Application of Maximum Principle for Time Minimization of Circuit Design Process

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Abstract: - The possibility of applying the maximum principle of Pontryagin to the problem of optimisation of electronic circuits is analysed. It is shown that in spite of the fact that the problem of optimisation is formulated as a nonlinear task, and the maximum principle in this case isn't a sufficient condition for obtaining a maximum of the functional, it is possible to obtain the decision in the form of local minima. The relative acceleration of the CPU time for the best strategy found by means of maximum principle compared with the traditional approach is equal to some orders of magnitude.

Key-Words: - Circuit optimisation, controllable dynamic system, optimisation strategies, maximum principle of Pontryagin.

1 Introduction

To improve the overall quality of electronic circuit designs, it is very important to reduce their design time. Many works devoted to this problem focus on how to reduce the number of operations when solving two main problems: circuit analysis and numerical optimisation. By solving these problems successfully, one can reduce the total time required for analogue circuit optimisation and this fact serves as a basis for improving design quality.

The methods used to analyse complex systems are being improved continuously. Some well-known ideas related to the use of a method of sparse matrixes [1] and decomposition methods [2] are used for the reduction of time for the analysis of circuits. Some alternative methods such as homotopy methods [3] were successfully applied to circuit analysis.

Practical methods of optimisation were developed for circuit designing, timing, and area optimisation [4]. However, classical deterministic optimisation algorithms may have a number of drawbacks: they may require that a good initial point be selected in the parameter space, they may reach an unsatisfactory local minimum, and they require that the cost function be continuous and differentiable. To overcome these issues, special methods were applied to determine the initial point of the process by centring [5] or applying geometric programming methods [6].

A more general formulation of the circuit optimisation problem was developed on a heuristic level some decades ago [7]. This approach ignored Kirchhoff's laws for all or part of a circuit during the optimisation process. The practical aspects of this idea were developed for the optimisation of microwave circuits [8] and for the synthesis of high-performance analog circuits [9] in an extreme case where all the equations of the circuit were not solved during the optimisation process.

In work [10] the problem of circuit optimisation is formulated in terms of the theory of optimal control. Thus, the process of circuit optimisation was generalised and defined as the dynamic controllable system. In this case, the basic element is the control vector that changes the structure of the equations of the system of optimisation process. Thus, there is a set of strategies of optimisation that have different number of operations and different computing times. The introduction and analysis of the function of Lyapunov of the optimisation process [11-12] allows comparison of various strategies of optimisation and choosing the best of them having minimum processor time. At the same time, the problem of searching for the optimal strategy and the corresponding optimal trajectory can be solved most appropriately by the maximum principle of Pontryagin [13].

The main complexity of application of the maximum principle consists of the search of initial values for auxiliary variables at the solution of the conjugate system of equations. Application of the maximum principle in case of linear dynamic systems is based on the creation of an iterative process [14-15].

In case of nonlinear systems, the convergence of this process is not guaranteed. However, application of the additional approximating procedures [16] allows constructing sequence of the solutions converging to a limit under certain conditions.

The first step in the problem of possibility of application of maximum principle for circuit optimisation was presented in [17]. In the present work, the application of the maximum principle for circuit optimization was investigated with a sufficient accuracy.

2 Problem Formulation

Let's analyse an example of the optimisation of the elementary nonlinear circuit for which the solution was obtained on the basis of the maximum principle. We will consider the simplest nonlinear circuit of a voltage divider in Fig. 1.

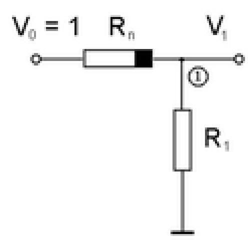

Fig. 1. Simplest nonlinear voltage divider

Let us consider that the nonlinear element has the following dependence:

$$R_n = a + b(V_1 - V_0), \qquad (1)$$

where a>0, b>0, a>b, V_0 and V_1 the voltages on an input and an output of circuit.

We will consider that V_0 is equal 1. We will define the variables x_1, x_2. $x_1 = R$, $x_2 = V_1$. Thus the vector of phase variables $\mathbf{X} \in R^2$. In this case the formula (1) can be replaced with the following expression:

$$R_n = a + b(x_2 - 1). \qquad (2)$$

We can present the equation of a circuit in the form:

$$g_1(x_1, x_2) \equiv x_2[x_1 + a + b(x_2 - 1)] - x_1 = 0 \qquad (3)$$

The circuit optimisation is formulated as a problem of obtaining at the exit of a circuit of the defined voltage w. We will determine the cost function of the optimisation process by the formula:

$$C(\mathbf{X}) = (x_2 - w)^2 . \qquad (4)$$

In this case, the problem of circuit optimisation is converted to minimisation of the cost function $C(\mathbf{X})$. Following theoretical bases that were developed in [10], we formulate the problem for circuit optimisation as a task of search of the optimisation strategy with a minimum possible CPU time. For this purpose, we define the functional, which is subject to minimisation, by the following expression:

$$J = \int_0^T f_0(\mathbf{X})\ dt \ , \qquad (5)$$

where $f_0(\mathbf{X})$ is the function that is conditionally determining the density of a number of arithmetic operations in a unit of time t. In that case, the integral (5) defines total number of operations necessary for circuit optimisation and is proportional to the total CPU time.

The structure of function $f_0(\mathbf{X})$ cannot be defined. However, we can compute CPU time using the possibilities of the compiler. We will further identify the integral (5) with CPU time, and therefore, the problem of minimisation of CPU time corresponds to a problem of minimisation of the integral (5).

According to [10], we introduce the control vector **U** that consists of only one component $u(t)$ for the reviewed example. This component has one of two possible values: 0 or 1. The control vector allows to generalise circuit optimisation process and to define a set of the optimisation strategies differing in operations number and CPU time. The generalised cost function is defined by the formula:

$$F(\mathbf{X}) = C(\mathbf{X}) + \varphi(\mathbf{X}), \qquad (6)$$

where $\varphi(\mathbf{X})$ is an additional penalty function, which can be determined, for example, by the following formula:

$$\varphi(\mathbf{X}) = \sum_{j=1}^{M} u_j \cdot g_j^2(\mathbf{X}), \tag{7}$$

M is the number of nodes of the circuit. In our case M=1.

Process of circuit optimisation thus can be described by the system (8) with restrictions (9):

$$\frac{dx_i}{dt} = f_i(x_1, x_2, u), \quad i=1, 2, \tag{8}$$

$$(1-u) g_1(x_1, x_2) = 0, \tag{9}$$

where functions $f_i(x_1, x_2, u)$ are defined by a concrete numerical method of optimisation. When using a gradient method, these functions are defined by the following formulas:

$$f_i(x_1, x_2, u) = -\frac{\delta}{\delta x_i} F(\mathbf{X}), \; i=1,2, \tag{10}$$

where the operator $\delta / \delta x_i$ is defined by the expression: $\frac{\delta}{\delta x_i}\sigma(\mathbf{X}) = \frac{\partial \sigma(\mathbf{X})}{\partial x_i} + \sum_{p=K+1}^{K+M} \frac{\partial \sigma(\mathbf{X})}{\partial x_p} \frac{\partial x_p}{\partial x_i}$.

The value $u(t)$=0 corresponds to the traditional strategy of optimisation (TSO). In this case in the system (8), there is only one equation for the independent x_1 variable, whereas the variable x_2 is defined from the equation (9). The value $u(t)$=1 corresponds to the modified traditional strategy of optimisation (MTSO) when both x_1 and x_2 variables are independent. In this case, the system (8) includes two equations for the independent variables x_1 и x_2, and the equation (9) disappears. A change in the value of function $u(t)$ with 0 on 1 and back can be made at any moment and generates a set of various strategies of optimisation. Two main strategies are defined as follows:

1) TSO, u=0. The equations (8)–(10) are replaced with the following equations:

$$\frac{dx_1}{dt} = -\frac{\partial C}{dx_2} \frac{dx_2}{dx_1}, \tag{11}$$

$$\frac{dx_2(x_1, t)}{dt} = \frac{\partial x_2}{\partial x_1} \frac{dx_1}{dt}, \tag{12}$$

where the derivative dx_2 / dx_1 is defined from the equation (9) and can be calculated by the formula:

$$\frac{dx_2}{dx_1} = \frac{1}{2b}\left[-1 + \frac{x_1 + c + 2b}{\sqrt{(x_1 + c)^2 + 4bx_1}}\right], \; c=a-b.$$

2) MTSO, u=1. The equations (8) are transformed to the next one:

$$\frac{dx_i}{dt} = -\frac{\delta}{\delta x_i}\left[C(\mathbf{X}) + g_1^2(\mathbf{X})\right], \quad i=1, 2. \tag{13}$$

In a general case, the right-hand parts of the equations (8) can be presented in the form:

$$\begin{aligned} f_1(x_1, x_2, u) &= (1-u) \cdot f_{11}(x_1, x_2) + u \cdot f_{12}(x_1, x_2), \\ f_2(x_1, x_2, u) &= (1-u) \cdot f_{21}(x_1, x_2) + u \cdot f_{22}(x_1, x_2), \end{aligned} \tag{14}$$

where the functions $f_{ij}(x_1, x_2)$ are determined by the following formulas:

$$\begin{aligned} f_{11}(x_1, x_2) &= \frac{(w - x_2)}{b}\left[-1 + \frac{x_1 + c + 2b}{\sqrt{(x_1 + c)^2 + 4bx_1}}\right] \\ f_{12}(x_1, x_2) &= -2(x_2 - 1)\{(x_2 - 1)x_1 + [a + b(x_2 - 1)]x_2\} \\ f_{21}(x_1, x_2) &= \frac{(w - x_2)}{2b^2}\left[-1 + \frac{x_1 + a + b}{\sqrt{(x_1 + c)^2 + 4bx_1}}\right]^2 \\ f_{22}(x_1, x_2) &= -2(x_2 - w) - 2(c + x_1 + 2bx_2) \\ &\quad \cdot [(x_2 - 1)x_1 + ax_2 + b(x_2 - 1)x_2] \end{aligned} \tag{15}$$

According to methodology of the maximum principle, the system of the conjugate equations for additional variables ψ_1, ψ_2 has the next form:

$$\begin{aligned} \frac{d\psi_1}{dt} &= -\frac{\partial f_1(x_1, x_2, u)}{\partial x_1} \cdot \psi_1 - \frac{\partial f_2(x_1, x_2, u)}{\partial x_1} \cdot \psi_2, \\ \frac{d\psi_2}{dt} &= -\frac{\partial f_1(x_1, x_2, u)}{\partial x_2} \cdot \psi_1 - \frac{\partial f_2(x_1, x_2, u)}{\partial x_2} \cdot \psi_2. \end{aligned} \tag{16}$$

The Hamiltonian is expressed by the following formula:

$$H = \psi_1 \cdot f_1(x_1, x_2, u) + \psi_2 \cdot f_2(x_1, x_2, u) \tag{17}$$

Substituting (14) in (17) and doing identical transformations, we obtain the following expression for the Hamiltonian:

$$H = \psi_1 \cdot f_{11}(x_1, x_2) + \psi_2 \cdot f_{21}(x_1, x_2) + u \cdot \Phi(x_1, x_2, \psi_1, \psi_2), \quad (18)$$

where

$$\Phi(x_1, x_2, \psi_1, \psi_2) = \psi_1 \cdot [f_{12}(x_1, x_2) - f_{11}(x_1, x_2)] + \psi_2 \cdot [f_{22}(x_1, x_2) - f_{21}(x_1, x_2)].$$

According to the maximum principle, we obtain the next main condition for the control function u:

$$u = \begin{cases} 0, & \Phi < 0 \\ 1, & \Phi > 0 \end{cases} \quad (19)$$

The behaviour of the control function *u(t)* that corresponds to the maximum principle is also defined by the functions $\psi_1(t)$ and $\psi_2(t)$, which are computed from the Eq. 1 (16).Please, leave two blank lines between successive sections as here.

3 Numerical Results

The solution of the equations (16) depends on the initial values ψ_{10} и ψ_{20}, which are defined within the precision of the common multiplier. One of these constants can be taken arbitrarily. Let us define the constant $\psi_{10} = -1$. The value of the constant ψ_{20}, which corresponds to the correct solution of a task in the conditions of the maximum principle ψ_{20c}, can be obtained by iterative procedure. We use the iterative procedure like a gradient method, which minimise the functional (5). The analysis of the process of optimisation for a similar example, which is carried out in work [18], showed that the TSO (u=0) is the optimal one when both initial values of variables x_1 and x_2, (x_{10}, x_{20}) are positive. In this case the number of iterations is equal to 3898, and CPU time is equal to 42.88 msec for the initial point x_{10} =1, x_{20} =2. At the same time, the negative initial values of the variable x_2 significantly lead to other results. In the case of negative initial values of the variable x_2, emergence of effect of acceleration of the process of circuit optimisation is possible [18]. This effect accelerates the optimisation process in many times. It is interesting to check if this result corresponds to the maximum principle.

Fig. 2 shows the trajectories of the process of circuit optimisation with the negative initial value of coordinate x_{20}, (x_{10} =1, x_{20} = –2).

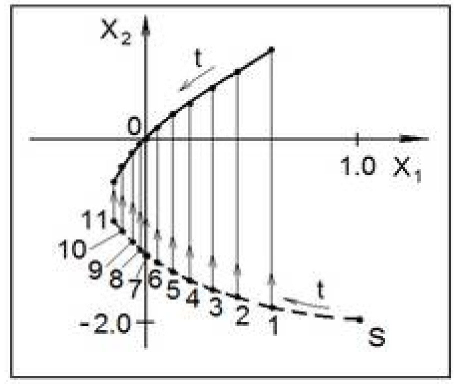

Fig. 2. Trajectories of optimisation process with initial point (x_{10} =1, x_{20} = –2) and different values of ψ_{20}.

The structure of function *u(t)* that was obtained automatically and corresponds to a condition of the maximum principle (19) has one or two points of a rupture that corresponds to switching from the trajectory corresponding to MTSO (u=1, a dotted curve) on trajectory corresponding to TSO (u=0, a continuous curve). Coordinates of a point of switching of t_{sw} depend on the value of ψ_{20}. The data corresponding to the different points of switching from 1 to 11 in Fig. 2 are presented in Table 1.

Table 1. Data of some strategies with different initial values of variable $\psi_2(t)$.

N	ψ_{20}	Control function structure	Switching points	Total iterations number	CPU time (msec)
1	7.27	1; 0; 1	198; 199	2606	14.34
2	7.265	1; 0; 1	200; 201	2464	13.56
3	7.26	1; 0; 1	202; 203	2274	12.52
4	7.255	1; 0; 1	203; 204	2148	11.82
5	7.25	1; 0; 1	205; 206	1759	9.68
6	7.245	1; 0	206	207	1.14
7	7.24	1; 0	209	620	5.67
8	7.235	1; 0	211	711	6.66
9	7.23	1; 0	214	785	7.46
10	7.225	1; 0	216	818	7.81
11	7.22	1; 0	219	855	8.21

A change in the value of ψ_{20} from 7.27 to 7.245 leads to reduction of iterations number and CPU time from 14.34 msec to 1.14 msec, but the CPU time is increasing later on. That is visible also in Fig. 3, where the dependence of CPU time of a task from initial value ψ_{20} is shown.

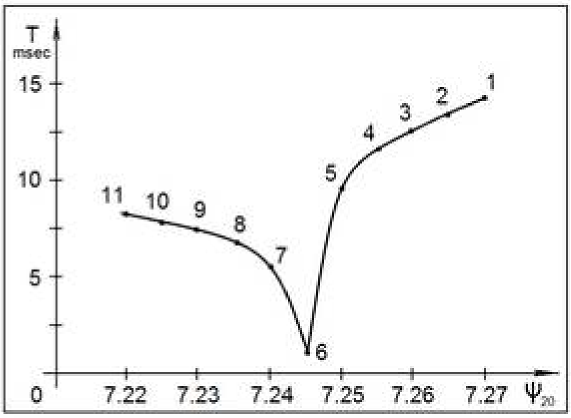

Fig. 3. CPU time for different initial values of $\psi_2(t)$

The value $\psi_{20\,opt}$ = 7.245 corresponds to the minimum CPU time $T_{\min}$ and in this case the integral J and the initial value of variable $\psi_2(t)$ provides the maximum value of a Hamiltonian according to the maximum principle. The gain in time computed as time relation for TSO by the minimum time of $T_{\min}$ thus equal to 37.6 times.

Let us define partial Hamiltonians $H_{(0)}$, $H_{(1)}$ by formulas:

$$H_{(0)} = \psi_1 \cdot f_1(x_1, x_2, 0) + \psi_2 \cdot f_2(x_1, x_2, 0), \quad (20)$$

$$H_{(1)} = \psi_1 \cdot f_1(x_1, x_2, 1) + \psi_2 \cdot f_2(x_1, x_2, 1). \quad (21)$$

Dependencies of the functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ for various values of parameter ψ_{20} are presented in Fig. 4 – Fig. 6. Optimum value of a constant ψ_{20} is equal to 7.245 and corresponds to the results presented in Fig. 4.

In this case the function $H_{(1)}(t)$ passes above the function $H_{(0)}(t)$ from the beginning of the process until the point T_{sw}. At this point both functions become equal, function $\Phi(t)$ changes a sign, and according to condition (19), value of the control function u is changing to 1 on 0. Then, the iterative process comes to the end because the criterion for the end of the optimisation process is satisfied.

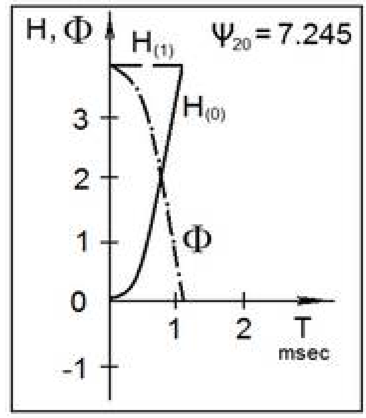

Fig. 4. Time dependency of functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ for optimal parameter ψ_{20}.

We can analyse the behaviour of the functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ with non-optimal initial value ψ_{20}. The point of switching of the control function u from 1 on 0 is not satisfying the optimum point. The behaviour of functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ is shown in Fig. 5 for ψ_{20}=7.249.

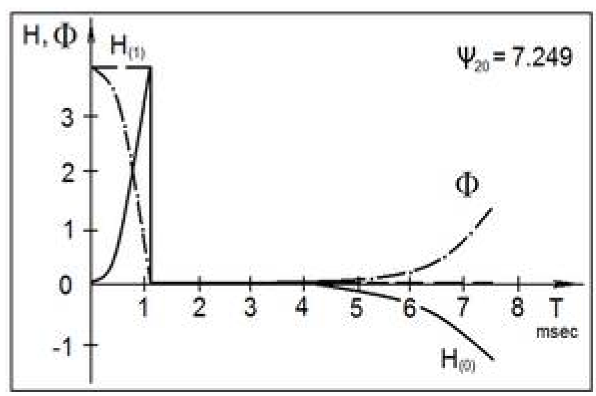

Fig. 5. Time dependency of functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ for non-optimal value of parameter ψ_{20},

$\psi_{20} > \psi_{20opt}$.

The control function switching happens before an optimum point and the computing time grows till 7.55 msec.

The behaviour of these functions is given in Fig. 6 at ψ_{20}= 7.24. In this case the control function switching happens after an optimum point and the time of computing grows again to 5.67 msec.

It is clear that when the point of switching differs from the optimal one, the value of the Hamiltonian is changing over time.

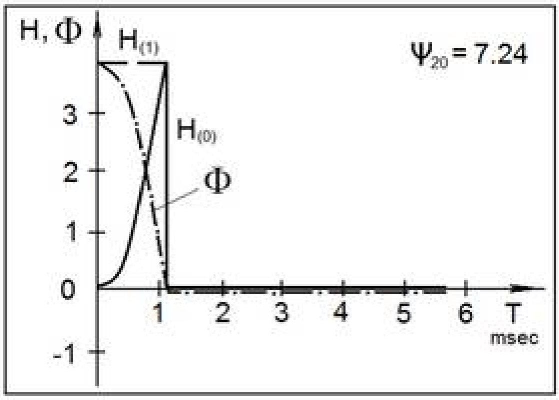

Fig. 6. Time dependency of functions $H_{(0)}(t)$, $H_{(1)}(t)$ and $\Phi(t)$ for non-optimal value of parameter ψ_{20}, $\psi_{20} < \psi_{20opt}$.

The analysis of the optimisation process for the considered circuit has shown that use of the maximum principle really allows for the finding of the optimum structure of the control function u(t) by means of the iterative procedure. At the same time the considerable reduction of the processor time in comparison with the traditional approach is observed.

4 Conclusion

The analysis of optimisation process of the presented circuit showed that application of the maximum principle really allows finding the optimum structure of the control function *u(t)* by means of iterative procedure. Thus, considerable reduction of CPU time in comparison with traditional approach is observed.

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