Mathematical Engineering in the Teaching of Linear Algebra to Address Complex Geometric Objects

EDUARDO A. GAGO, PAOLA A. SZEKIETA, NICOLAS DE BRITO
Computer and Multidisciplinary Laboratory of Basic Sciences
Universidad Tecnológica Nacional – Facultad Regional Rosario
Zeballos 1341
ARGENTINA

Abstract: - Teaching Mathematics in the engineering programs at university implies learning conditions, which can allow students to acquire knowledge through the development of simple models to new ways of thinking and reasoning. This paper describes an experience carried out in the Algebra and Analytic Geometry course of Chemical Engineering program where Lineal Algebra is introduced by the use of specific software. The proposal aims at giving students a computational approximation to Lineal Algebra in the analysis of a cellular automaton and its relation to the fractal known as Sierpinski's Triangle. The adopted approach is carried out through theoretical-practical lessons where teaching and learning situations prompt students to carry out analytic procedures to apply theoretical concepts presented in class, and enhance the applications of current approaches in mathematics and engineering.

Key-Words: - Simulation, cellular automaton, models, Sierpinski's Triangle, competences, Algebra.

Received: May 25, 2021. Revised: March 16, 2022. Accepted: April 13, 2022. Published: May 5, 2022.

1 Introduction

Digital tools have changed the ways and strategies at university engineering teaching, as they are currently responsible of the elaboration of didactic approaches unthinkable a few years ago.

There are several factors that may hinder the teaching and learning process of the Linear Algebra (LA) content, among them the insufficient basic knowledge that students bring about when they start university. Therefore, it becomes essential to formulate simple models that can trigger motivational situations leading to the acquisition of new ways of thinking and reasoning [1].

In this case, the teaching approach should provide a computational view of LA to tackle the conceptual difficulties that this mathematical field presents to students during engineering applications, mainly due to the high level of abstraction of some concepts involved.

Also, it is frequently observed that the Mathematics teaching in the Engineering programs is usually based on continuous dynamic models, although, even in the professional practice, many of these models are ruled by the Discreet Variable use.

Therefore, it is considered relevant to train students into the analysis of discrete and simple models, as in the case of fractals and Cellular Automaton (CA), which can facilitate the development of competencies by the use of online computer packages using digital devices.

Situations are proposed so that students are prompted to develop analytical procedures to use the theoretical concepts presented in the classroom and to enhance their competencies.

Fractals and CA are mathematical models that describe and study objects and frequent phenomena in Nature that can't be explained by the classic theories but can be obtained by simulations of the creation process.

If previous mathematical concepts of the development of any subject have been internalized properly, the systems simulations with engineering applications will promote fertile ground for students to develop their critical, creative and reflexive sense, integrating knowledge from the subjects of the first years of their study.

In the present article an experience in the Algebra and Analytical Geometric subject of the first level of the Chemical Engineering Program is described. The proposal intends students to integrate their knowledge of Algebra matrix when the construction of fractals objects is developed from an interdisciplinary perspective so that they can relate their knowledge to other curriculum subjects. The activities have the objective of developing procedures of algorithm to relate the theoretical knowledge with applications than can be interesting in the engineering field.

2 Methodology

The Informatics and Multidisciplinary Basic Sciences Laboratory is the university physical room where the experience is carried out. In this environment, a collaborative work area is generated by a theoretical-practical-technological workshop where concepts of geometric lineal transformation on a plane and CA are applied. This is achieved by introducing models relevant to the engineering field and related to the generation of objects of current mathematics.

The theoretical-practical lessons characteristics rely on establishing guidelines for an interactive work of engineering analysis that results into an activity that generates new ideas.

This teaching and learning methodology allows us first to carry out the theoretical investigation of the subject, creating an environment where the student takes an active role, overcoming the spectator role, and can construct the concepts by experimentation and elaboration of conclusions.

In this type of experiences, the teacher takes a guiding role in the learning process and acts as a facilitator towards the comprehension of the target subject, including questions to trigger a mechanism of continuity in the extension and further investigation of different lines of research on the target problem [2].

The laboratory workshop aims at training students to use specific software so that they can develop their wit and dexterity through the use of computer resources, and thus be able to conceptualize the proposed content. [1].

The curricular activities are premised on the dynamic analysis of discrete systems that involve developing competencies and fostering the inter-multidisciplinary character that should prevail in the curricula of engineering programs.

Appropriate strategies are developed to aim at a favourable learning situations, taking advantage of numerical, symbolic and graphic calculation aspects with digital tools, which implies an intense creative activity on the part of the student and a teacher's real commitment in terms of directionality, coordination and pedagogical assistance [2], [3].

In addition, the widespread availability of new interactive technologies exhibits an immense amount of possibilities that are achieved with the development of new models of teaching and learning. Taking into account that the engineering analysis work is based on a computer-mediated system and communication, it is considered important to generate an environment where a set of activities, interaction and communicative relationships are produced as the fundamental axis of the educational process.

3 Motivation

The present article tackles the need to find an alternative way to the teaching learning process that takes into account not only the program requirements but also students' motivation, by adding application problems than can be connected to the theory of the target subject.

In some cases, the students' low level ok previous required knowledge makes hard to find models on the LA area than can be understood. However, it is the teacher's challenge to try to find problematic situations that can spark students' curiosity and interest.

Epistemological obstacles generated by the Linear Transformations concept itself when it is initially presented to students also pose a restriction. Students need to have previous organizers to be able to approach the contents presented in order to establish relationships as complex and rich as possible to increase the meaning of their learning. [4].

Due to all these factors, it is convenient to first help the student to remember, reorganize and assimilate previous knowledge needed in the proposed content to achieve the planned learning successfully.

A way to carry out the experience is to design cognitive links between new content and the knowledge structure that the student has, and elaborate the right strategies to guide them towards a favorable learning situation [1].

This may imply intense activity from the students and a real teacher's commitment onto the directionality, coordination and pedagogical help.

Using a valuable tool as for example the symbolic calculus, serves as a connection of mathematical functions applied to real situations, which permits to validate the proper student skills, as the development of basic capacities [5].

The interconnection between LA concepts, real cases in the engineering environment, and the anticipation of knowledge that will later become more complex in the subjects that deal with the basic and applied technologies provide students versatility when they have to approach increasingly sophisticated models in the specific program subjects.

4 Objectives

The current technological representation and calculus possibilities require the development of common sense, the critical review of the results emanating from computer programs, sensibility to different alternatives, the capacity for analysis and the rational selection of proposals such as the detection of the relevant variables in a problem.

The need to develop new working styles and the implementation of alternative methodologies is then required, as well as the need to rethink the interaction between the different areas [6].

The activity aims to integrate computational mathematics in technological areas of engineering programs within the basic sciences, and to incorporate new working styles based on organizing principles that allow knowledge to be connected and to be meaningful by transforming what disciplinary borders generate.

Thus, multidisciplinary work is considered a necessity in this proposal as an unavoidable professional attitude according to new trends in the world of science and work [4], [6]. It seeks to promote scientific experimentation within the disciplines that make up the Mathematics in the engineering area, to address problems that require mathematical modeling, to apply different solution methods appropriately, to select the algorithm to apply and identify the tools developed in each method.

The goal is to ensure that students acquire generic and specific skills, in particular the generic ones which have a transverse nature in the curricular map, to work on conceptualization and theorization based on concrete practice, overcoming the theoretical-practical dichotomy.

It is essential to understand that interdisciplinary work generates change processes initiated from the potential offered by the different computational resources in the construction of knowledge.

The implementation of classroom strategies from the mathematical-informatics relation allows articulating basic and technological disciplines giving the needed skills to the future professional to comprehend and suggest answers, especially when they face different alternatives and take decisions from informed answers.

5 Experience Development

A technological-practical-theoretical class is designed in sessions of three-hour lectures each, whose aim is to guide students into a meaningful process of learning the proposed topic, through the analysis of the linear transformations and the CA and the relationship that is set with geometric visualizations.

In the Laboratory workshops, we work with groups of students that do not exceed three members, and in order to carry out the planned task, they are presented with a case of fractal geometry to support their use of informatics tools. Students are free to get in groups according to their own criteria and choose which group they want to join in, but a collaborative teamwork environment is set as an essential condition. Each group has at least one desktop computer, but they are also welcome to bring their own personal computers to work.

It is necessary to foster communication among students as an essential part of human and social development. In fact, communicative competencies are indispensable in individuals and society, especially in a world of wide, diverse and multiple information that circulates throughout various media.

Information is required to understand what is happening, but at the same time, information is needed to make the right decisions. [1].

In the first part of this experience, simple models are presented by generating the fractal called Sierpinski's Triangle (ST) applying the concepts of the Related Transformations theme, within the Linear Transformations study unit.

In the second class, CA Rule 90 is studied, where the concepts of matrix algebra are applied, and then the same ST fractal is generated from this new content.

5.1 Fractals Objects Generation

In the first work session the fractal objects generation is proposed. The designed stages to carry out all lesson sessions in class are:

- Assembling working groups
- Setting problematic situation.
- Subject theory investigation.
- Reviewing bibliographical available resources about the subject and subsequent selecting.
- Reviewing developed theory lecture content.
- Situation modeling.
- Solving proposed models.
- Conclusions.

It is proposed as a learning outcome to familiarize students with fractal sets from science.

The first problem that arises is to explain the concept of fractal object. The explanation will be limited to showing how it is possible to generate objects from a triangle that are repeated within the same structure.

The fractal word, expressed in relation to mathematical concepts, was shown up for first time in year 1977 when Benoit Mandelbrot used it to refer a certain sets with all or some of the next properties:

- They have details in all scales, understanding than they are observed at any scale level, they

manifest the previous observed details at global level.
- They are self-similar, i.e., they are made up of parts that are similar to the whole.
- They have a simple algorithmic description, understanding that its construction is based in a simple algorithm.

First, it is intended that the students create the fractal object named ST, analysing the affine transformations to be used for that process [7].

Applying random iterated algorithm to generate fractal, the principle is clear and simple; it is easy to implement in any programming language and the results it generates are spectacular and may be used for any graphical goal, not only for mathematical reasons [8].

In this context, if an orthogonal axis system is fixed, the experience begins from an isosceles triangle whose vertices possesses the plane coordinates: $A(1,0)$ y.$B(0,1)$.

Through the application of an iterated function system (IFS), which is a mathematical structure consisting in this case of three interrelated functions.

This set is symbolized as IFS: with the IFS students proceed to generate the desired geometric structure.

The related transformation laws that generates IFS are [9]:

$$f_1: R^2 \to R^2 / f_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1)$$

$$f_2: R^2 \to R^2 / f_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad (2)$$

$$f_3: R^2 \to R^2 / f_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \quad (3)$$

Equations (1), (2) and (3) defining the SFI. When they perform the first iteration on the triangle applying the IFS, it is obtained the generator that is a triangle divided in four smaller triangles, as show in Fig. 1 [2], [9], [10].

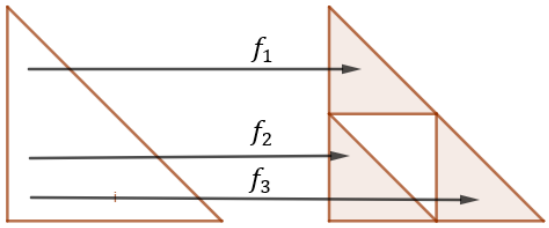

Fig.1 IFS to OAB application.

From there and as consequence of continuing IFS application, the students can find the next two ST graphics observed in Fig.2.

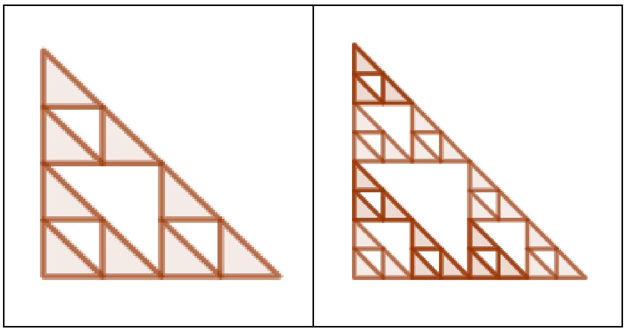

Fig.2 ST First iterations.

The iteration going to infinity when applying the IFS allows students to generate a fractal named ST, Fig.3 shows the final result of the experience [1].

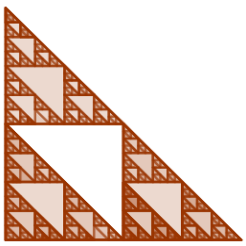

Fig.3 ST Infinite representation.

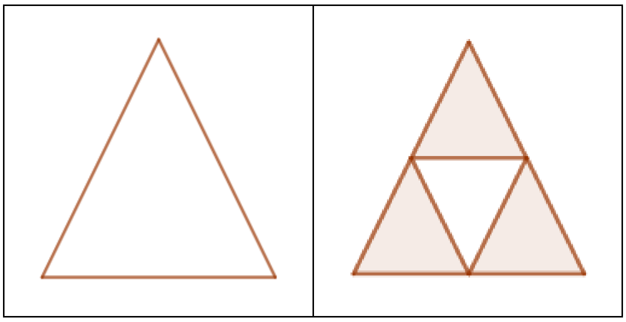

Fig.4 ST Early Phases.

Once the students build Fig.3 ST, and if an orthogonal axis system is fixed, students are asked to create a new ST graphic, but now starting off from a triangle whose vertexes have as coordinates $O(0,0)$,

$A(1,0)$ y $C(0.5,1)$.

If students carry out the same stages as in the case before, they can obtain a new ST but in another plane position. If they start off from graphic triangle OAC as initial phase and build one more stage in the process, both situations are observed in Fig.4.

After graphing the triangles in Fig.4, they perform two more steps like the triangles seen in Fig.5.

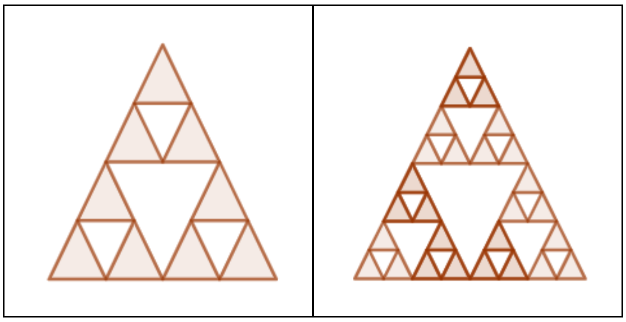

Fig.5 2 and 3 ST process phases.

Last, Fig. 6 presents the final graphic of the final triangle after performing many divisions of the internal triangles.

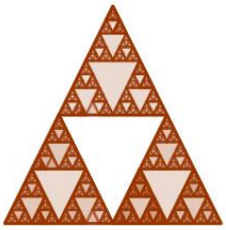

Fig.6 ST Final Phase.

5.2 CA Generation

The designed pedagogical stages to carry out the CA generation in class are:

- Assembling working groups.
- Setting problematic situation.
- Theoretical Subject Investigation.
- Reviewing bibliographical available resources about the subject and subsequent selecting.
- CA Analysis, denominated Rule 90
- CA Modeling.
- Interpretation results.

5.2.1 Theoretical Investigation

A brief orientation guide is proposed so that the students can learn the fundamental concepts of the CA that will be dealt with during the experience. The proposal presented considers the following aspects:

- What is a CA?
- How is an elemental CA defined?
- What are the CA Cells?
- What is the definition of neighborhood?
- Is the term neighbour the same as neighbourhood, and are there differences between the two concepts?
- What are the states of CA?
- What are the periodic limits in an elementary CA?
- How does the CA called rule 90 develop?

The students investigate in the laboratory with the classmates assigned to the class chosen by them, to finally write a single document with the answers to the proposed questionnaire. The students arrived at the following considerations:

A CA is a mathematical and computational model that describes the dynamic system behavior model that evolves in different stages at discrete time intervals.

A CA is suitable for modelling all kinds of complex systems that can be described as a massive collection of simple objects that interact locally with each other.

Elementary Cellular Automata (ECA) are one-dimensional structures made up of a single-row matrix or vector where each cell can have two possible states (0 or 1), and a rule to determine its evolution in the next state. Each number (0 or 1) that makes up the automaton is called a cell or cell of ECA [11].

The Matrix:

$$A = \left(a_{11}a_{12}\dots a_{1k-1}a_{1k}a_{1k+1}\dots a_{1j-1}a_{1j}\right)$$

is an example of a ECA where can be 0 or 1 randomly, so $1 \leq k \leq j$.

The neighbourhood of a a_{1k} cell consists on analysing that cell properly and its two adjacent cells (a_{1k-1} and a_{1k+1}), that is called Neighbors

In the case of a_{1k} cell, it will be convenient to call the left neighbour to a_{1k-1}, and the right neighbour to a_{1k+1}. In the case of a_{11} cell the left neighbour will be a_{1j}, and to the a_{1j}, cell, the right neighbour will be a_{11}. To the a_{11} and a_{1j} cells are denominated periodic limits of ECA.

To determinate an ECA evolution into their next state, each cell is individually analyzed and their neighbors in a time $t = k$, and this update is the one that promotes the passage to a new state $t = k + 1$, $\forall k \in N_0$.

90 Rule, one of the CA proposed by Stephen Wolfram, is a model based in the binary numbering system (BNS) work, from 8 $(8 = 2^3)$ first set

numbers N_0 of decimal numbering system (DNS).

DNS	BNS
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Table 1. Relation by DNS and BNS for 0 to 7 numbers

This considerations promote the existence of 8 $(8 = 2^3)$ possible configurations for a cell and its two immediate Neighbors. The rule that defines the CA must specify the resulting state for each of these possibilities, that is, there are 256 ECA possible [11], [12].

Using the scheme proposed by Wolfram, called the Wolfram code, it is possible to assign a number from 0 to 255 to each rule.

The relationship that exists between SND and SNB to the first 8 set numbers N_0 is observed in Table 1.

Knowing that the relationship that exists between the number 90 in the SND with respect to SNB are:

$$(90)_{10} = (0101110100)_2$$

allows to build the transition rule expressed in Table 2 that defines the Rule 90

Current cell pattern	New state to central cell
000	000
001	001
010	010
011	011
100	100
101	101
110	110
111	111

Table 2. Transition rule for the evolution of states applying Rule 90.

5.2.2 Room for learning. Results

Students are proposed to apply Rule 90 to the matrix A, and find the evolution of the three states following it, then graph them by mean of a computational tool. Being:

$$A = (0 \quad 00 \quad 1 \quad 00 \quad 0)$$

The students indicate that cell of A corresponds to $t = 0$ of the ACE, whose graphic can be observed on Fig.7.

The Black color on Fig.7 graphically represents to number 1, while the white color represents number 0.

Fig.7 A in $t = 0$ Graphic.

The next states that determine the evolution of A are:

For $t = 1$, it is observed that the matrix changes its dimensions to 2 rows

$$\begin{pmatrix} 0 & 00 & 1 & 00 & 0 \\ 0 & 01 & 0 & 10 & 0 \end{pmatrix}$$

The state $t = 1$ is graphically represented as observed in Fig.8.

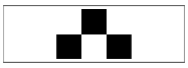

Fig.8 A for $t = 1$ Graphic Evolution

For $t = 2$, the matrix changes its dimension again and now there are 3 rows.

$$\begin{pmatrix} 0 & 00 & 1 & 00 & 0 \\ 0 & 01 & 0 & 10 & 0 \\ 0 & 10 & 0 & 01 & 0 \end{pmatrix}$$

The graphic for state $t = 2$ is shown in Fig. 9

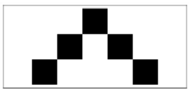

Fig 9: Evolution graphic of A for $t = 2$.

In $t = 3$, the dimension of the new matrix grows by one row, already having 4 rows.

$$\begin{pmatrix} 0 & 00 & 1 & 00 & 0 \\ 0 & 01 & 0 & 10 & 0 \\ 0 & 10 & 0 & 01 & 0 \\ 1 & 01 & 0 & 10 & 1 \end{pmatrix}$$

The evolution graphic of A after applying three times the 90 rule $(t = 3)$ is shown in Fig. 10.

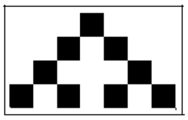

Fig.10 Evolution graphic of A for $t = 3$.

The students interpret that the graph of Fig. 10 begins to present similarities with the ST, this being a fractal that they worked on in the previous assignment about Linear Transformations.

From this observation, students are asked to analyse the behaviour of a row matrix with an odd number of columns, such that the central cell is 1, and the rest of the elements are all 0, as shown in the following structure:

$$B = (00....010.....00)$$

Once this analysis is done, they are asked to state some row matrices with a high number of columns associated with the pattern proposed by B, and using rule 90 to analyze and graph successive states of the evolution of matrices mentioned above.

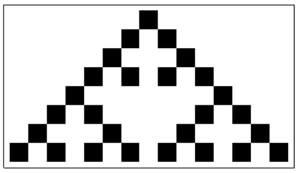

Fig.11 The evolution of various states of type B matrix, applying the 90 Rule.

Fig.11 shows the evolution of a matrix that in its initial state has 15 columns developing 8 stages.

An original matrix of 27 columns developed in 15 steps is observed in Fig.12,

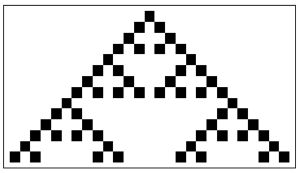

Fig. 12 The evolution of various states of type B matrix, applying the 90 Rule.

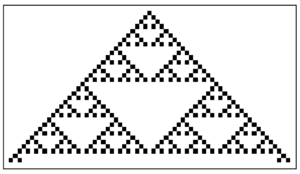

Fig.13 The evolution of various states of type B matrix, applying the 90 Rule.

Fig. 13 shows that the matrix evolution that in its initial state has 65 columns developed in 30 steps.

While Fig. 14 shows an original matrix of 227 columns developed in 130 steps.

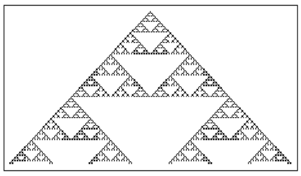

Fig.14 The evolution of various states of type B matrix, applying the 90 Rule.

The ACE observed in Figures 1 to 8 have been generated by the students, through the specific software application. From there, they may conclude that applying 90 Rule to a pattern B matrix, a fractal object known as ST is generated.

4 Conclusion

The CA has been used in different disciplines successfully. Current attention is directed towards the development of models that are capable of carrying out complex tasks such as cryptography, artificial intelligence, image processing and turbulence analysis, among other.

This is why it is considered important to introduce concepts and applications in engineering education, thus proposing a starting point for the study of mathematical models that take into account the resolution of discrete variable problems.

Through the incorporation of activities such as the present proposal, carried out in the Algebra and Analytical Geometry subject, the aim is to promote a change of perspective that leads to seeing the generation of algorithms as a mathematical activity per excellence, and computer science, as complementary knowledge for the execution of those algorithms and management of their outputs.

The students who participated in the experience have expressed their interest in carrying out the application of matrices using a mathematical model such as the CA that uses discrete variables.

In addition, the students had the possibility of contrasting the construction of the ST from two different positions, since in this experience they did so from the principles stipulated by the Linear Transformations and rule 90 of the CA.

These teaching strategies stimulated students' interest in exploring new concepts, using digital tools, and proposing solutions.

The approach of multidisciplinary activities from the Basic Sciences through the exploration of new knowledge encourage creativity from the analysis and management of information, which results in an innovative teaching-learning process.

References:

[1] A. Tinnirello, E. Gago, P. Szekieta, *Algorithmic Mathematics in Linear Algebra Applications*, Proceedings of 12th International Technology, Education and Development Conference, Vol. 1, pp. 2018.

[2] R. Posada Álvarez, *Formación Superior Basada en Competencias, Interdisciplinariedad y Trabajo Autónomo del Estudiante*, Revista Iberoamericana de Educación, Vol. 1, 2011. pp. 6-8.

[3] G. Bischof, E. Bratschitsch, A. Casey, D. Rubesa, *Facilitating Engineering Mathematics Education by Multidisciplinary Projects*, Journal of American Society for Engineering Education, 2007.

[4] M. Rosen, *Engineering Education: Future Trends and Advances*, Proceedings of the 6th. WSEAS International Conference on Engineering Education, Vol. 1, 2009, pp. 44-52.

[5] J. Beckers, J. Crinon, G. Simons, Aprendizaje por competencias: Reducir las desigualdades de aprendizaje entre los alumnos, Publisher Popular, 2014.

[6] V. Macchiarola, *Currículum basado en competencias: Sentidos y críticas*, Revista Argentina de Enseñanza de la Ingeniería, Vol. 8, No. 14, 2007, pp. 39-46.

[7] V. Spinadel, J. Perera. Geometría Fractal, Editorial Nueva Librería, 2007.

[8] Z. Zhu, E. Dong, Geometric Modelling of General Sierpinski Fractals using iterated function system in Matlab, *Proceedings of WSEAS Transactions on Advances in Engineering Education*, Vol. 14, 2017, pp. 50.

[9] G. Nakos, D. Joyner. Álgebra lineal con aplicaciones, Publisher International Thomson, 1999.

[10] B. Kolman, D. Hill, Álgebra lineal con aplicaciones y MATLAB. Publisher Pearson Addison-Wesley, 2006.

[11] M. Calegaris, G. Rodriguez, *Simulaciones Computacionales: Autómatas Celulares*. Actas del congreso: Los métodos numéricos en la enseñanza, la ingeniería y las ciencias, Vol. 1, 2010, pp. 25-35.

[12] R. Gaylord, K. Nishidate, Modeling Nature: Cellular Automata Simulations with Mathematica, Springer-Verlag, 1996., Vol. 1, 2010, pp. 25-35.

[13] R. Gaylord, K. Nishidate, Modeling Nature: Cellular Automata Simulations with Mathematica, Springer-Verlag, 1996.