Rapid Prototyping for Optimal Control of Electrical Drives

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Abstract: In this paper the optimal control of electrical drives like elevators and conveyor machines is approached. Optimal control for moving components means better energy efficiency and also mechanical shock free operation provides a better component lifetime. In the first part of the paper the concepts of optimal control for stationary and dynamic regime together with the optimizing methods such as Gradient algorithm, Lagrange multipliers, Pontryaghin theory are emphasized. For each kind of regime and methods, examples used in electrical drives for transport are presented. In the last part of the paper, an optimal control of movement of a robotic arm is designed and an other for mining elevators is modeled and simulated using Rapid Prototyping methods in the Simulink-dSpace platform.

Key-words: optimal control, electrical drives, modeling, simulation, Simulink-dSPACE, application.

1 Principles and methods of optimal control

1.1. The need for optimal control

The purpose of automated control is driving a process without the intervention of human operators by a controller that generates output values based on information about the status of the process.[5]
The controller drives the technological process with a variety of commands, from which the ones that fulfill a certain criteria are chosen, according to a mathematical performance index (IP), objective, cost, criteria, etc.[7]
If a system is controllable and observable, it is defined by it's state equations[3], the limitations imposed by the process are known, these are called restrictions, and they define the performance index IP and allow us to write the equations of the optimal system:

$$\begin{aligned}
&\dot{x}(t)=f[x(t),u(t)];\, x\in R^n, u\in R^P, f\in C^1\\
&y(t)=g[x(t),u(t)];\, y\in R^q, g\in C^1\\
&h[x(t),u(t)]=0;\, h\in R^t, t<n\\
&IP=min(max)\,F[x(t),u(t)]=0;\, F\in R^1
\end{aligned}\tag{1}$$

Achieving control of $y(t)$ is done by passing the state $x(t)$ from x_0 to x_n using an input $u_{[t_0,t_n]}$ so that the performance index (IP) is extreme (minimum or maximum) and fulfills the restriction $h[x(t),u(t)]$. $u(t)$ is called an optimal command, and the state $x(t)$ is called optimal trajectory.
Determining the extreme can be done by considering the system to be in a stationary or dynamic state. If the system is considered to be stationary, the optimization is stationary, and dynamic if otherwise.
In the case of control in electrical transportation equipment, the value controlled is speed $\omega(t)$ and the optimal trajectory is called an optimal control tachogram. Optimization methods differ for the cases of dynamic and stationary working regimes.

1.2. Optimal control principles

Stationary optimization

Stationary regime is the case in which the state values of the system stay constant, meaning $\dot{x}\equiv 0$.
This means that the function $f[(t),u(t)]\equiv 0$ and the system reaches an equilibrium point or it gets to move in a limited cycle. The function $f[x(t),u(t)]=0$ can be included anyway in the restriction function $h[x(t),u(t)]$.
The optimal system equations are obtained:

$$\begin{aligned}
y(t)&=g[x_0,u(t)]\\
h[x_0,u(t)]&=0\\
IP=min(max)\,F[x_0,u(t)];&\ u(t)\in U
\end{aligned}\tag{2}$$

The stationary optimization defined be equations (2) is also called mathematical programming. Even thou stationary optimization is easier in general, it can only be solved for particular cases of restriction functions $h[x_{0,}u(t)]$ and performance index $F[x_0, u(t)]$:

- when h, F are linear, the optimization is done using linear equation programming: $Au=b\,; IP=min(max)\{c^T u\}$ where $A\in R^n\times R^m, m<n\,; u, b, c\in R^n$.
- When h functions are linear, and F is a quadratic function, the optimization is done using quadratic programming: $Au=b\,; IP=\min_{u\in U}\{\frac{1}{2}u^T Cu+p^T u\}$, where $C\in R^n x R^n$ is symmetric and positive and $p\in R^n$
- when h functions are linear, and F a convex function, optimization is done using convex programming.

Dynamic optimization

When the variation of state values is noticeable $\dot{x}\neq 0$ and the optimal system is:

$$\begin{gathered}\dot{x}(t)=f[x(t), u(t)]\\ h[x(t), u(t)]=0\\ IP=\underset{u(t)\in U}{min(max)} F[x(t), u(t)]\end{gathered} \tag{3}$$

Unlike equations (1) here we consider the output value to be included in the restrictions. Dynamic optimization in the general case can no longer be solved, except for the cases presented earlier. In this case, IP becomes an integral function.

Working with the above statements, the dynamic optimization problem can be solved in certain particular cases of the equations (3):

- when the system is linear and the functional also linear the dynamic optimization can be written:

$$\dot{x}=Ax+bu\,;$$
$$IP=min(max)\int_{t_0}^{t}[u^T(\tau)Q(\tau)x(\tau)]d\,\tau$$

- when the system is linear and the functional quadratic, the quadratic optimization is written:

$$\dot{x}=Ax+bu\,;$$
$$IP=min(max)$$
$$\int_{t_0}^{t}[u^T(\tau)P(\tau)u(\tau)+x^T(\tau)Q(\tau)x(\tau)]d\,\tau$$

1.3. Optimal stationary control methods

There are many methods of stationary optimization, but two of them are most widely used: the gradient and Lagrange multipliers.

Lagrange multipliers

The stationary optimization where there are no imposed sign restrictions on the input allows us to write equations (2) like this:

$$\begin{gathered}y=f(u)\\ h(u)=0\\ IP=min(max)\Phi(u)\end{gathered}$$

The solution to this problem is based on Lagrange's theorem which states that the solutions are given by the equation system solutions:

$$\frac{\partial\Phi(u, x)}{\partial u}=0\,;$$
$$\frac{\partial\Phi(u, x)}{\partial x}=0\,;\Phi(u, x)=F(u)+x^T a(u)$$

where the function $a(u)=[f(u)-y, h(u)]$ represents restrictions and $\Phi(u, x)$ is called the Lagrange function. The solution will be maximum or minimum, according to the H (Hessian) matrix which can be positive or negative defined:

$$H=\begin{bmatrix}\frac{\partial^2\Phi(x, u)}{\partial u^2} & \frac{\partial^2\Phi(u, x)}{\partial u\partial x}\\ \frac{\partial^2\Phi(u, x)}{\partial x\partial u} & \frac{\partial^2\Phi(u, x)}{\partial x^2}\end{bmatrix} \tag{4}$$

The gradient method

This method seeks the functions minimum, closing to it based on the maximum slope.

Using a vectorial function $F(u)$ with a differential:

$$dF=\frac{\partial F}{\partial u}du=\frac{\partial F}{\partial u_1}du_1+\frac{\partial F}{\partial u_2}du_2+...+\frac{\partial F}{\partial u_n}du_n=$$
$$=grad\,F(u)\cos\beta$$

then $\bar{du}\cos\beta=\bar{dl}$ and the gradient can be represented: grad $F=\frac{dF}{d\,l}$ which is the maximum variation of the function F according to the direction $d\,l$ that is the maximum slope.

In conclusion, the vector *F* with the components:

$$\nabla F=\left[\frac{\partial F}{\partial u_1};\frac{\partial F}{\partial u_2};...\frac{\partial F}{\partial u_n};\right]$$

gives us the maximum direction of growth of the function *F,* and $-\Delta F$ the direction of the maximum slope. Determining the the minimum of $F(u)$ is done starting from an initial point u_0 and solving grad $F(u_0)$ after which a new point

$u_1 = u_0 - r \cdot grad\ F(u)$ is chosen. The values $F(u_1) = F(u_0 - r\ grad\ F)$ are determined for different values of the ratio *r* determined with the relation $r_i = r_{i-1} + \alpha, i = 1, 2, \ldots, k$ as long as $F(r_i) < F(r_{i-1})$. When $F(r_{k+1}) \geq F(r_k)$, the last value of the point that is stored is $u_k = u_0 - r_k\ grad\ F$. For this point, the new gradient is determined and a new cycle of iterations begins. The minimum will be reached when the gradient cancels, or when $grad\ F \leq \epsilon$, where ϵ is the set precision.

The presented method has the disadvantage that close to the minimum extra cycles are generated without ever reaching minimum. To eliminate this inconvenient, the Fletcher-Reevs method of determination is used.

This method starts from a random point u_0, grad $F(u_0)$ is determined and the direction $S_1 = -grad\ F(u_0)$ and after that the new vector $u_1 = u_0 + \alpha_1 S_0$ is chosen, where α_1 is the value for which the function $F(\alpha_1)$ is minimal.

The new gradient grad $F(u_1)$ is later determined, $\gamma_1 = \left[\frac{grad\ F(u_1)}{grad\ F(u_0)}\right]$ and the new direction $S_1 = -grad\ F(u_0) + \gamma_1 S_0$ will be determined.

Vector $u_2 = u_1 + \alpha_2 S_1$ results, where α_2 is determined with the functions minimum, $F(\alpha_2)$. The iterations continue until the set precision is reached with $grad\ F(u_n) \leq \epsilon$.

1.4. Dynamic optimal control methods

Dynamic optimization, more difficult that stationary, benefits of more methods of which, used more often are two: variational computation and Pontryagin maximum.

The optimization of systems in dynamic regime means determining u(t) that makes an optimal performance index, and if it's linear, it means solving the problem described by equations (5)

$$\begin{gathered} \dot{x}(t) = f[x(t), u(t)] \\ h[x(t), u(t)] = 0; \\ IP = min(max)\{\int_{t_0}^{t} F[x(t), u(t)]dt\} \end{gathered} \tag{5}$$

Variational computation

If no restrictions are imposed upon the systems state values, the problem is solved by extracting ***u*** from the first equation and substituting it in IP, that will give us the following:

$$IP = min(max)\{\int_{t_0}^{t_1} F[t, x(t), \dot{x}(t)]dt\} \tag{6}$$

The x(t) curve, that optimizes IP, is called extremal and it's obtained as a solution of the Euler equation of the extremals:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$$

The extremal creates a maximum or a minimum of the index IP, according to Legendre's conditions:

$\frac{\partial^2 F}{\partial \dot{x}^2} \leq 0$ for maximum and $\frac{\partial^2 F}{\partial \dot{x}^2} \geq 0$ for minimum.

If restrictions are imposed, then the optimization problem works for an IP built using Lagrange multipliers (λ)

For example, if we impose the following restriction $\int_{t_0}^{t} h(x, u)dt = k$ the new performance index will be given by the integral applied over the Lagrange function:

$\phi(u, x) = F(x(t), u(t)) + \lambda h(x(t), u(t))$, and so:

$$IP = \min_{u(t) \in U}(max) \int_{t_0}^{t} [F(x(t), u(t)) + \lambda h(x(t), u(t))]dt \tag{7}$$

For finding the extremal, Euler's formula is applied on the composed functional, $\phi(u, x)$.

In the case of using Euler's formula, a few specific cases appear:

- If the integral function is like $\phi(t, x, \dot{x})$ then the Euler formula is known:
$$\frac{\partial \phi}{\partial x} - \frac{d}{dt}\left[\frac{\partial \phi}{\partial x}\right] = 0$$
- If the function is like $\phi(t, x_1, x_2, \ldots, x_n; \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$ then we have an Euler equation system like this:
$$\frac{\partial \phi}{\partial x_1} - \frac{d}{dt}\left[\frac{\partial \phi}{d\dot{x}_1}\right] = 0$$
$$\frac{\partial \phi}{\partial x_2} - \frac{d}{dt}\left[\frac{\partial \phi}{d\dot{x}_2}\right] = 0$$
- If the function is like $\phi(t, x, \dot{x}, \ldots, x^{(n)})$ then Euler equation is:

$$\frac{\partial \phi}{\partial x_1} - \frac{d}{dt}\left[\frac{\partial \phi}{d\dot{x}}\right] + \frac{d^2}{dt^2}\left[\frac{\partial \phi}{d\ddot{x}}\right] + \ldots$$
$$\ldots + (-1)^n \frac{d^n}{dt^n}\left[\frac{\partial \phi}{d x^{(n)}}\right] = 0$$

- If the function is like $\phi(u, x, \dot{x}, \ldots, x^{(n)})$, where u is an independent variable, Euler's equation is like:

$$\frac{\partial \phi}{\partial x_1} - \frac{d}{du}\left[\frac{\partial \phi}{d\dot{x}}\right] + \frac{d^2}{du^2}\left[\frac{\partial \phi}{d\ddot{x}}\right] + \ldots$$
$$\ldots + (-1)^n \frac{d^n}{du^n}\left[\frac{\partial \phi}{d x^{(n)}}\right] = 0$$

- If the function is like $\phi(t, \dot{x})$, and doesn't explicitly depend on variable x, the Euler equation is:

$$\frac{\partial \phi}{\partial x} = C_1 = const$$

- If the function is like $f(x, \dot{x})$ and doesn't explicitly depend on independent variables $(t; u)$, the Euler equation is obtained with a variable change and the result is:

$$\phi - \dot{x}\frac{\partial \phi}{\partial \dot{x}} = C_2 = const$$

Pontryaghin's maximum
In this method's case, the system is considered to be of this form:

$$\dot{x}_1 = f_1(x_1, \ldots, x_n; u_1, \ldots, u_p),$$
$$\dot{x}_2 = f_2(x_1, \ldots, x_n; u_1, \ldots, u_p), \ldots,$$
$$\dot{x}_n = f_n(x_1, \ldots, x_n; u_1, \ldots, u_p)$$

and has this vectorial form: $\dot{x} = f(x, u)$ or:
$dx_i / dt = f_i(\ldots, x_i, \ldots, u_j, \ldots); i = 1,2,\ldots n$

the performance index $IP = \max_{u(t)\in U} \int_{t_0}^{t_1} f_0(x, u)\, dt$

The maximum principle developed by Pontryagin requires the optimal command $u(t)$ to be determined and also the optimal trajectory $x(t); t\in[t_0, t_1]$, that maximizes the IP functional. This is built using auxiliary convergent functions. The method is used less frequently so we will skip its detailed presentation.

2 Dynamic optimization of cinematic movement

2.1. Optimizing robot arm movement

In the example of an industrial robot arm, who's gripper needs to move to a given point, using a minimal motion curve.
The following are known:

H – horizontal distance
T – movement duration
v – tangential speed to trajectory
x – trajectory length

The optimal control problem is the following:
The gripper motion tachogram needs to be determined so that the x length of the trajectory to be minimum and the traveled space and time kept constant.
We have the following relations:

$$\int_0^T v \cdot dt = H\, ; x = \int_0^T \sqrt{1+\dot{v}^2} \cdot dt$$

Mathematically, the problem can be written like:

$$G(v) = \int_0^T v\, dt$$

$$IP = min\{\int_0^T \sqrt{1+\dot{v}^2}\, dt + G(v)\}$$

Lagrange's formula is applied for the function $f(v, \dot{v})$ under the integral:

$$f(v, \dot{v}) - \dot{v}\frac{\partial f}{\partial \dot{v}} = const$$

$$\sqrt{1+\dot{v}^2} + \lambda v - \dot{v}\frac{\dot{v}}{\sqrt{1+\dot{v}^2}}$$

By separating the variables we get:

$$v = \int \frac{c_2 - \lambda v}{\sqrt{1-(c_2-\lambda v)^2}} dv + c_1$$

A circle equation is obtained after integration:

$$\left(v - \frac{c_2}{\lambda}\right)^2 + \left(t + \frac{c_1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

We impose that the circle pass through the points (0,0); (o,T) and we get the optimal speed:

$v_{opt} = \sqrt{\frac{T^2}{4} - \left(t - \frac{T}{2}\right)^2}$ The speed curve is a circle that has an equal range with $\frac{T}{2}$ and the semicircle surface is the traveled space H

$$H = \frac{\pi T^2}{8}, T = \sqrt{\frac{8}{\pi}} \cdot \sqrt{H} = 1{,}59 \cdot \sqrt{H}$$

If H=1m, T_optim = 1,59 s , v_max = 0,795 m/s

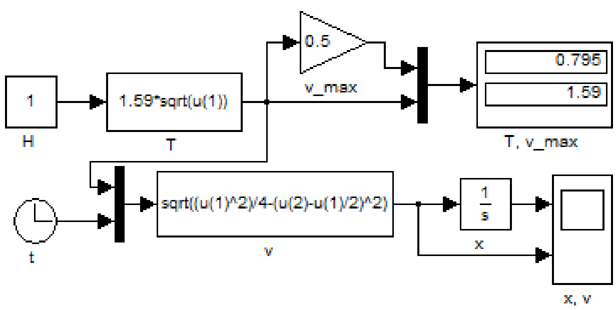

Fig 1: Robot arm movement model

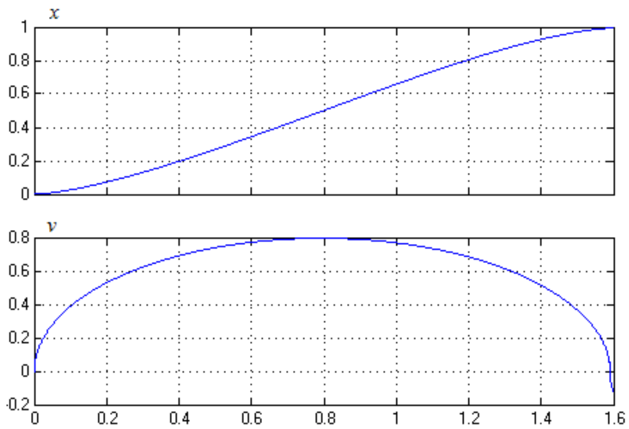

Fig 2: Robot arm movement – distance and speed

2.2. Optimization of a mining elevator

The same problem is accurate for an extracting installation (mining elevator)[6] that for H=400m will have T_optim = 22,49 s , and v_max = T/2 = 11,24 m/s

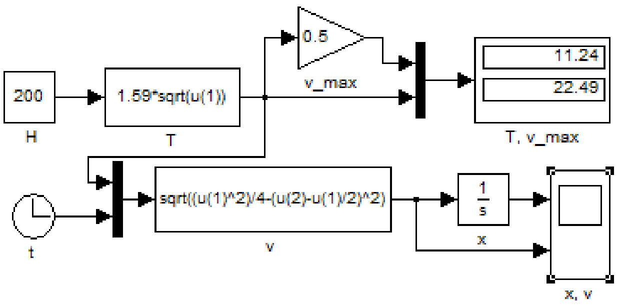

Fig 3: Movement model of mining elevator

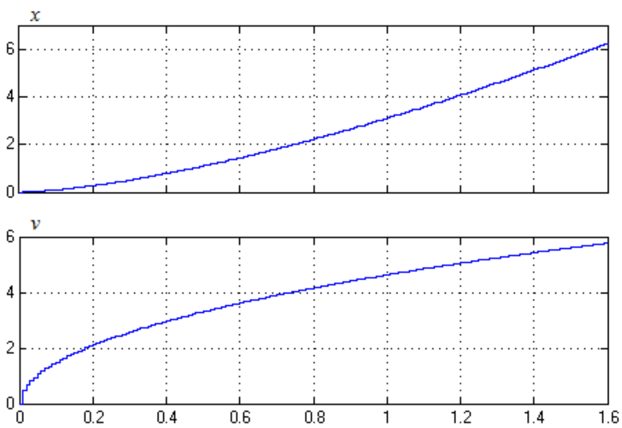

Fig 4: Mining elevator simulation - distance and speed

2.3. Dynamic optimization with minimal shock of a traction installation

Let's consider the case of a pulling installation that needs to travel in the time T on a distance L a mass (load) so that the quadratic integral of the shock during its travel to be minimal. The following symbols will be use for speed, acceleration, shock and space: v(t); a(t); s(t), x(t). Between these there are the known relations:

$$s(t)=\frac{d\,a(t)}{dt}=\dot{a}(t)\,;a(t)=\frac{d\,v(t)}{dt}=\dot{v}(t)\,;\dot{x}(t)=v(t)$$

The quadratic integral is $F=\frac{1}{T}\int_0^T[\dot{a}(t)]^2\cdot dt$

The dynamic optimization problem is the following: The law of speed command needs to be determined so that the quadratic integral of shock be minimal during travel L and duration T.

The problem's equations:

$$IP=f(\dot{a})=min\{\int_0^T\frac{1}{T}[\dot{a}(t)]^2\cdot dt\} \qquad (8)$$

and restriction: $L=\int_0^T v(t)\cdot dt$

We apply Lagrange's formula for IP when it's not dependent of a and t.

$$\frac{\partial f}{\partial\dot{a}}=C_0;\ f=\frac{1}{T}(\dot{a})^2;\ \frac{\partial f}{\partial\dot{a}}=\frac{2}{T}\dot{a}=C_0;$$

$$a(t)=\frac{T}{2}C_0\cdot t+C_1;\ \frac{T}{2}C_0=C_2$$

$$\frac{d\,v(t)}{dt}=C_2\cdot t+C_1;\ v(t)=C_2\frac{t^2}{2}+C_1\cdot t$$

The constants C_1, C_2 are determined from the conditions:

$$t=T\,;v(t)=0 \quad\text{and}\quad \int_0^T v(t)\cdot dt=L$$

$$C_1=\frac{6\mathrm{L}}{T^2};C_2=\frac{-12\mathrm{L}}{T^3};v_{max}=1.5\frac{L}{T}$$

The command law is parabolic:

$$v(t)=\frac{6\mathrm{L}}{T^2}(t-\frac{t^2}{T})$$

Acceleration and shock have the following equations:

$$a(t)=\frac{6\mathrm{L}}{T^2}(1-\frac{2}{T}t) \qquad (9)$$

$$s(t)=\frac{12\mathrm{L}}{T^3}=const$$

We notice that $a_{max}=\frac{6\mathrm{L}}{T^2};\ v_{max}=1{,}5\frac{L}{T}$

In conclusion, the cinematic equations of optimal control with minimum shock are:

$$
\begin{aligned}
x(t) &= a_{max}\cdot\left(\frac{t^2}{2}-\frac{t^3}{3T}\right)\\
v(t) &= a_{max}\cdot\left(t-\frac{t^2}{T}\right)\\
a(t) &= a_{max}\cdot\left(1-2\frac{t}{T}\right)\\
s(t) &= -a_{max}\frac{2}{T}
\end{aligned}
\qquad (10)
$$

As an example, let's consider a locomotive pulling rolling stock on a distance of L=80m, in T=20s. The model is built and simulation results are obtained in the following figure.

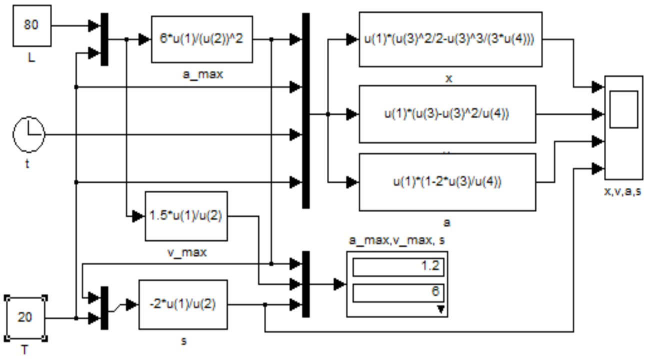

Fig 5: Locomotive movement model

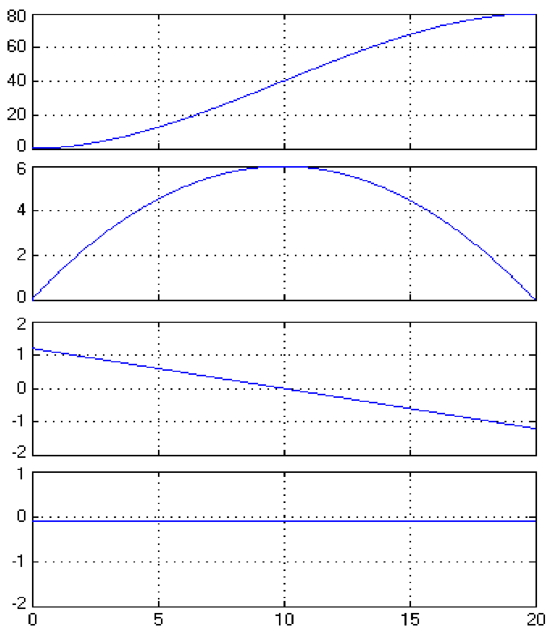

Fig 6: Locomotive movement simulation results: distance (x), speed (v), acceleration (a), shock (s)

3 Conclusions

Optimal control for moving components means better energy efficiency and also mechanical shock free operation provides a better component lifetime. The paper studies concepts of optimal control for stationary and dynamic regime together with optimizing methods like Gradient algorithm, Lagrange multipliers, Pontryaghin theory.

Using MATLAB-Simulink to design a model of a robotic arm and different transportation systems and simulating their control and movement, simulation data show the correct and expected results. Acceleration and deceleration is smooth, with no noticeable spikes, meaning a shock free operation, from both an electrical and mechanical standpoint. Rapid Prototyping methods in the Simulink-dSpace platform can be used to implement the control algorithm on a real controller to be used with any of these examples in the real world as is, with minimal testing on the real plants, because of the accurate modeling of the systems.

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