisfies a representation theorem in the line of Artzner *et al.* (1999) or Rockafellar *et al.* (2006). More precisely, consider the sub-gradient of ρ

$$\Delta_\rho = \left\{z \in L^2; -\mathbb{E}(yz) \leq \rho(y), \forall y \in L^2\right\} \subset L^2 \tag{1}$$

composed of those linear expressions lower than ρ. Δ_ρ will be convex and weakly$-$compact (Schaeffer, 1970) and ρ will be its envelope, in the sense that

$$\rho(y) = Max\ \left\{-\mathbb{E}(yz);\ z \in \Delta_\rho\right\} \tag{2}$$

will hold for every $y \in L^2$. Furthermore, we will also assume that

$$\{1\} \subset \Delta_\rho \subset \left\{z \in L^2;\ \mathbb{E}(z) = 1\right\} \tag{3}$$

and

$$\Delta_\rho \subset \left\{z \in L^2;\ \mathbb{P}(z \geq 0) = 1\right\}. \tag{4}$$

These assumptions are equivalent to the usual properties of continuity, sub-additivity, homogeneity, mean dominance, translation invariance and monotonicity. To sum up, we have:

Assumption 2 $\rho : L^2 \longrightarrow \mathbb{R}$ *is continuous, sub-additive* ($\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2)$ *if* $y_1, y_2 \in L^2$*), homogeneous* ($\rho(\alpha y) = \alpha\rho(y)$ *if* $y \in L^2$ *and* $\alpha \geq 0$*), mean dominating* ($\rho(y) \geq -\mathbb{E}(y)$ *if* $y \in L^2$*), translation invariant* ($\rho(y + k) = \rho(y) - k$ *if* $y \in L^2$ *and* $k \in \mathbb{R}$*) and decreasing* ($\rho(y_1) \leq \rho(y_2)$ *if* $y_1, y_2 \in L^2$ *and* $\mathbb{P}(y_1 - y_2 \geq 0) = 1$*).* $\square$

The closed sub-space $Y \subset L^2$ generated by the $m + 1$ available assets will represent the set of pay-offs which can be attained by means of a buy and hold (or static) strategy, and the pricing rule

$$\Pi\left(\sum_{j=0}^{m} y_j S_j\right) = \sum_{j=0}^{m} y_j p_j. \tag{5}$$

will provide us with the current price of Portfolio $(y_j)_{j=0}^{m} \in \mathbb{R}^{m+1}$ (or Pay-off $\sum_{j=0}^{m} y_j S_j$). Hence, $\Pi : Y \to \mathbb{R}$ may be understood as a linear and continuous real valued function on Y.

3 Portfolio choice problem

A usual in the Markowitz approach and in recent studies about portfolio selection involving risk measures (Agarwal and Naik, 2004, Stoyanov *et al.*, 2007, Balbás *et al.*, 2010, Dupacová and Kopa, 2014, Zhao and Xiao, 2016, etc.), the optimal investment strategy will simultaneously maximize the expected return and minimize the global risk. Thus, our main problem will be

$$\left\{ \begin{array}{ll} Min\ \rho\left(\sum_{j=0}^{m} y_j S_j\right) & \\ \sum_{j=0}^{m} y_j p_j \leq 1 & \\ \mathbb{E}\left(\sum_{j=0}^{m} y_j S_j\right) \geq R & \\ y_j \in \mathbb{R}, & j = 0,1,...,m \end{array} \right. \tag{6}$$

$R > 1$ denoting the desired expected return. Problem (6) is convex due to Assumption 2. Bearing in mind this assumption, (1), (2), (3) and (4), and proceeding as in Balbás *et al.* (2013), one can prove the existence of a linear dual problem characterizing the solutions of (6). Hence, let us present the result below whose proof will be omitted because a similar one is available in the cited reference.

Theorem 1 *Consider Problem*

$$\left\{ \begin{array}{l} Max\ R\mu - \lambda \\ \mathbb{E}\left(\dfrac{S_j}{p_j}(z+\mu)\right) = \lambda, \quad j = 0,1,...,m \\ \lambda \geq 0,\ \mu \geq 0,\ z \in \Delta_\rho \end{array} \right. \tag{7}$$

$(\lambda, \mu, z) \in \mathbb{R} \times \mathbb{R} \times \mathrm{L}^2$ *being the decision variable.*

$a)$ *If Problem (6) is feasible bounded then Problem (7) is feasible, bounded and solvable, and the optimal values of (6) and (7) coincide.*

$b)$ *Suppose that* y^* *is (6)-feasible and* (λ^*, μ^*, z^*) *is (7)-feasible. Then,* y^* *solves (6) and* (λ^*, μ^*, z^*) *solves (7) if and only if the complementary slackness conditions below*

$$\left\{ \begin{array}{l} \sum_{j=0}^{m} y_j^* \mathbb{E}\left(S_j z\right) \geq \sum_{j=0}^{m} y_j^* \mathbb{E}\left(S_j z^*\right), \quad \forall z \in \Delta_\rho \\ \lambda^*\left(1 - \sum_{j=0}^{m} p_j y_j^*\right) = 0 \\ \mu^*\left(\sum_{j=0}^{m} y_j^* \mathbb{E}\left(S_j\right) - R\right) = 0 \end{array} \right.$$

hold. □

The first constraint of Problem (6) allows us to simplify the dual problem.

Corollary 2 *Consider Problem*

$$\left\{ \begin{array}{l} Min\ (R-1)\mu - 1 \\ \mathbb{E}\left(\dfrac{S_j}{p_j}(z+\mu)\right) = 1+\mu, \quad j = 0,1,...,m \\ \mu \geq 0,\ z \in \Delta_\rho \end{array} \right. \tag{8}$$

$(\mu, z) \in \mathbb{R} \times \mathrm{L}^2$ *being the decision variable.*

$a)$ *If Problem (6) is feasible and bounded then Problem (8) is feasible, bounded and solvable, and the optimal values of (6) and (8) coincide.*

$b)$ *Suppose that* y^* *is (6)-feasible and* (μ^*, z^*) *is (8)-feasible. Then,* y^* *solves (6) and* (μ^*, z^*) *solves (8) if and only if the complementary slackness conditions below*

$$\left\{ \begin{array}{l} \sum_{j=0}^{m} y_j^* \mathbb{E}\left(S_j z\right) \geq \sum_{j=0}^{m} y_j^* \mathbb{E}\left(S_j z^*\right), \quad \forall z \in \Delta_\rho \\ \sum_{j=0}^{m} p_j y_j^* = 1 \\ \mu^*\left(\sum_{j=0}^{m} y_j^* \mathbb{E}\left(S_j\right) - R\right) = 0 \end{array} \right. \tag{9}$$

hold.

Proof. Suppose that $\lambda = 1 + \mu$ must hold for every (7)-feasible element $(\lambda, \mu, z) \in \mathbb{R} \times \mathbb{R} \times \mathrm{L}^2$. Then, the result will trivially follows from Theorem 1 above. Besides, the relationship between λ and μ is an obvious consequence of (3) and the first constraint of (7) for $j = 0$. □

Remark 3 *Notice that the solution* (μ^*, z^*) *of (8) does not depend on* $R > 1$*, and therefore this optimization problem essentially remains the same if the objective function is replaced by* μ*. Furthermore, due to (3), the first constraint of (8) becomes obvious if* $j = 0$*, and therefore straightforward manipulations imply that (8) and Problem*

$$\left\{ \begin{array}{l} Min\ \mu \\ \mathbb{E}\left(S_j \dfrac{z+\mu}{1+\mu}\right) = p_j, \quad j = 1,...,m \\ \mu \geq 0,\ z \in \Delta_\rho \end{array} \right. \tag{10}$$

are equivalent.

If (μ^*, z^*) *solves (10), then Corollary* $2a$ *implies that*

$$\rho^*(R) = (R-1)\mu^* - 1 \tag{11}$$

will be the relationship between the desired return $R > 1$*, and the associated optimal risk level* $\rho^*(R)$*. Notice that (11) is an affine expression. This result was already pointed out by Balbás et al. (2010) in a more general setting. Since* μ^* *is the slope of the straight line (11), this parameter will be called the market price of risk.* □

As said in Section 2, S_1 is a risky index (or share) and the underlying asset of S_j, $j = 2,3,...,m$. Since the (null) riskless rate may be attained with the riskless asset, it is natural to assume that investors will not buy S_1 for a similar or lower expected return. Therefore, the assumption $\mathbb{E}(S_1) > p_1$ is natural too. Besides, a positive relationship (slope) between the desired expected return and the optimal risk is also consistent with many classical findings in portfolio theory. Thus, bearing in mind (11), it seems natural to impose that $\mu^* > 0$. To sum up;

Assumption 3 *Inequalities* $\mathbb{E}(S_1) > p_1$ *and* $\mu^* > 0$ *hold.* □

Conditions (9) enable us to find the optimal strategy when there are no available derivatives.

Lemma 4 *Suppose that (6) is feasible, bounded and solvable. Suppose that* $m = 1$*. Then, the solution of (6) is*

$$\begin{cases} y_1^* = (R-1) / (\mathbb{E}(S_1) - p_1) \\ y_0^* = 1 - p_1 y_1^* \end{cases} \tag{12}$$

whereas every solution of (10) is characterized by

$$\begin{cases} \mu^* = (p_1 + \rho(S_1)) / (\mathbb{E}(S_1) - p_1) \\ \mathbb{E}(z^* S_1) = -\rho(S_1) \ and\ z^* \in \Delta_\rho \end{cases} \tag{13}$$

Proof. The second and third condition of (9), along with Assumption 3, trivially lead to (12). The first condition of (9) implies that

$$y_1^* \mathbb{E}(S_1 z) \geq y_1^* \mathbb{E}(S_1 z^*), \quad \forall z \in \Delta_\rho.$$

Since the first equality of (12) implies that $y_1^* > 0$, we have

$$\mathbb{E}(S_1 z) \geq \mathbb{E}(S_1 z^*), \quad \forall z \in \Delta_\rho,$$

and (2) trivially leads to the second equality of (13). Lastly, the first equality in (13) trivially follows from the first constraint in (10) and the equality $\mathbb{E}(z^* S_1) = -\rho(S_1)$.

Conversely, if (13) holds then it is easy to see that (μ^*, z^*) is (10)-feasible and and therefore it solves the problem because (10) is solvable (Corollary 2 and Remark 3) and the proved implication shows that its optimal value is μ^*. □

Theorem 5 *Suppose that* $m > 1$*. Suppose that (6) is feasible, bounded and solvable. Consider a solution* (μ^*, z^*) *of (10). The solution* y^* *contains no derivatives (*$i.e.$*,* $y_j^* = 0$*,* $j = 3, 4, ..., m$*) if and only if*

$$\mathbb{E}\left(S_j \frac{z^* + \mu^*}{1 + \mu^*}\right) = p_j, \quad j = 2, ..., m. \tag{14}$$

If so, (y_0^*, y_1^*) *is given by (12) and* (μ^*, z^*) *is given by (13).*

Proof. If $y^* = (y_0^*, y_1^*, 0, ..., 0)$ solves (6) then (y_0^*, y_1^*) solves the same problem when $S_2, S_3, ..., S_m$ are removed. Thus, Lemma 4 implies that (12) and (13) must hold. Furthermore, the first constraint in (10) implies (14).

Conversely, (14) implies the fulfillment of the first constraint of (10), and the rest of constraints of (6) and (10), along with the conditions in (9) become obvious if we bear in mind that (6) is solvable due to the Theorem assumptions. □

Remark 6 *Suppose that the pricing rule* Π *of (5) can be extended to the whole space* L^2*, and the extension* $\Pi : L^2 \longrightarrow \mathbb{R}$ *is still denoted by* Π *and it is an increasing and linear (and therefore continuous, Schaeffer, 1974) function.*[3] *Then, there exists a unique* $z_\Pi \in L^2$ *such that*

$$\Pi(y) = \mathbb{E}(y z_\Pi), \tag{15}$$

holds for every $y \in L^2$*,*

$$\mathbb{P}(z_\Pi > 0) = 1 \tag{16}$$

and

$$\mathbb{E}(z_\Pi) = 1.$$

z_Π *is usually called stochastic discount factor (*SDF*, Duffie,1988). Bearing in mind (15), if* $m > 1$ *then Condition (14) in Theorem* 5 *holds if and only if*

$$\mathbb{E}(S_j z_\Pi) = \mathbb{E}\left(S_j \frac{z^* + \mu^*}{1 + \mu^*}\right), \quad j = 2, ..., m. \tag{17}$$

However, there will be many practical cases making (17) infeasible and, consequently, implying that the optimal buy and hold strategy will contain derivatives. Indeed, suppose for instance that $Log(z_\Pi)$ *is not essentially bounded for below.*[4] *Since (4) and Assumption* 3 *imply that*

$$\mathbb{P}\left(\frac{z^* + \mu^*}{1 + \mu^*} \geq \frac{\mu^*}{1 + \mu^*} > 0\right) = 1,$$

then we will have that

$$z_\Pi \neq \frac{z^* + \mu^*}{1 + \mu^*},$$

[3]Interesting particular cases arise if the set of market quotations $(p_j)_{j=0}^m$ perfectly fit the theoretical prices generated by a complete pricing model (binomial model, Black and Scholes model, etc.). If $(p_j)_{j=0}^m$ fits the theoretical prices generated by an incomplete pricing model then the extension $\Pi : L^2 \longrightarrow \mathbb{R}$ *often exists as well. For instance, it exists if* the set Ω only contains finitely many states (Harrison and Kreps, 1979). Therefore, cases such as the usual trinomial models are also included in or analysis. If Ω contains infinitely many states then the existence of Π is also possible. For instance, though "formally" stochastic volatility models are incomplete, in practice it is assumed the existence of volatility dependent assets making them complete. Otherwise it would be impossible to use these models so as to give a unique price of the usual derivatives. Further details about the existence of Π under general conditions for Ω may be found in Luenberger (2001).

[4]Expression (16) shows that $Log(z_\Pi)$ makes sense. Moreover, $Log(z_\Pi)$ is bonded from below if and only if

$$\mathbb{P}(z_\Pi < \varepsilon) > 0$$

for every $\varepsilon > 0$, and it often holds in practive. For example, it holds if z_Π has a log-normal distribution (Black and Scholes model, Wang, 2000) or a heavier tailed one (stochastic volatility pricing models).

and therefore the set

$$\left\{ y \in L^2;\ \mathbb{E}\left(z_\Pi y\right) = \mathbb{E}\left(\frac{z^* + \mu^*}{1+\mu^*} y\right)\right\}$$
$$= \left\{ y \in L^2;\ \mathbb{E}\left(\left(z_\Pi - \frac{z^* + \mu^*}{1+\mu^*}\right) y\right) = 0\right\}$$

will be a closed proper sub-space (hyperplane) of L^2*. In other words, the inequality*

$$\mathbb{E}\left(z_\Pi y\right) \neq \mathbb{E}\left(\frac{z^* + \mu^*}{1+\mu^*} y\right)$$

will hold for most of the derivatives $y \in L^2$ *of* S_1*, and therefore (17) will very easily fail. Moreover, notice that the existence of an unbounded from below* $Log\left(z_\Pi\right)$ *only involves the pricing rule* Π *of (15), and therefore the failure of (17) will hold for every risk measure satisfying Assumption* 2 *(for example, the* $CVaR$*, the* $WCVaR$ *or the* $RCVaR$*).*

To sum up, Theorem 5 *implies that the buy and hold optimal strategy will contain derivatives if the real quotations of the derivative market respect the predictions of some important pricing model of Financial Economics, and this finding is independent of the selected risk measure* ρ *satisfying Assumption* 2*.* □

4 Numerical experiment

Let us illustrate the results of Section 3 with a very simple example. We will deal with an arbitrage free and almost model-independent option market. As above, suppose that $S_0 = 1$ is a riskless asset and consider a security S_1 whose behavior is given by a geometric Brownian motion (GBM) with a current price, drift and volatility equaling one dollar, 2% and 40%, respectively. Consider also a derivative market where European calls and puts can be traded. The unique maturity is one year, and the available strikes are

$$\begin{pmatrix} Calls & Puts \\ 0.5 & 1.2 \\ 0.7 & 1.5 \\ 1 & 1.9 \end{pmatrix}$$

Suppose that the market quotations perfectly fit the Black and Scholes model, $i.e.$, all of the market prices equal the theoretical ones given by the Black and Scholes formula. Accordingly, they become

$$\begin{pmatrix} Calls & Puts \\ 0.504700865 & 0.291880947 \\ 0.333711519 & 0.539261689 \\ 0.158519419 & 0,912489695 \end{pmatrix} \qquad (18)$$

Obviously, since the Black and Scholes model is arbitrage free, this market is arbitrage free as well. Consider an investor who is interested in composing an efficient portfolio. The selected risk measure ρ is the $CVaR_\alpha$, α being the level of confidence. Suppose that $\alpha = 85\%$. Despite the fact that this investor can verify that the two matrices above lead to a constant implied volatility $\sigma = 0.4$, and therefore the data confirm in this case the Black and Scholes model, let us assume that he/she is still very ambiguous with respect to that. Accordingly, he/she will accept deviations between the predictions of the log-normal distribution and the realized value of S_1 in one year. He/she considers that the error between the probabilities of the log-normal distribution and the real probabilities may become 100%. In other words, for every Borel subset $B \subset \mathbb{R}$, the real probability $Q\left(S_1 \in B\right)$ of the event $S_1 \in B$ will be laying within the spread $[0, 2\mathbb{P}\left(S_1 \in B\right)]$, where $\mathbb{P}\left(S_1 \in B\right)$ is the theoretical probability under log-normality. In such a case, instead of the $CVaR_{85\%}$ risk measure, the investor will use the robust risk measure $RCVaR_{85\%}$. In general,

$$\left\{ \begin{array}{l} RCVaR_\alpha\left(y\right) := \\ Max\ \left\{CVaR_{(Q,\alpha)}\left(y\right);\ 0 \le \frac{dQ}{d\mathbb{P}} \le 2\right\}, \end{array}\right. \qquad (19)$$

where Q is a $\mathbb{P}-$continuous probability measure and $CVaR_{(Q,\alpha)}\left(y\right)$ is the $CVaR_\alpha$ of y under Q. Balbás *et al.* (2016b) have shown that the $RCVaR_\alpha\left(y\right)$ above is well defined for every $y \in L^2$, along with the fulfillment of Assumption 2. Moreover the sub-gradient (1) is composed of those random variables $z \in L^2$ satisfying the conditions

$$\left\{ \begin{array}{l} \mathbb{E}\left(z\right) = 1 \\ 0 \le \frac{dQ}{d\mathbb{P}} \le 2 \\ 0 \le z \le \frac{1}{1-\alpha}\left(\frac{dQ}{d\mathbb{P}}\right) \end{array}\right. .$$

It is easy to see that the set above coincides with

$$\left\{ \begin{array}{l} \left\{z \in L^2;\ 0 \le z \le \frac{2}{1-\alpha},\ \mathbb{E}\left(z\right) = 1\right\} = \\ \left\{z \in L^2;\ 0 \le z \le \frac{1}{1-(1+\alpha)/2},\ \mathbb{E}\left(z\right) = 1\right\}. \end{array}\right.$$

Since this is the sub-gradient of the $CVaR_{(1+\alpha)/2}$ risk measure (Rockafellar *et al.,* 2006),

$$RCVaR_\alpha = CVaR_{(1+\alpha)/2} \qquad (20)$$

and the high ambiguity level of this example only implies that the level of confidence must properly increase. In particular, for $\alpha = 85\%$ one has $(1+\alpha)/2 = 92.5\%$, and our investor will optimize the portfolio for the $CVaR_{92.5\%}$ risk measure.

Though the existence of ambiguity only implies a larger level of confidence, it is important to point out that we are dealing with an ambiguous setting. Expression (19) implies "a worst case approach", and therefore the risk level guaranteed by the optimal buy and hold strategy will be guaranteed by every $CVaR_{(Q,85\%)}$, and Q does not have to be known. In this sense, the optimal buy and hold strategy will not depend on the Black and Scholes model, since a huge error of this model (up to 100%) is accepted.

Recovering the level of confidence $(1+\alpha)/2 = 92.5\%$ implied by (20), and the log-normal distribution for S_1, one can consider the equality

$$S_1(\omega) = Exp\ \left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\Phi^{-1}(\omega)\right) \tag{21}$$

for $\omega \in (0,1)$, and with $r = 2\%$, $\sigma = 40\%$ and $T = 1$ denoting the selected drift, volatility and maturity (Wang 2000, Balbás *et al.*, 2016a, etc.). Therefore, bearing in mind that (21) generates an increasing function of $\omega \in (0,1)$, Expression (2) leads to

$$\rho(S_1) = RCVaR_{85\%}(S_1) = CVaR_{92.5\%}(S_1) =$$

$$\frac{\int_0^{1-0.925} \left(Exp\ \left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\Phi^{-1}(\omega)\right)\right) d\omega}{0.925 - 1}.$$

Computing the integral with 10^6 Monte Carlo simulations one can estimate

$$\rho(S_1) = -0.448939385$$

Consequently, if one solves Problems (6) and (10) with only two securities (S_0 and S_1,$i.e.$, $m = 1$), then Lemma 4, (12) and (13) lead to

$$\left\{ \begin{array}{l} y_1^* = \dfrac{R-1}{0.02020134} \\ \\ y_0^* = 1 - \dfrac{R-1}{0.02020134} \end{array} \right. \tag{22}$$

and

$$\left\{ \begin{array}{l} \mu^* = 27.27841887 \\ z^* = \left\{ \begin{array}{ll} \dfrac{1}{1-0.925}, & \omega \leq 1 - 0.925 \\ 0, & otherwise \end{array} \right. \end{array} \right.$$

According to Theorem 5, the necessary and sufficient condition guaranteeing that the solution above remains the same if one also involves the six available European options ($i.e.$, $m = 7$) will be given by (14). Since

$$\mathbb{E}\left(y\frac{z^* + \mu^*}{1+\mu^*}\right) = \int_0^1 y(\omega)\,\frac{z^*(\omega) + \mu^*}{1+\mu^*} d\omega$$

for every random variable y, after estimating the six integrals

$$\mathbb{E}\left(S_j\frac{z^* + \mu^*}{1+\mu^*}\right), \quad j = 2, ..., 7$$

with 10^6 Monte Carlo simulations, the obtained results are

$$\begin{pmatrix} Calls & Puts \\ 0.50567611 & 0.296603733 \\ 0.338486228 & 0.542180941 \\ 0.164017017 & 0.913931802 \end{pmatrix} \tag{23}$$

Matrices (18) and (23) show that none of the given options satisfy (14), $i.e.$, (22) will not be the optimal solution any more if the investor adds a single available option.[5] Actually, the investor will improve her/his portfolio $(risk, return)$ as much as possible if he/she adds the six available options and then solves (10) and (6). In the first step the linear optimization problem (10) may be solved with several algorithms (Anderson and Nash, 1987). In the second step (6) may be solved by means of (9). We will not address this question in order to shorten the paper exposition.

5 Conclusion

We have been dealing with a portfolio choice problem maximizing the expected return and simultaneously minimizing a general (and frequently coherent) risk measure. Since derivative securities usually generate an asymmetric pay-off, the use of risk measures beyond the variance is justified by the lack of compatibility between this risk measure and the usual utility functions (or the second order stochastic dominance). Moreover, if the selected risk measure also incorporates the investor ambiguity (i.e., if one deals with a robust risk measure), then the presented approach becomes model-independent.

Theorem 5 and Remark 6 have shown that every stock index is often outperformed by a buy and hold strategy containing the index and some of its derivatives. As a consequence, every investment only containing international benchmarks will not be efficient,

[5] Notice that this numerical finding is consistent with Remark 6.

and the investor should properly add some derivatives. Though there is still a controversy, this finding had been pointed out in dynamic frameworks, but a main novelty is that one does not need to rebalance any position before the expiration date of the incorporated derivatives. This is very important in practice because transaction costs and other market imperfections may provoke significant capital losses when trading derivatives.

Acknowledgments

Research partially supported by "*Ministerio de Economía*" (grant $ECO2012-39031-C02-01$, Spain). The usual caveat applies.

References

Agarwal, V. and N. Naik (2004). Risks and portfolio decisions involving hedge funds. *Review of Financial Studies*, 17, 63-98.

Ahn, D., J. Boudoukh, M. Richardson and R.F. Whitelaw (1999). Optimal risk management using options. *The Journal of Finance*, 54, 359 - 375.

Anderson, E.J. and P. Nash (1987). *Linear programming in infinite-dimensional spaces*. John Wiley & Sons.

Artzner, P., F. Delbaen, J.M. Eber and D. Heath (1999). Coherent measures of risk. *Mathematical Finance*, 9, 203 - 228.

Aumann, R.J. and R. Serrano (2008). An economic index of riskiness. *Journal of Political Economy*, 116, 810-836.

Balbás, A., B. Balbás and R. Balbás (2010). $CAPM$ and $APT-$like models with risk measures. *Journal of Banking & Finance*, 34, 1166–1174.

Balbás, A., B. Balbás and R. Balbás (2013). Good deals in markets with friction. *Quantitative Finance*, 13, 827–836.

Balbás, A., B. Balbás and R. Balbás (2016a). Outperforming benchmarks with their derivatives: Theory and empirical evidence. *The Journal of Risk*, 18, 4, 25 - 52.

Balbás, A., B. Balbás and R. Balbás (2016b). Good deals and benchmarks in robust portfolio selection. *European Journal of Operational Research*, 250, 666 - 678.

Bondarenko, O. (2014). Why are put options so expensive? *Quarterly Journal of Finance*, 4, 3, 1450015.

Constantinides, G.M., M. Czerwonko, J.C. Jackwerth and S. Perrakis (2011). Are options on index futures profitable for risk-averse investors? Empirical evidence. *The Journal of Finance*, 66, 1407-1437.

Davis M.H.A., V.G. Panas and T. Zariphopoulou (1993). European option pricing with transaction costs. *Siam Journal of Control and Optimization*, 31, 470 - 493.

Duffie, D. (1988). *Security markets: Stochastic models*. Academic Press.

Dupacová, J. and M. Kopa (2014). Robustness of optimal portfolios under risk and stochastic dominance constraints. *European Journal of Operational Research*, 234, 434 - 441.

Goovaerts, M.J. and R. Laeven (2008). Actuarial risk measures for financial derivative pricing. *Insurance: Mathematics and Economics*, 42, 540-547.

Guan, G. and Z. Liang (2014). Optimal reinsurance and investment strategies for insurer under interest rate and inflation risks. *Insurance: Mathematics and Economics*, 55, 105 – 115.

Harrison, J. and D. Kreps (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20, 381 - 408.

Haugh, M.B. and A.W. Lo (2001). Asset allocation and derivatives. *Quantitative Finance*, 1, 45–72.

Kalichenko, K., S. Uryasev and R.T. Rockafellar (2012). Calibrating risk preferences with the generalized capital asset pricing model based on mixed conditional value-at-risk deviation. *The Journal of Risk*, 15, 1, 45 – 70.

Luenberger, D.G. (2001). Projection pricing. *Journal of Optimization Theory and Applications*, 109, 1 - 25.

Ogryczak, W. and A. Ruszczynski (1999). From stochastic dominance to mean risk models: Semideviations and risk measures. *European Journal of Operational Research*, 116, 33-50.

Peng, X. and W. Wang (2016). Optimal investment and risk control for an insurer under inside information. *Insurance: Mathematics and Economics*, 69, 104 - 116.

Rockafellar, R. T. and S. Uryasev (2000). Optimization of conditional-value-at-risk. *The Journal of Risk*, 2, 21 – 42.

Rockafellar, R. T., S. Uryasev and M. Zabarankin (2006). Generalized deviations in risk analysis. *Finance and Stochastics*, 10, 51 – 74.

Schaeffer, H.H. (1970). *Topological vector spaces*. Springer.

Schaeffer, H.H. (1974). *Banach lattices and positive operators*. Springer.

Stoyanov, S.V., S.T. Rachev and F.J. Fabozzi (2007). Optimal financial portfolios. *Applied Mathematical Finance*, 14, 401-436.

Wang, S.S. (2000). A class of distortion operators for pricing financial and insurance risks. *Journal of Risk and Insurance*, 67, 15-36.

Zhao, P. and Q. Xiao (2016). Portfolio selection problem with Value-at-Risk constraints under non-extensive statistical mechanics. *Journal of Computational and Applied Mathematics*, 298, 74-91.

Zhuang, S.C., C. Weng, K.S. Tan and H. Assa (2016). Marginal indemnification function formulation for optimal reinsurance. *Insurance: Mathematics and Economics*, 67, 65-76.