The Continuous-Time $\mathcal{H}_\infty$ Model Matching Problem: 1 DOF Static State Feedback with Integral Control Approach

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Abstract: The aim of this paper is to develop a new approach for a solution of the continuous-time model matching problem with a static state feedback in the sense of $\mathcal{H}_\infty$ optimality criterion. The main contribution is to reformulate the $\mathcal{H}_\infty$ model matching problem in linear matrix inequality settings, to present the solvability conditions and to give a design procedure for a one degree of freedom static state feedback with integral control law. The results are applied to an example system.

Key–Words: Model Matching Problem, Linear Matrix Inequalities, $\mathcal{H}_\infty$ Optimal Control Problem, One Degree of Freedom Static State Feedback, Integral Control.

1 Introduction

The model matching problem is one of the most familiar problems in the control theory [16]. The continuous-time $\mathcal{H}_\infty$ model matching problem (MMP) is to find a controller transfer matrix $R(s)$ which is stable and causal, that is $R(s)\in\mathbb{R}\mathcal{H}_\infty$, which minimizes the $\mathcal{H}_\infty$ norm of $G_m(s)-G(s)R(s)$ where $G_m(s)$ and $G(s)$ are the model and the system transfer matrices, respectively. Moreover, $G_m(s)$ and $G(s)$ are stable and proper transfer matrices. That is to say, the closed-loop performance $G(s)R(s)$ approximates the desired performance $G_m(s)$ such that,

$$\gamma_{opt} = \inf_{R(s)\in\mathbb{R}\mathcal{H}_\infty} \|G_m(s) - G(s)R(s)\|_\infty.$$

In the literature, there are some results on the $\mathcal{H}_\infty$ MMP: [6, 8, 9]. Moreover, the solutions of the continuous- and discrete-time $\mathcal{H}_\infty$ MMP via linear matrix inequality (LMI) optimization are given in [1, 2, 3, 4, 13]. However, in none of them, one degree of freedom static state feedback with integral control structure is used for feedback configuration.

In this study, a special formulation is developed to solve the continuous-time $\mathcal{H}_\infty$ MMP by a one degree of freedom (1 DOF) static state feedback with integral control. One degree of freedom controller means that there is only one controller block in the closed system, [14]. This formulation enables us to use the methods which are presented for the solution of the continuous-time $\mathcal{H}_\infty$ optimal control problem (OCP) and so the continuous-time $\mathcal{H}_\infty$ MMP can completely be solved by the LMI-based numerical optimization.

The paper is organized in the following way: In Section 2, a special formulation for the continuous-time $\mathcal{H}_\infty$ MMP by a 1 DOF static state feedback with integral control is presented in LMIs. In Section 3, the main result is given by a theorem which provides have the existence conditions of the solution. In Section 4, the problem is examined for the strictly proper case. In Section 5, the 1 DOF static state feedback with integral control is constructed by using the synthesis theorem. A numerical example and the conclusions are finally given in Section 6 and 7, respectively.

Notations

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^{n\times m}$	The set of $n\times m$ real matrices.
I_n	Identity matrix of $n\times n$ dimension.
$0_{n\times m}$	The matrix which has $n\times m$ dimension, and all elements are zero.
$KerM$	The null space of the linear operator M.
ImM	The range of the linear operator M.

N^T The transpose of the matrix N.

$P > 0$ The matrix P is positive definite.

$\lambda_{max}(A)$ The maximal eigenvalue of the matrix A.

$\sigma_{max}(A)$ The maximal singular value of the matrix A which is defined

$$\sigma_{max}(A) = \sqrt{\lambda_{max}(A^T A)}.$$

$\|G(s)\|_\infty$ The $\mathcal{H}_\infty$ norm of the transfer matrix $G(s)$ is defined as

$$\|G(s)\|_\infty = \sup_{\omega \in [0,\infty]} \sigma_{max}[G(j\omega)].$$

2 The Continuous-time $\mathcal{H}_\infty$ MMP by a 1 DOF Static State Feedback with Integral Control in LMI Optimization

In order to solve the continuous-time $\mathcal{H}_\infty$ MMP via LMI approach, the problem should be reformulated as the standard continuous-time $\mathcal{H}_\infty$ OCP. First of all, I will take any state-space equations of the given system $G(s)$ and the model system $G_m(s)$ as follows:

$$G(s): \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) && (1)\\ y_s(t) &= Cx(t) + Dv(t) && (2) \end{aligned}$$

$$G_m(s): \quad \begin{aligned} \dot{q}(t) &= Fq(t) + Gw(t) && (3)\\ y_m(t) &= Hq(t) + Jw(t) && (4) \end{aligned}$$

where $x(t) \in \mathbb{R}^{n_s}$, $q(t) \in \mathbb{R}^{n_m}$; $v(t)$, $w(t)$, $y_s(t)$ and $y_m(t) \in \mathbb{R}^m$. The control input $u(t)$ is generated by a static state feedback controller:

$$u(t) = Kx(t).$$

In Figure 1, the block diagram of a continuous-time $\mathcal{H}_\infty$ MMP by a static state feedback with integral control is given. In this formulation the steady-state value of the output $y_s(t)$ will follow a step function input with zero error. In this paper, a 1 DOF control structure is proposed, [14].

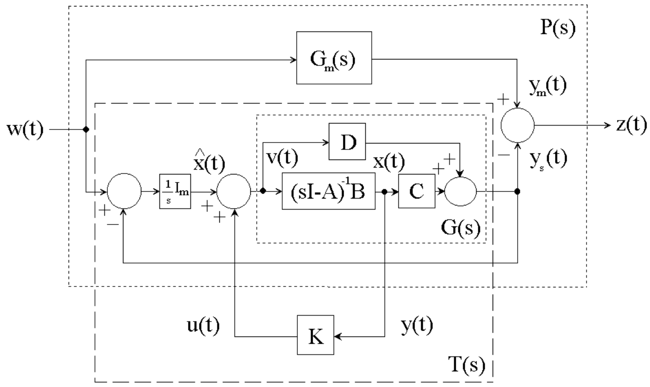

Figure 1. The block diagram of model matching system with 1 DOF static state feedback in the integral control.

The $P(s)$ shown in Figure 1 can be given as,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ -C & -D & 0 \\ 0 & 0 & F \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \\ G \end{bmatrix} w(t) + \begin{bmatrix} B \\ -D \\ 0 \end{bmatrix} u(t) \quad (5)$$

$$z(t) = \begin{bmatrix} -C & -D & H \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ q(t) \end{bmatrix} + Jw(t) - Du(t) \quad (6)$$

$$y(t) = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ q(t) \end{bmatrix}. \quad (7)$$

From the above equations, let us define some matrices as follows:

$$\underline{A} = \begin{bmatrix} A & B & 0 \\ -C & -D & 0 \\ 0 & 0 & F \end{bmatrix} \quad (8)$$

$$B_1 = \begin{bmatrix} 0_{n_s \times m} \\ I_m \\ G \end{bmatrix} \quad (9)$$

$$B_2 = \begin{bmatrix} B \\ -D \\ 0_{n_m \times m} \end{bmatrix} \quad (10)$$

$$C_1 = \begin{bmatrix} -C & -D & H \end{bmatrix} \quad (11)$$

$$C_2 = \begin{bmatrix} I_{n_s} & 0_{n_s \times m} & 0_{n_s \times n_m} \end{bmatrix} \quad (12)$$

$$D_1 = J \quad (13)$$

$$D_2 = -D. \quad (14)$$

As a result, the continuous-time $\mathcal{H}_\infty$ MMP by a 1 DOF static state feedback with integral control is equivalent to the continuous-time $\mathcal{H}_\infty$ OCP. Figure 2 shows this idea:

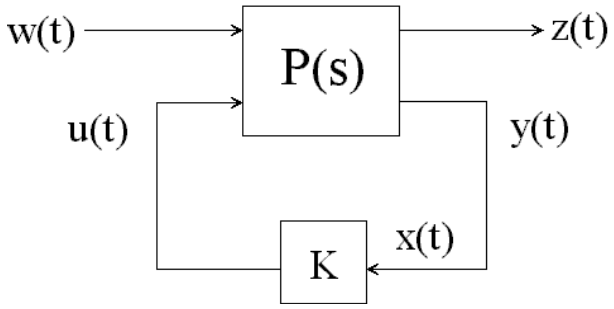

Figure 2. The block diagram of the general form of $\mathcal{H}_\infty$ OCP with a static controller.

The closed-loop transfer matrix from $w(t)$ to $z(t)$ is

$$T_{zw}(s) = D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl} \tag{15}$$

where

$$\begin{aligned} A_{cl} &= \underline{A} + B_2KC_2 & (16)\\ B_{cl} &= B_1 & (17)\\ C_{cl} &= C_1 + D_2KC_2 & (18)\\ D_{cl} &= D_1. & (19) \end{aligned}$$

If the matrix K which makes stable the matrix $A+BK$, can be found out, it is said that the matrix pair (A, B) is stabilizable.

The following lemma can be given for the internal stability of the closed-loop system:

Lemma 1 *For the system in* (5), (6) *and* (7), *there is a matrix K such that the matrix $A_{cl} = \underline{A} + B_2KC_2$ is Hurwitz if and only if the matrix pair*

$$\left(\begin{bmatrix} A & B \\ -C & -D \end{bmatrix}, \begin{bmatrix} B \\ -D \end{bmatrix} \right) \tag{20}$$

is stabilizable and the matrix F is Hurwitz.

Proof: When $\underline{A}$, B_2, C_2 and K are used in A_{cl}, the following relation is obtained:

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A & B & 0 \\ -C & -D & 0 \\ 0 & 0 & F \end{bmatrix} \\ &+ \begin{bmatrix} B \\ -D \\ 0 \end{bmatrix} K \begin{bmatrix} I & 0 & 0 \end{bmatrix} & (21)\\ &= \begin{bmatrix} A+BK & B & 0 \\ -C-DK & -D & 0 \\ 0 & 0 & F \end{bmatrix}. & (22) \end{aligned}$$

Therefore, the matrix A_{cl} is Hurwitz if and only if the matrix

$$\begin{bmatrix} A+BK & B \\ -C-DK & -D \end{bmatrix} \tag{23}$$

and the matrix F are Hurwitz. The matrix

$$\begin{bmatrix} A+BK & B \\ -C-DK & -D \end{bmatrix} \tag{24}$$

can be rewritten as

$$\begin{aligned} \begin{bmatrix} A+BK & B \\ -C-DK & -D \end{bmatrix} &= \begin{bmatrix} A & B \\ -C & -D \end{bmatrix} \\ &+ \begin{bmatrix} B \\ -D \end{bmatrix} K \begin{bmatrix} I & 0 \end{bmatrix}. \end{aligned}$$

If we take

$$L = K \begin{bmatrix} I & 0 \end{bmatrix} \tag{25}$$

, since the matrix K can always be determined by

$$K = L \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{26}$$

, the matrix

$$\begin{bmatrix} A+BK & B \\ -C-DK & -D \end{bmatrix} \tag{27}$$

is asymptotically stable if and only if the matrix pair

$$\left(\begin{bmatrix} A & B \\ -C & -D \end{bmatrix}, \begin{bmatrix} B \\ -D \end{bmatrix} \right) \tag{28}$$

is stabilizable. [15] ■

For a synthesis theorem on the LMI-based solution of the continuous-time $\mathcal{H}_\infty$ MMP with integral control, let us give the following lemmas. They will be used to prove the theorem which will be presented later. The first lemma is well known as **The Bounded Real Lemma** and can be used to turn the continuous-time $\mathcal{H}_\infty$ OCP into an LMI:

Lemma 2 *Consider a continuous-time transfer matrix $T(s)$ of (not necessarily minimal) realization*

$$T(s) = D + C(sI - A)^{-1}B. \tag{29}$$

The following statements are equivalent:
i)

$$\|D + C(sI - A)^{-1}B\|_\infty < \gamma \tag{30}$$

and the matrix A is Hurwitz,
ii) *there is a solution $X > 0$ to the LMI:*

$$\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \tag{31}$$

Proof: See [7]. ■

Lemma 3 *Suppose P, Q and H are matrices and the matrix H is symmetric. The matrices N_P and N_Q are full rank matrices satisfying $ImN_P = KerP$ and $ImN_Q = KerQ$. Then there is a matrix J such that,*

$$H + P^T J^T Q + Q^T J P < 0 \tag{32}$$

if and only if the inequalities

$$N_P^T H N_P < 0 \qquad and \qquad N_Q^T H N_Q < 0 \tag{33}$$

are both satisfied.

Proof: See [10]. ■

Lemma 4 *The block matrix*

$$\begin{bmatrix} P & M \\ M^T & N \end{bmatrix} < 0 \tag{34}$$

if and only if

$$N < 0 \qquad and \qquad P - MN^{-1}M^T < 0. \tag{35}$$

In the sequel, $P - MN^{-1}M^T$ will be referred to as the **Schur complement** *of N.*

Proof: See [5]. ■

3 Main Result

A synthesis theorem can be presented on the LMI-based solution of the problem now:

Theorem 5 *A 1 DOF static state feedback plus integral controller $K \in \mathbb{R}^{m \times n_s}$ exists for the continuous-time $\mathcal{H}_\infty$ MMP if and only if there is a matrix*

$$X_{cl} = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0 \tag{36}$$

such that,

$$\begin{bmatrix} \begin{pmatrix} B^T \\ 0_{n_m \times n_s} \end{pmatrix} X_2 + \begin{pmatrix} -D^T & 0 \\ 0 & F^T \end{pmatrix} X_3 + X_3 \begin{pmatrix} -D & 0 \\ 0 & F \end{pmatrix} + X_2^T \begin{pmatrix} B & 0_{n_s \times n_m} \end{pmatrix} & X_3 \begin{pmatrix} I_m \\ G \end{pmatrix} & \begin{pmatrix} -D^T \\ H^T \end{pmatrix} \\ \begin{pmatrix} I_m & G^T \end{pmatrix} X_3 & -\gamma I_m & J^T \\ \begin{pmatrix} -D & H \end{pmatrix} & J & -\gamma I_m \end{bmatrix} < 0 \tag{37}$$

$$\begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix}^T \cdot \begin{bmatrix} \begin{pmatrix} A & B & 0 \\ -C & -D & 0 \\ 0 & 0 & F \end{pmatrix} X_{cl}^{-1} + X_{cl}^{-1} \begin{pmatrix} A & B & 0 \\ -C & -D & 0 \\ 0 & 0 & F \end{pmatrix}^T & X_{cl}^{-1} \begin{pmatrix} -C^T \\ -D^T \\ H^T \end{pmatrix} & \begin{pmatrix} 0_{n_s \times m} \\ I_m \\ G \end{pmatrix} \\ \begin{pmatrix} -C & -D & H \end{pmatrix} X_{cl}^{-1} & -\gamma I_m & J \\ \begin{pmatrix} 0_{m \times n_s} & I_m & G^T \end{pmatrix} & J^T & -\gamma I_m \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix} < 0 \tag{38}$$

where N_c is a full rank matrix with

$$ImN_c = Ker \begin{bmatrix} B^T & -D^T & 0_{m \times n_m} & -D^T \end{bmatrix}. \tag{39}$$

Proof: From The Bounded Real Lemma, $K \in \mathbb{R}^{m \times n_s}$ is a 1 DOF static state feedback controller in Figure 2 if and only if the LMI

$$\begin{bmatrix} A_{cl}^T X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \tag{40}$$

holds for some $X_{cl} > 0$ in $\mathbb{R}^{(n_s+n_m+m) \times (n_s+n_m+m)}$. Using the expressions A_{cl}, B_{cl}, C_{cl} and D_{cl} in (16), (17), (18) and (19), this LMI can also be written as:

$$H_{X_{cl}} + P_{X_{cl}}^T K Q + Q^T K^T P_{X_{cl}} < 0 \tag{41}$$

where

$$H_{X_{cl}} = \begin{bmatrix} \underline{A}^T X_{cl} + X_{cl} \underline{A} & X_{cl} B_1 & C_1^T \\ B_1^T X_{cl} & -\gamma I_m & D_1^T \\ C_1 & D_1 & -\gamma I_m \end{bmatrix} \tag{42}$$

$$Q = \begin{bmatrix} C_2 & 0_{n_s \times m} & 0_{n_s \times m} \end{bmatrix} \tag{43}$$

$$P_{X_{cl}} = \begin{bmatrix} B_2^T X_{cl} & 0_m & D_2^T \end{bmatrix}. \tag{44}$$

I can use Lemma 3 to eliminate the matrix K in the LMI (41). Therefore, the LMI (41) holds for some K if and only if

$$N_{P_{X_{cl}}}^T H_{X_{cl}} N_{P_{X_{cl}}} < 0 \quad and \quad N_Q^T H_{X_{cl}} N_Q < 0 \tag{45}$$

where

$$ImN_{P_{X_{cl}}} = KerP_{X_{cl}} \tag{46}$$

$$ImN_Q = KerQ \tag{47}$$

$$X_{cl} > 0. \tag{48}$$

Then, the first inequality in (45) can be rewritten as $N_P^T T_{X_{cl}} N_P$ where the matrix N_P denotes any basis of $KerP$ and

$$P = \begin{bmatrix} B_2^T & 0_m & D_2^T \end{bmatrix}. \tag{49}$$

I can take as

$$P_{X_{cl}} = P \begin{bmatrix} X_{cl} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix} \tag{50}$$

hence

$$N_{P_{X_{cl}}} = \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix} N_P. \tag{51}$$

Consequently,

$$N_{P_{X_{cl}}}^T H_{X_{cl}} N_{P_{X_{cl}}} < 0 \tag{52}$$

is equivalent to

$$N_P^T \left\{ \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix} H_{X_{cl}} \begin{bmatrix} X_{cl}^{-1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix} \right\}$$

$$.N_P = N_P^T T_{X_{cl}} N_P < 0 \tag{53}$$

where

$$T_{X_{cl}} = \begin{bmatrix} \underline{A} X_{cl}^{-1} + X_{cl}^{-1} \underline{A}^T & B_1 & X_{cl}^{-1} C_1^T \\ B_1^T & -\gamma I_m & D_1^T \\ C_1 X_{cl}^{-1} & D_1 & -\gamma I_m \end{bmatrix}. \tag{54}$$

Meanwhile, from (49) follows that bases of $KerP$ are

$$N_P = \begin{bmatrix} V_1 & 0 \\ 0 & I_m \\ V_2 & 0 \end{bmatrix} \tag{55}$$

where

$$N_c = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{56}$$

is any basis of the null space of $\begin{bmatrix} B_2^T & D_2^T \end{bmatrix}$. So the condition

$$N_P^T T_{X_{cl}} N_P < 0 \tag{57}$$

can be reduced to

$$\begin{bmatrix} V_1 & 0 \\ 0 & I_m \\ V_2 & 0 \end{bmatrix}^T \begin{bmatrix} \underline{A} X_{cl}^{-1} + X_{cl}^{-1} \underline{A}^T & B_1 & X_{cl}^{-1} C_1^T \\ B_1^T & -\gamma I_m & D_1^T \\ C_1 X_{cl}^{-1} & D_1 & -\gamma I_m \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & I_m \\ V_2 & 0 \end{bmatrix} < 0 \tag{58}$$

or equivalently

$$\begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix}^T \begin{bmatrix} \underline{A} X_{cl}^{-1} + X_{cl}^{-1} \underline{A}^T & X_{cl}^{-1} C_1^T & B_1 \\ C_1 X_{cl}^{-1} & -\gamma I_m & D_1 \\ B_1^T & D_1^T & -\gamma I_m \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix} < 0. \tag{59}$$

Similarly, in (45) the condition

$$N_Q^T H_{X_{cl}} N_Q < 0 \tag{60}$$

is equivalent to

$$\begin{bmatrix} N_o & 0 \\ 0 & I_m \end{bmatrix}^T \begin{bmatrix} \underline{A}^T X_{cl} + X_{cl} \underline{A} & X_{cl} B_1 & C_1^T \\ B_1^T X_{cl} & -\gamma I_m & D_1^T \\ C_1 & D_1 & -\gamma I_m \end{bmatrix}$$

$$. \begin{bmatrix} N_o & 0 \\ 0 & I_m \end{bmatrix} < 0 \tag{61}$$

where

$$ImN_o = Ker \begin{bmatrix} C_2 & 0_{n_s \times m} \end{bmatrix}. \tag{62}$$

Hence the matrix X_{cl} satisfies the LMI (41) if and only if the matrix X_{cl} satisfies the LMIs (59) and (61). To complete the proof, it sufficies to use (8), (9) and (10) into the LMI (61):

$$\begin{aligned} ImN_o &= Ker \begin{bmatrix} C_2 & 0_{n_s \times m} \end{bmatrix} \\ &= Ker \begin{bmatrix} I_{n_s} & 0_{n_s \times m} & 0_{n_s \times n_m} & 0_{n_s \times m} \end{bmatrix} \end{aligned}$$

and

$$N_o = \begin{bmatrix} 0_{n_s \times m} & 0 & 0 \\ I_m & 0 & 0 \\ 0 & I_{n_m} & 0 \\ 0 & 0 & I_m \end{bmatrix}. \tag{63}$$

Therefore, the following inequality can be derived,

$$\begin{bmatrix} 0_{n_s \times m} & 0 & 0 \\ I_m & 0 & 0 \\ 0 & I_{n_m} & 0 \\ 0 & 0 & I_m \end{bmatrix}^T$$

$$. \begin{bmatrix} \begin{pmatrix} A & B & 0 \\ -C & -D & 0 \\ 0 & 0 & F \end{pmatrix}^T X_{cl} + X_{cl} \begin{pmatrix} A & B & 0 \\ -C & -D & 0 \\ 0 & 0 & F \end{pmatrix} & X_{cl} \begin{pmatrix} 0_{n_s \times m} \\ I_m \\ G \end{pmatrix} & \begin{pmatrix} -C^T \\ -D^T \\ H^T \end{pmatrix} \\ \begin{pmatrix} 0_{m \times n_s} & I_m & G^T \end{pmatrix} X_{cl} & -\gamma I_m & J^T \\ \begin{pmatrix} -C & -D & H \end{pmatrix} & J & -\gamma I_m \end{bmatrix}$$

$$\cdot \begin{bmatrix} 0_{n_s \times m} & 0 & 0 \\ I_m & 0 & 0 \\ 0 & I_{n_m} & 0 \\ 0 & 0 & I_m \end{bmatrix} < 0 \quad (64)$$

and the first condition (37) is obtained as,

$$\left[\begin{array}{cc} \begin{pmatrix} B^T \\ 0_{n_m \times n_s} \end{pmatrix} X_2 + \begin{pmatrix} -D^T & 0 \\ 0 & F^T \end{pmatrix} X_3 + X_3 \begin{pmatrix} -D & 0 \\ 0 & F \end{pmatrix} + X_2^T \begin{pmatrix} B & 0_{n_s \times n_m} \end{pmatrix} & X_3 \begin{pmatrix} I_m \\ G \end{pmatrix} \quad \begin{pmatrix} -D^T \\ H^T \end{pmatrix} \\ \begin{pmatrix} I_m & G^T \end{pmatrix} X_3 & -\gamma I_m \quad\quad J^T \\ \begin{pmatrix} -D & H \end{pmatrix} & J \quad\quad -\gamma I_m \end{array} \right] < 0. \quad (65)$$

Finally, the condition (66) can easily be derived when (8), (9) and (10) are used in the LMI (59):

$$\begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix}^T \cdot \left[\begin{array}{cc} \begin{pmatrix} A & B & 0 \\ -C & -D & 0 \\ 0 & 0 & F \end{pmatrix} X_{cl}^{-1} + X_{cl}^{-1} \begin{pmatrix} A & B & 0 \\ -C & -D & 0 \\ 0 & 0 & F \end{pmatrix}^T & X_{cl}^{-1} \begin{pmatrix} -C^T \\ -D^T \\ H^T \end{pmatrix} \quad \begin{pmatrix} 0_{n_s \times m} \\ I_m \\ G \end{pmatrix} \\ \begin{pmatrix} -C & -D & H \end{pmatrix} X_{cl}^{-1} & -\gamma I_m \quad\quad J \\ \begin{pmatrix} 0_{m \times n_s} & I_m & G^T \end{pmatrix} & J^T \quad\quad -\gamma I_m \end{array} \right] \begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix} < 0 \quad (66)$$

■

4 The Strictly Proper Model System Case

Since the system is generally strictly proper in the real life, $D = 0$ is taken. Moreover the model system can generally be chosen as strictly proper, that is $J = 0$. Therefore (37) and (66) LMI's can be reduced to more simple form:

Theorem 6 *A 1 DOF static state feedback plus integral controller $K \in \mathbb{R}^{m \times n_s}$ exists for the continuous-time $\mathcal{H}_\infty$ MMP if and only if there is a matrix*

$$X_{cl} = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0 \quad (67)$$

such that,

$$\begin{pmatrix} B^T \\ 0_{n_m \times n_s} \end{pmatrix} X_2 + \begin{pmatrix} 0 & 0 \\ 0 & F^T \end{pmatrix} X_3 + X_3 \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} + X_2^T \begin{pmatrix} B & 0_{n_s \times n_m} \end{pmatrix} + \frac{1}{\gamma} X_3 \begin{pmatrix} I & G^T \\ G & G^T.G \end{pmatrix} X_3 + \frac{1}{\gamma} \begin{pmatrix} 0 & 0 \\ 0 & H^T.H \end{pmatrix} < 0 \quad (68)$$

$$\begin{bmatrix} W_c & 0 \\ 0 & I \end{bmatrix}^T \cdot \left[\begin{array}{cc} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix} X_{cl}^{-1} + X_{cl}^{-1} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix}^T + \frac{1}{\gamma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & G^T \\ 0 & G & G.G^T \end{pmatrix} & X_{cl}^{-1} \begin{pmatrix} -C^T \\ 0 \\ H^T \end{pmatrix} \\ \begin{pmatrix} -C & 0 & H \end{pmatrix} X_{cl}^{-1} & -\gamma I_m \end{array} \right] \cdot \begin{bmatrix} W_c & 0 \\ 0 & I \end{bmatrix} < 0 \quad (69)$$

where W_c is a full rank matrix with

$$Im W_c = Ker \begin{bmatrix} B^T & 0 & 0_{m \times n_m} \end{bmatrix}. \quad (70)$$

Proof: Let us write the LMI (37) for $D = 0$ and $J = 0$:

$$\left[\begin{array}{cc} \begin{pmatrix} B^T \\ 0_{n_m \times n_s} \end{pmatrix} X_2 + \begin{pmatrix} 0 & 0 \\ 0 & F^T \end{pmatrix} X_3 + X_3 \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} + X_2^T \begin{pmatrix} B & 0_{n_s \times n_m} \end{pmatrix} & X_3 \begin{pmatrix} I_m \\ G \end{pmatrix} \quad \begin{pmatrix} 0 \\ H^T \end{pmatrix} \\ \begin{pmatrix} I_m & G^T \end{pmatrix} X_3 & -\gamma I_m \quad\quad 0 \\ \begin{pmatrix} 0 & H \end{pmatrix} & 0 \quad\quad -\gamma I_m \end{array} \right] < 0. \quad (71)$$

When the Schur complement argument is used, above LMI can be reduced following form:

$$\begin{pmatrix} B^T \\ 0_{n_m \times n_s} \end{pmatrix} X_2 + \begin{pmatrix} 0 & 0 \\ 0 & F^T \end{pmatrix} X_3 + X_3 \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$$

$$+X_2^T \begin{pmatrix} B & 0_{n_s \times n_m} \end{pmatrix} + \frac{1}{\gamma} X_3 \begin{pmatrix} I & G^T \\ G & G^T.G \end{pmatrix} X_3$$

$$+\frac{1}{\gamma} \begin{pmatrix} 0 & 0 \\ 0 & H^T.H \end{pmatrix} < 0. \qquad (72)$$

On the other hand, if J=0 is written in (66),

$$\begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix}^T$$

$$. \left[\begin{matrix} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix} X_{cl}^{-1} + X_{cl}^{-1} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix}^T \\ \begin{pmatrix} -C & 0 & H \end{pmatrix} X_{cl}^{-1} \\ \begin{pmatrix} 0_{m \times n_s} & I_m & G^T \end{pmatrix} \end{matrix} \right.$$

$$\left. \begin{matrix} X_{cl}^{-1} \begin{pmatrix} -C^T \\ 0 \\ H^T \end{pmatrix} & \begin{pmatrix} 0_{n_s \times m} \\ I_m \\ G \end{pmatrix} \\ -\gamma I_m & 0 \\ 0 & -\gamma I_m \end{matrix} \right] \begin{bmatrix} N_c & 0 \\ 0 & I_m \end{bmatrix} < 0 \qquad (73)$$

is written. When the Schur complement argument is used, above LMI can be reduced following form:

$$N_c^T . \left[\begin{matrix} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix} X_{cl}^{-1} \\ \begin{pmatrix} -C & 0 & H \end{pmatrix} X_{cl}^{-1} \end{matrix} \right.$$

$$+X_{cl}^{-1} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix}^T + \tfrac{1}{\gamma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & G^T \\ 0 & G & G.G^T \end{pmatrix}$$

$$\left. \begin{matrix} X_{cl}^{-1} \begin{pmatrix} -C^T \\ 0 \\ H^T \end{pmatrix} \\ -\gamma I_m \end{matrix} \right] .N_c < 0. \qquad (74)$$

From the equation (39)

$$ImN_c = Ker \begin{bmatrix} B^T & 0 & 0_{m \times n_m} & 0 \end{bmatrix} \qquad (75)$$

or

$$N_c = \begin{bmatrix} W_c & 0 \\ 0 & I \end{bmatrix} \qquad (76)$$

where

$$ImW_c = Ker \begin{bmatrix} B^T & 0 & 0_{m \times n_m} \end{bmatrix} \qquad (77)$$

are written. That is, if the equation (77) is used, the LMI (69) is obtained:

$$\begin{bmatrix} W_c & 0 \\ 0 & I \end{bmatrix}^T . \left[\begin{matrix} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix} X_{cl}^{-1} \\ \begin{pmatrix} -C & 0 & H \end{pmatrix} X_{cl}^{-1} \end{matrix} \right.$$

$$+X_{cl}^{-1} \begin{pmatrix} A & B & 0 \\ -C & 0 & 0 \\ 0 & 0 & F \end{pmatrix}^T + \tfrac{1}{\gamma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & G^T \\ 0 & G & G.G^T \end{pmatrix}$$

$$\left. \begin{matrix} X_{cl}^{-1} \begin{pmatrix} -C^T \\ 0 \\ H^T \end{pmatrix} \\ -\gamma I_m \end{matrix} \right] . \begin{bmatrix} W_c & 0 \\ 0 & I \end{bmatrix} < 0. \qquad (78)$$

■

5 Controller Construction

Although Theorem 5 is about the solvability conditions of the continuous-time $\mathcal{H}_\infty$ MMP by the 1 DOF static state feedback with integral control, it also provides a controller construction procedure. Moreover The MATLAB LMI Control Toolbox [11] can be used to solve LMIs. The controller construction procedure can be summarized as follows:

Step 1: Find a solution $X_{cl} > 0$ to the LMIs (37) and (66) for γ_{opt} which is the minimal of γ.

Step 2: Obtain a 1 DOF static state feedback control law $K \in \mathbb{R}^{m \times n_s}$ in the LMI (41).

In the following section, Theorem 6 and the controller construction algorithm will used to design a controller to achieve model matching.

6 Numerical Example

Consider the second-order unstable system

$$G(s) = \frac{s + 0.5}{(s-1)(s+0.2)}.$$

The model system is taken as

$$G_m(s) = \frac{1}{s+1}.$$

The state-space equations of $G(s)$ are obtained as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \qquad (79) \\ y_s(t) &= \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \qquad (80) \end{aligned}$$

The state-space equations of $G_m(s)$ are obtained as

$$\begin{aligned} \dot{q}(t) &= -q(t) + w(t) \qquad (81) \\ y_m(t) &= q(t). \qquad (82) \end{aligned}$$

The matrix F is Hurwitz. Since the matrix pair

$$\left(\begin{bmatrix} A & B \\ -C & -D \end{bmatrix}, \begin{bmatrix} B \\ -D \end{bmatrix} \right) = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0.2 & 0.8 & 1 \\ -0.5 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \qquad (83)$$

is controllable, it is stabilizable. Therefore because of Lemma 1, there is a solution for the continuous-time $\mathcal{H}_\infty$ MMP by a 1 DOF static state feedback with integral control. The state-space equations of $P(s)$ in Figure 2 can be given as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{\hat{x}}(t) \\ \dot{q}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.2 & 0.8 & 1 & 0 \\ -0.5 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \hat{x}(t) \\ q(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t) \qquad (84) \\ z(t) &= \begin{bmatrix} -0.5 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \hat{x}(t) \\ q(t) \end{bmatrix} \qquad (85) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \hat{x}(t) \\ q(t) \end{bmatrix}. \qquad (86) \end{aligned}$$

When I search for a controller, γ_{opt}, the matrix X_{cl} and the 1 DOF static state feedback controller are obtained as follows:

$$\gamma_{opt} = 1.144$$

$$X_{cl} = \begin{bmatrix} 0.5123 & 0.1992 & -0.1094 & -0.1501 \\ 0.1992 & 0.5126 & -0.1407 & -0.1587 \\ -0.1094 & -0.1407 & 0.5294 & -0.1310 \\ -0.1501 & -0.1587 & -0.1310 & 0.9079 \end{bmatrix} > 0$$

$$K = \begin{bmatrix} -1.8655 & -4.1419 \end{bmatrix}.$$

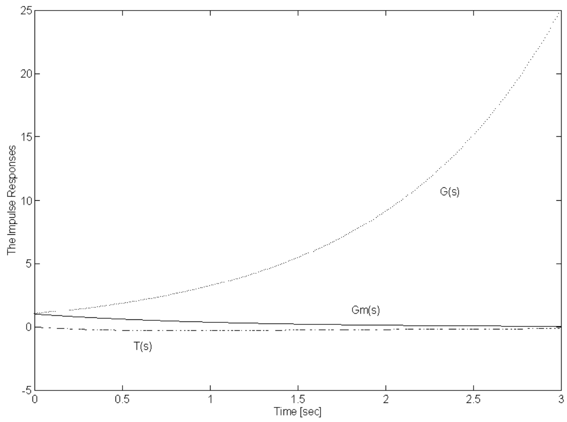

Figure 3. The impulse responses of $G(s)$: ..., $G_m(s)$: $- - -$ and $T(s)$: $-.-$

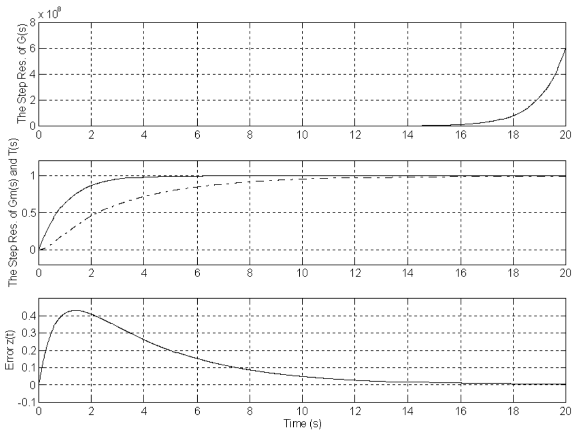

Figure 4. The step responses of $G_m(s)$: $- - -$, $T(s)$: $-.-$ and the error function.

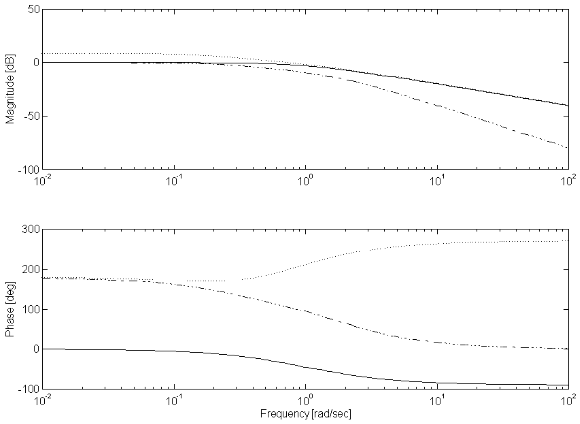

Figure 5. The Bode diagrams of $G(s)$: ..., $G_m(s)$: $---$ and $T(s)$: $-.-$

$T(s)$ is the closed-loop transfer matrix, i.e. $G(s)$ with a 1 DOF static state feedback plus integral controller as it is seen in Figure 1. Figure 3 and Figure 4 illustrate the impulse responses and the unit step responses of $G(s)$, $G_m(s)$ and $T(s)$. In Figure 5, the Bode diagrams of $G(s)$, $G_m(s)$ and $T(s)$ are shown. They are matched over γ_{opt}. As the figures indicate, the controlled system follows the dynamics of the target system.

7 Conclusions

In this paper, the continuous-time $\mathcal{H}_\infty$ model matching problem by the one degree of freedom static state feedback with an integral controller is investigated. In the previous studies, the $\mathcal{H}_\infty$ model matching problem was not solved by one degree of freedom static state feedback plus integral control which makes zero to the steady-state error.

State feedback control with the integral block is well known, [12]. But in this approach there is no zero assignment. System zeros affect the response of a system a little also. The model matching approach contains poles and zeros assignments. Moreover lots of control problem (The disturbance rejection, robust stability etc...) can be solved by using the LMI theory, [7]. In these problems, the solutions are LMIs. If the disturbance rejection and the model matching problem are wanted to solve simultaneously, the matrix $X > 0$ must be found out for all LMI conditions. Therefore it is important to find the LMI conditions of solution of the continuous-time $\mathcal{H}_\infty$ model matching problem by the one degree of freedom static state feedback with an integral controller.

Before the problem is not solved, a block diagram in Figure 1 which is reduced the problem to $\mathcal{H}_\infty$ optimal control problem is proposed and then a synthesis theorem is found out. According to the numerical example, the model matching is really done and the steady-state error is zero. However, the model matching performance can be improved, if two LMIs in Theorem 5 have to be simplified in future.

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