

Phase-Space Analysis of a Test Particle Dynamics in TRAPPIST-1 Using a Gravitational Model

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Abstract: - Using the Circular Restricted Three-Body Problem (CRTBP) framework, this study examines the dynamics of a small test particle in the TRAPPIST-1 system, concentrating on the gravitational effect between the star and the planet TRAPPIST-1e. The equations of motion are represented using dimensionless coordinates in a rotating reference frame in both Lagrangian and Hamiltonian forms, beginning with the general Newtonian N-body equations. The Runge–Kutta–Fehlberg (RKF45) method and root-finding techniques are used to solve the nonlinear equations in extensive numerical simulations carried out in MATLAB in order to determine equilibrium points. The intricate nonlinear nature of the system and the sensitivity of motion to initial circumstances are further highlighted by phase-plane analysis. The findings show how resonance structures, stability regions, and orbital confinement in compact exoplanetary systems can be better understood by combining classical CRTBP theory with contemporary numerical integration methods. As a result, the TRAPPIST-1 system functions as a representative dynamical laboratory for researching resonant motion and nonlinear gravitational interactions in densely populated planetary structures.

Key-Words: - Circular Restricted Three-Body Problem; TRAPPIST-1 system; Lagrange points; Jacobi constant; Resonant and periodic orbits

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1 Introduction

A broad class of planetary systems with orbital topologies significantly different from the Solar System has been uncovered by the explosive growth of exoplanet finds. Strong mutual gravitational interactions are particularly evident in compact multi-planet systems orbiting low-mass stars, which makes them perfect testbeds for analytical and numerical Celestial Mechanics research. A prime example of how mean-motion resonances and three-body Laplace relations interact to control resonant dynamics and long-term stability is the TRAPPIST-1 system, which consists of seven Earth-sized planets contained within a few hundredths of an astronomical unit [1].

Located around 39 light-years away in the constellation Aquarius, TRAPPIST-1 is an ultra-cool red dwarf star with roughly 12% of the Sun's mass. The system is an important target for researching planetary formation and habitability because it contains at least seven exoplanets the size of Earth, all of which orbit relatively near to their star. Three (e, f, and g) of the planets, designated b through h, are located in the star's habitable zone, where conditions

may permit liquid water to exist on their surfaces at the right temperature and pressure. These planets have short orbital periods and easily observable transits because of their close proximity to the host star. TRAPPIST-1 offers a special lab for studying system evolution, planetary temperatures, and the possibility of extraterrestrial life [2].

Theoretically, reduced models derived from the general N-body problem can be used to study the dynamical development of individual planets in such densely packed systems. The Circular Restricted Three-Body Problem (CR3BP) is a helpful first-order model for studying a planet's motion that is mostly affected by the gravity of its host star and a nearby planet. The Jacobi constant, a conserved quantity in the system, is essential for researching orbital stability and locating dynamically inaccessible areas of phase space [3], [4].

Because the planetary orbits are coplanar and nearly circular, and because nearby planets are close to low-order mean-motion commensurabilities, the CR3BP is relevant to the TRAPPIST-1 system. These commensurabilities lead to resonant Hamiltonians in canonical variables that are derived using canonical

transformations and averaging methods. The basic characteristics of resonant motion, such as libration of resonant angles and confinement of trajectories near stable equilibrium points similar to the Lagrange points of the CR3BP, are captured by the resulting reduced Hamiltonian systems [5].

The TRAPPIST-1 planets participate in a series of three-body Laplace resonances in addition to two-body resonances, where linear combinations of their mean longitudes meet approximate integral relations. By specifying suitable action-angle variables and resonant combinations of angular coordinates, a canonical framework can be used to conveniently characterize these resonances. Even in the presence of significant gravitational interactions between the planets, the existence of such Laplace-type constraints restricts the available phase space and helps to maintain the system's long-term stability [6].

The stability of the TRAPPIST-1 resonant chain is extremely sensitive to the masses and starting orbital phases of the planets, as demonstrated by numerical N-body simulations. On timeframes far shorter than the system's projected age, even little departures from precise resonance might cause chaotic behavior and destabilization. By directly connecting dynamical models with observational data, this sensitivity has been utilized to constrain planetary masses and densities through transit timing changes [7].

TRAPPIST-1e has a particularly significant role in this resonant architecture. Its orbital history is substantially influenced by both two-body and three-body resonance interactions since it is integrated into the resonant chain and situated in the habitable zone. A comprehensive examination of stability zones, resonance overlap, and the impact of minor perturbations such as tidal dissipation or weak non-Keplerian forces is made possible by modeling its motion using CR3BP-inspired and resonant Hamiltonian techniques [5].

The TRAPPIST-1 system is particularly well-suited for research based on symplectic numerical integrators and canonical perturbation theory due to its exceptional compactness, low orbital eccentricities, and high degree of coplanarity. For testing theoretical predictions of resonance dynamics, long-term stability, and phase-space structure in multi-planet systems governed by Hamiltonian mechanics, the system thus offers a remarkable natural laboratory [3], [4], [5], [6], [7].

Five equilibrium points, referred to as Lagrange points, were identified by Euler and Lagrange's analytical investigations of the CRTBP [8], [9]. The two main bodies' orbital planes contain all five

locations. While the triangular points L4 and L5 are situated 60° ahead of and behind the less massive body along its orbit, creating equilateral triangles with the primaries, the points L1, L2, and L3 are collinear with the line joining the two huge bodies.

In this paper, we first introduce the general N-body problem before concentrating on the restricted (circular) three-body problem, which is expressed in both Hamiltonian and Lagrangian formulations. In particular, the study looks at a test particle's behavior in the TRAPPIST-1 exoplanetary system. This study was previously released as a preprint on Research Square (Hysa, 2023) [10]. The article has been enhanced in terms of theory as well as technical and numerical computations. Presenting the results graphically has increased the number of calculations and interpretations of the findings.

Additionally, we examined the velocity of a test particle in the Kepler 22 exoplanetary system using the CRTBP gravitational model in another earlier study [11].

In this work, we used numerical techniques to study a test particle's behavior. To simulate particle motions, a computer model was created and evaluated using MATLAB. For numerical integration, methods including Newton's technique and the Runge–Kutta–Fehlberg (RKF45) algorithm were used. The nonlinear equations of motion for the Circular Restricted Three-Body Problem (CRTBP) were solved in both Lagrangian and Hamiltonian formulations in order to conduct extensive MATLAB simulations. The behavior of a test particle in the TRAPPIST-1 exoplanetary system was better understood thanks to these simulations.

Also, in this work, we use numerical simulations to investigate the dynamical behavior of the TRAPPIST-1 planetary system. According to our analysis, a chain of three-body resonances connects the planets, limiting their motions and structuring the system into a cohesive dynamical structure. Despite the system's small shape, we examine how these resonances affect long-term stability and discover that their combined interactions preclude broad chaotic behavior. This points to a type of self-organization in which, despite intense gravitational interactions, ordered motion is maintained by resonant couplings.

All things considered, our findings offer fresh perspectives on how resonant mechanisms allow densely populated exoplanetary systems to maintain stability over extended periods of time.

2 Fundamental Equations of Motion

By developing his three principles of motion, Isaac Newton laid the groundwork for classical mechanics in his seminal work *Philosophiæ Naturalis Principia Mathematica*, which was published in 1687 [15]. These principles are essential to comprehending how objects behave when subjected to forces and are still fundamental to modern physics. Newton's second law, which is commonly stated as follows, governs the dynamics of a material point:

$$\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m}, (1)$$

where \mathbf{r} denotes the position vector, \mathbf{F} the total force acting on the object, and m its mass. This is a second-order differential equation, the solution of which provides the time evolution of the system, and in general, may not admit an analytical solution depending on the complexity of the applied force.

$$\mathbf{F} = G \frac{m_1 m_2}{r^2} \frac{\mathbf{r}}{r}, (2)$$

where $G = 6.67 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}$ – is the gravitational constant. This law has provided the basis for our understanding of celestial mechanics and planetary motion.

Within *Principia*, Newton also qualitatively addressed the problem of gravitational interaction among three bodies, particularly in Proposition 66, which later evolved into what is now known as the three-body problem. This classical problem involves determining the future motion of three masses under mutual gravitational attraction, given their initial conditions. Despite centuries of research, no general analytical solution exists for this problem, which has been proven to exhibit chaotic behavior in many configurations.

The three-body problem is not limited to celestial mechanics; it also emerges in quantum physics and electromagnetism, where similar mathematical complexities are encountered, and exact solutions remain elusive. The importance of this problem lies in its rich dynamical behavior and its implications in the stability analysis of natural and artificial celestial systems, including planetary, stellar, and exoplanetary systems.

In this paper, we focus on the application of the three-body problem to gravitational interactions within an exoplanetary system, highlighting the necessity for numerical approaches in contemporary astrophysical simulations.

Let us consider the motion of N bodies in a d -dimensional Euclidean space \mathbb{R}^d . The evolution of each body's state of motion is governed by the mutual gravitational interactions among all bodies. According to Newton's second law and the law of

universal gravitation, the dynamics of this system are described by the second-order nonlinear differential equation:

$$m_i \ddot{\mathbf{r}}_i = -G \sum_{j=1, j \neq i}^N \frac{m_i m_j (\mathbf{r}_i - \mathbf{r}_j)}{r_{ij}^3}, (3)$$

where, $i = 1, \dots, N$, and m_i denotes the mass of the i -th body, $\mathbf{r}_i(t) \in \mathbb{R}^d$ is its positions vectors, and r_{ij} is the mutual distance between masses i and j [12].

We focus particularly on the planar case ($d = 2$) or on three-dimensional space for $d = 3$. However, as shown by Poincaré, no explicit analytical solution exists for $N > 2$, due to the system's intrinsic complexity and sensitivity to initial conditions [13].

To facilitate both theoretical treatment and numerical integration, the second-order system can be rewritten as a system of first-order differential equations, separating positions and velocities:

$$\begin{aligned} \dot{\mathbf{r}}_i &= \mathbf{v}_i \\ \dot{\mathbf{v}}_i &= G \sum_{j=1, j \neq i}^N m_j \frac{(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \end{aligned} (4)$$

for $i = 1, 2, \dots, N$. Since each vector has 3 components in 3D space, the complete system consists of $6N$ nonlinear first-order differential equations, requiring $6N$ integration constants to uniquely determine the solution [14].

Two coupled equations describe the time evolution of each body in the system. According to the first equation, a body's velocity equals the rate at which its location changes. To put it another way, the position changes over time based on the body's speed and direction of motion.

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The acceleration of the body is represented by the second equation, which explains how the velocity varies over time. The total gravitational pull of all the other bodies in the system determines this acceleration. In particular, everybody is subject to an attractive force from every other body, and the sum of these individual contributions yields the overall acceleration.

Another body's gravitational pull is directly proportional to its mass and inversely proportional to the cube of the distance between the two bodies. This effect points in the direction of the attracting mass along the line that connects the two bodies. Because

of this, bodies that are closer and heavier have a greater impact than those who are farther away.

Because first-order systems are better suited for numerical integration techniques like Euler or Runge-Kutta schemes, this formulation is especially beneficial. Particularly when studying complex dynamical systems such as the N -body gravitational issue, the system becomes more manageable for computational simulations when the problem is expressed in terms of locations and velocities.

To fully solve the N -body problem, a total of $6N$ integration constants are needed; since only ten are provided by conserved quantities, $6N - 10$ additional constants must be determined from initial conditions. For $N = 2$, the system is fully integrable, as shown historically by Kepler and Newton [15],

[15]. However, for $N = 3$ and beyond, the system lacks general analytical solutions and is commonly studied through numerical integration techniques.

By solving the system of nonlinear differential equations for the general three-body problem (3) using MATLAB software, the results shown in Figure 1 are obtained. The number of bodies we have considered in this numerical simulation is 12 and it is clear that their motion is chaotic as shown in Figure 1a. The trajectories of each body are shown in different colors. We have also calculated the moment of inertia, energy error and energy over time and the results are graphically presented in Figure 1b, c and d, respectively. The results shown in Figure 1 are original.

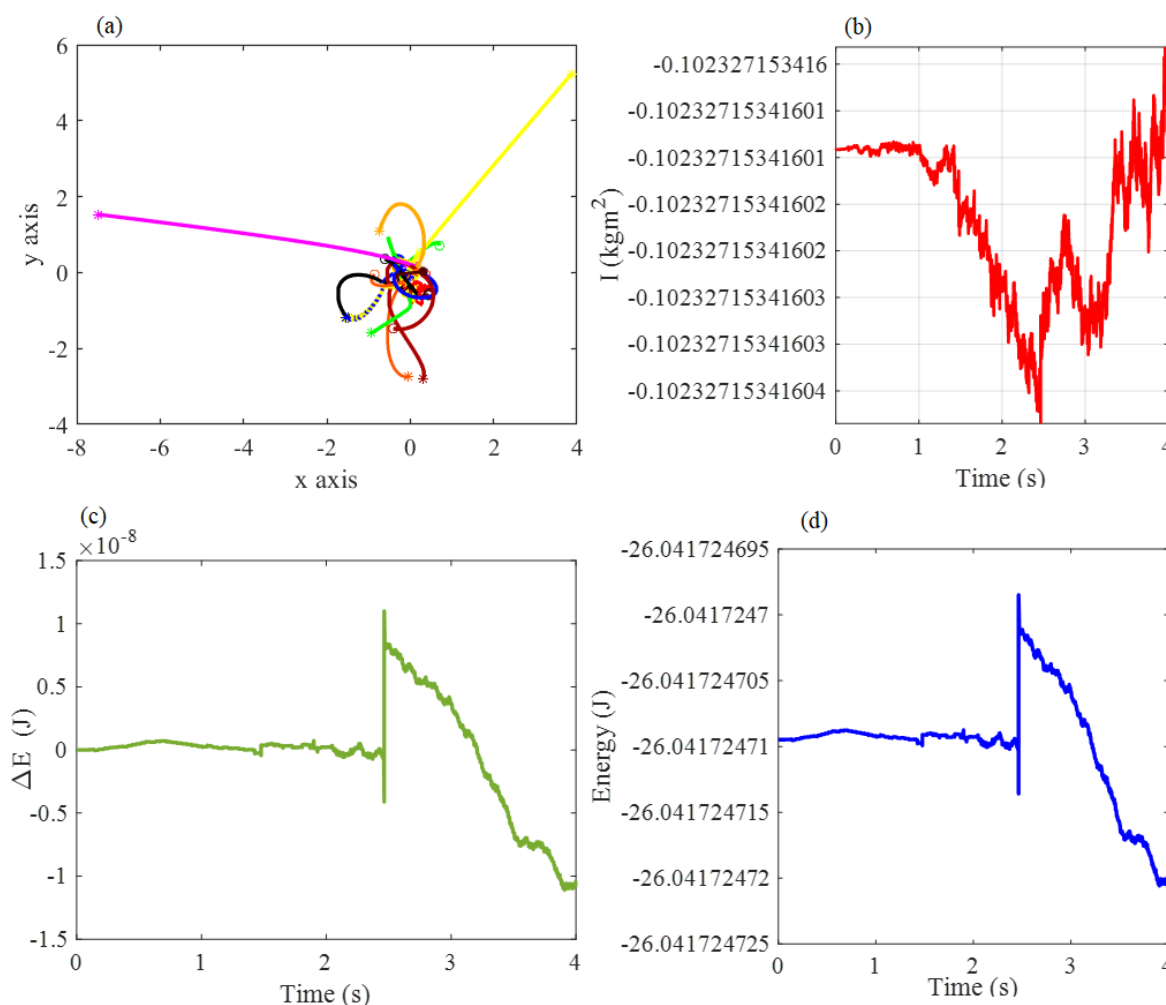


Fig. 1: (a) The trajectories of the 12 bodies (o is start, * is end). We see that the trajectories of the 12 bodies become chaotic. (b) moment of inertia as a function of time. The moment of inertia of the 12-body system changes very little over time. (c) The energy error as a function of time. The energy error of the 12-body system remains very small, on the order of $10e-8$ and changes very little over time. (d) The energy as a function of time. The energy changes very little over time and this means that this integral of motion is conserved

The Circular Restricted Three-Body Problem (CRTBP) is a particularly significant simplification of the ordinary three-body problem. Subject to two fundamental presumptions that greatly simplify the system, this model takes into account the motion of three bodies under mutual gravitational pull. First, it is assumed that two of the bodies, called the primaries, have finite masses and orbit around their shared center of mass in a circle. Since the third body has no effect on their motion, it may be ascertained separately. Large astronomical objects, such a planet and a star or a planet and its moon, are usually represented by these primaries.

Second, it is believed that the mass of the third body is insignificant in relation to the primaries. It is therefore regarded as a test particle and has no effect on their motion. But the gravitational field created by the two primary affects its motion.

These presumptions make the problem easier to solve. The dynamics of the third body can be more easily explained by using a rotating reference frame in which the two major bodies stay stationary. The combined gravitational attraction of the primaries and other apparent forces resulting from the coordinate system's rotation, such as centrifugal and Coriolis effects, control the third body's motion in this frame.

The existence of equilibrium points, or Lagrange points, where the gravitational and inertial effects balance each other, is a crucial aspect of the CRTBP. The third body can remain stationary in relation to the two primaries at these points. Astrophysics and the planning of space missions are greatly interested in these points.

For many real-world systems, including satellite motion in the Earth-Moon system or spacecraft dynamics in the Sun-planet system, the CRTBP offers a helpful approximation. It is a fundamental model in celestial mechanics and captures key aspects of nonlinear gravitational dynamics despite its simplifying assumptions.

A rotating reference frame can be used to represent the motion of a test particle (of negligible mass) under the gravitational force of two fundamental bodies in the Circular Restricted Three-Body Problem (CRTBP). The following Lagrangian controls the test particle's behavior in this frame,

The equations of motion (7) for the CRTBP display specific symmetry properties, such as invariance under the transformations $(t, x, y) \rightarrow (-t, x, -y)$, $(t, \dot{x}, \dot{y}) \rightarrow (-t, -\dot{x}, \dot{y})$, $(t, \ddot{x}, \ddot{y}) \rightarrow (-t, \ddot{x}, -\ddot{y})$ with respect to the x-axis in the rotating coordinate system. These

where the two primaries stay constant on the x-axis [16], [17]:

$$L = \frac{(v_x^2 + v_y^2 + v_z^2)}{2} + (xv_y - yv_x) + \frac{(x^2 + y^2)}{2} + \left(\frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2 + z^2}} \right) + \frac{1}{2}\mu(1-\mu), \quad (5)$$

where x, y, z are the coordinates of the test particle in the rotating frame, v_x, v_y, v_z are its velocity components, and μ is the mass ratio of the two primary bodies. Each term of the Jacobi integral has a clear physical meaning: $\frac{(v_x^2 + v_y^2 + v_z^2)}{2}$ represents the particle's translational kinetic energy in the rotating frame, $(xv_y - yv_x)$ results from the kinetic energy in the rotating frame being expressed. The apparent Coriolis forces are explained by it, $\frac{(x^2 + y^2)}{2}$ represents the centrifugal potential due to the rotation of the reference frame, $\left(\frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2 + z^2}} \right)$ corresponds to the gravitational attraction of the two primary bodies and the term $\frac{1}{2}\mu(1-\mu)$ is added for normalization, simplifying the Lagrangian's use in deriving the equations of motion.

The Lagrangian allows the derivation of the equations of motion via the Euler-Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad j = x, y, z.$$

Naturally, these equations take into account the gravitational forces from the two primaries as well as the effects of centrifugal and Coriolis forces. The test particle's trajectory in the spinning frame can be obtained by numerically solving these equations. Using this Lagrangian formulation we can write the equations of motion that describe the motion in the CRTBP.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} = 0,$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v_y} \right) - \frac{\partial L}{\partial y} = 0, \quad (7)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v_z} \right) - \frac{\partial L}{\partial z} = 0.$$

symmetries provide necessary conditions for the existence of periodic orbits [18].

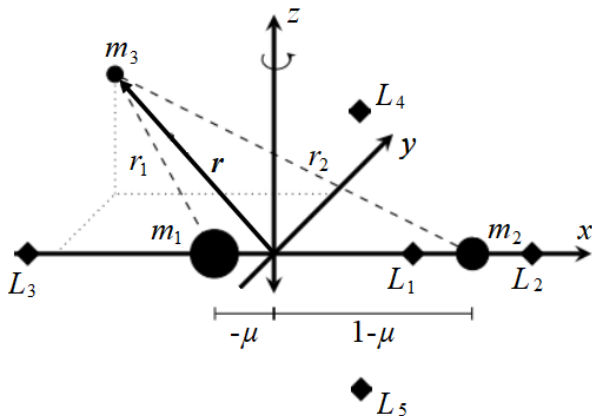


Fig. 2: Rotating coordinate system (nondimensional form) in the CRTBP and the positions of the Lagrange points

The system of the CRTBP admits five equilibrium solutions, referred to as Lagrange points (see Figure 2). These consist of three collinear points (L_1 , L_2 , and L_3) that lie along the line connecting the two primary bodies, and two triangular points (L_4 and L_5) that form equilateral triangles with the primaries. These points play a central role in the study of orbital dynamics and the planning of space missions. Since closed-form analytical solutions of the full system of equations are not available, trajectory computation in the CRTBP typically relies on numerical integration techniques. These methods enable the study of both short-term evolution and long-term dynamical stability of orbits in multi-body systems.

The Circular Restricted Three-Body Problem can be described as a Hamiltonian system, meaning that its nonlinear equations of motion can be expressed in Hamiltonian form. This formulation is obtained by applying a Legendre transformation to the Lagrangian, which introduces conjugate momenta defined in terms of the generalized coordinates and velocities. In this framework, the momenta are expressed as functions of the generalized positions and their corresponding velocities.

$$\begin{aligned} p_x &= \frac{\partial L}{\partial v_x} = v_x - y, \\ p_y &= \frac{\partial L}{\partial v_y} = v_y + x, \\ p_z &= \frac{\partial L}{\partial v_z} = v_z. \end{aligned} \quad (7)$$

The Hamiltonian of the system is defined

$$H = p\dot{q} - L = p_x v_x + p_y v_y + p_z v_z - L \quad (8)$$

So, the Hamiltonian of the system in the CRTBP is:

$$H = -\frac{1}{2} C(x, y, z, v_x, v_y, v_z) \quad (9)$$

From equations (10) we have

$$\begin{aligned} v_x &= p_x + y, \\ v_y &= p_y - x, \\ v_z &= p_z, \end{aligned} \quad (10)$$

The Hamiltonian corresponding to the nonlinear system (7) can be written [21]:

$$H = \frac{1}{2} (p_x^2 + p_y^2 + p_z^2) + y p_x - x p_y - \frac{(1-\mu)}{\sqrt{(x+\mu)^2 + y^2 + z^2}} - \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2 + z^2}}. \quad (11)$$

Periodic orbit in a $2n$ -dimensional Hamiltonian dynamical system are characterized by the following equations [19]:

$$\begin{aligned} q(T) &= q_0, \\ p(T) &= p_0, \end{aligned} \quad (12)$$

where T is the period of the orbit, (q_0, p_0) are the initial conditions at time $t_0 = 0$ and $(q(t), p(t))$, verifies Hamiltonian's equations [19]:

$$\frac{dr}{dt} = \begin{pmatrix} \frac{dq}{dt} \\ \frac{dp}{dt} \end{pmatrix} = \begin{pmatrix} + \frac{\partial H(q, p, t)}{\partial p} \\ - \frac{\partial H(q, p, t)}{\partial q} \end{pmatrix}, \quad (13)$$

Here, r represents the full state vector, consisting of both generalized coordinates and momenta, $(q, p) \in \mathbb{R}^6$.

For a system governed by the Lagrange equations (6), the equivalent form of Hamiltonian's equations are [20]:

$$\begin{aligned} v_x &= \frac{\partial H}{\partial p_x}; & \frac{dp_x}{dt} &= -\frac{\partial H}{\partial x}, \\ v_y &= \frac{\partial H}{\partial p_y}; & \frac{dp_y}{dt} &= -\frac{\partial H}{\partial y}, \\ v_z &= \frac{\partial H}{\partial p_z}; & \frac{dp_z}{dt} &= -\frac{\partial H}{\partial z}. \end{aligned} \quad (14)$$

The Jacobi constant determines the so-called Hill's region, which, in turn, defines the allowable region of motion in the rotating frame for a given energy level.

3 Results

All results presented graphically in the Figures of this section are original and calculated using MATLAB software. These calculations were performed for the TRAPPIST-1 exoplanetary system. This is a very interesting exoplanetary system consisting of a red dwarf star and 7 terrestrial planets orbiting it and which are very close to each other compared to the planets of the solar system. Also, there are no gas giant planets in this exoplanetary system.

First we consider the TRAPPIST-1 - TRAPPIST-1e system and a test particle. With the updated initial conditions, the trajectory continues to exhibit bounded librational motion within the triangular equilibrium region of the rotating frame. Compared to the previous orbit, the particle now starts slightly closer to the primary in the \square -direction, which

reduces the libration amplitude around the triangular points (Figure 3). The orbit remains confined within the exterior energetic domain, avoiding the collinear Lagrange gateways, and still does not become captured around the secondary nor follow a purely Keplerian path. Physically, this orbit represents a Trojan-type motion of the tadpole or transitional horseshoe variety, similar in nature to the previous case but more tightly confined around a single triangular point. The comparison highlights how small variations in the initial position can influence the libration amplitude and the spatial extent of the orbit, while the overall dynamical regime remains stable and regular over long integration times.

For the given initial conditions, the particle is placed along the \square -axis near the opposite side of the secondary, resulting in a trajectory that is energetically confined to a smaller domain and remains close to the primary (Figure 4). Unlike the previous case, which exhibited large-amplitude librational motion around the triangular Lagrange points, this orbit follows a more Kepler-like path around the primary, with minimal influence from the secondary. The motion is regular and bounded over the shorter integration interval, demonstrating stability, but the spatial extent is significantly reduced compared to the earlier Trojan-type orbits. Comparing all three cases highlights how the initial position strongly affects the orbit type: small displacements near the triangular points produce large-amplitude libration, while a position along the opposite axis favors a circumprimary, planet-type orbit with limited libration.

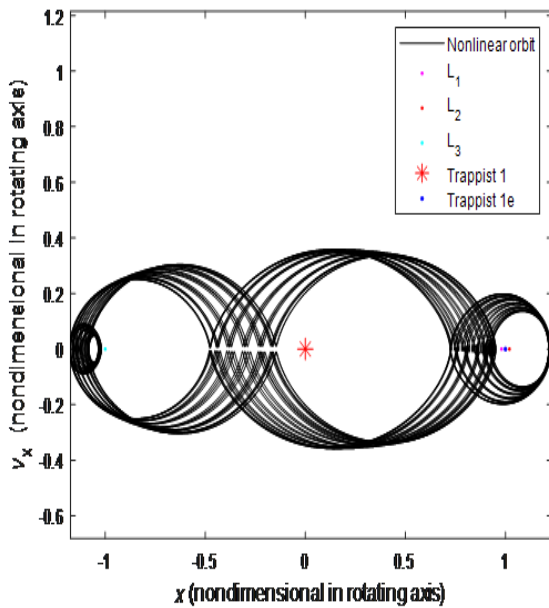


Fig. 3: An additional nonlinear trajectory is shown in the phase plane (x, v_x)

The computed trajectory in Figure 5 represents a short-period, bounded motion in the planar CR3BP. In this configuration, the particle starts just outside the secondary's orbit along the \square -axis, resulting in an orbit that remains closely aligned with the primary–secondary axis. Unlike the previously analyzed Trojan-type orbits with large-amplitude libration around the triangular Lagrange points, this orbit behaves more like a quasi-Keplerian circumprimary motion, with minor perturbations from the secondary. The trajectory is confined by the zero-velocity surfaces and remains regular and stable over the short integration interval. Physically, this orbit illustrates how the particle's proximity to the secondary modifies the motion: the librational component is suppressed, and the orbit remains nearly planar and aligned along the primary–secondary axis rather than exploring the triangular regions.

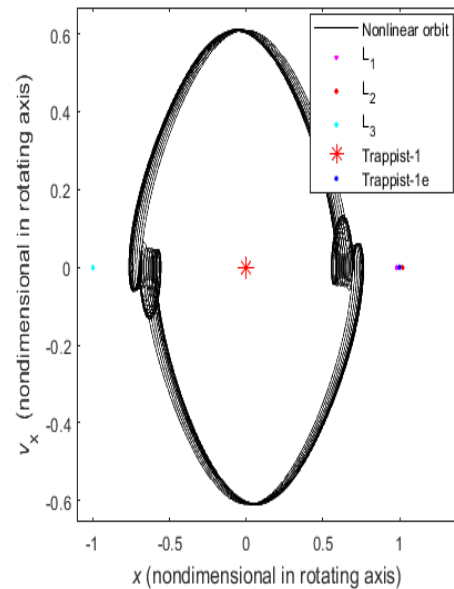


Fig. 4: On-orbit in the phase plane (x, v_x)

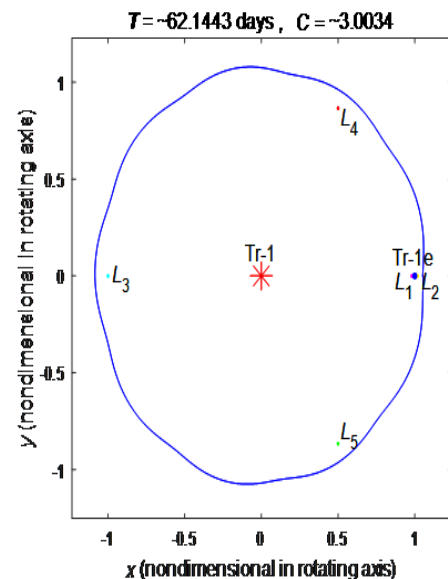


Fig. 5: Short-period, planar orbit. The trajectory remains closely aligned with the primary–secondary axis and exhibits a quasi-Keplerian character, contrasting with the previous large-amplitude Trojan-type librational orbits. The primary, secondary, and all five Lagrange points are indicated, highlighting the confined and regular nature of the motion within the rotating frame

Figure 6 shows another orbit in the exoplanetary system Trappist-1. Its period is approximately 6 days. The Figure 7 show the temporal evolution of the x and y positions of a small particle in the planar circular restricted three-body problem. The particle starts with an initial position offset from the primary–secondary axis and a small initial velocity along the x -direction. The top panel displays the x -coordinate over time, illustrating oscillations around a mean value influenced by the rotating-frame dynamics and the gravitational pull of both massive bodies. The bottom panel shows the y -coordinate, which oscillates around zero with smaller amplitude, reflecting motion near the plane of the primary–secondary axis. Together, these plots indicate that the particle undergoes bounded, quasi-periodic motion, where the combination of Coriolis forces and gravitational interactions leads to oscillatory trajectories that remain confined over the simulated interval. The motion resembles a planar librational orbit that does not approach the Lagrange points closely, maintaining energetic stability within the allowed region defined by the mass ratio and initial energy.

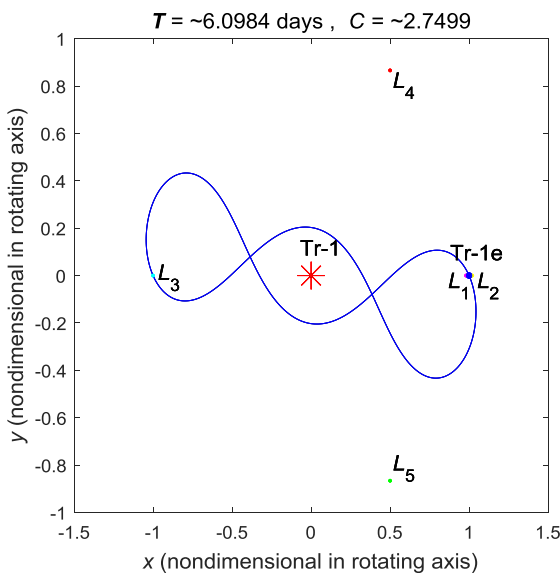


Fig. 6: Short-period, planar orbit in the TRAPPIST-1–TRAPPIST-1e CR3BP system. Its period is approximately 6 days

The figure 8 shows the trajectory of a small third body in the TRAPPIST-1–TRAPPIST-1e system

within the planar circular restricted three-body problem. The particle starts near a 2:1 resonance configuration, resulting in a curved path that reflects the combined gravitational influence of the two massive bodies and the Coriolis forces of the rotating reference frame. The primary star and the planet are shown as markers, while the third body follows a bounded orbit that periodically approaches and recedes from the secondary. The orbit demonstrates a resonant behavior, where the motion is quasi-periodic and remains confined within the energetically allowed region defined by the mass ratio. This configuration illustrates the typical dynamics of small bodies in resonant orbits around low-mass exoplanetary systems.

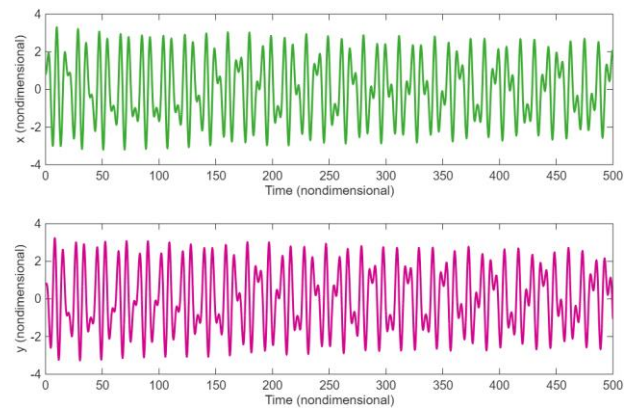


Fig. 7: Time evolution of the x (top) and y (bottom) positions of a small particle. The particle exhibits bounded, quasi-periodic motion in the rotating frame, oscillating around the primary–secondary axis due to the combined gravitational effects and Coriolis forces. The figures highlight the regular, stable nature of the orbit over the integration interval

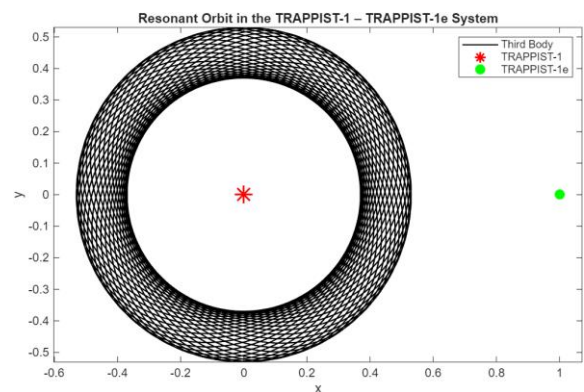


Fig. 8: Resonant orbit of a small third body. The trajectory, computed in the rotating frame, shows quasi-periodic motion near a 2:1 orbital resonance. The positions of the primary star (black) and the planet (red) are indicated. The orbit remains bounded and reflects the influence of both gravitational forces and Coriolis effects in the system

For successive triplets of planets, the Laplace-type resonant angles were calculated as linear combinations of the instantaneous mean longitudes [21],

$$\varphi_i(t) = \lambda_i(t) - 3\lambda_{i+1}(t) + 2\lambda_{i+2}(t). \quad (15)$$

The planar coordinates of each planet, $(x_i, y_i) = \text{atan2}(y_i, x_i)$, were used to calculate mean longitudes. The standard deviation offered a numerical indicator of resonant confinement, and the evolution of φ_i over the integration interval was examined to ascertain libration or circulation [22].

The evolution of φ_i over the integration interval was analyzed to determine libration or circulation, with the standard deviation providing a quantitative measure of resonant confinement [23]:

$$\sigma(\varphi_i) = \sqrt{\frac{1}{T} \int_0^T (\varphi_i(t) - \bar{\varphi}_i) dt}, \quad (16)$$

where $\bar{\varphi}_i = \frac{1}{T} \int_0^T \varphi_i(t) dt$ denotes the time-averaged values. This metric offers a numerical assessment of resonant confinement, where lower values suggest more intense libration and improved dynamical stability. A small $\sigma(\varphi_i)$ indicates strong libration, i.e. greater orbital stability.

We ran a complete N-body simulation of the TRAPPIST-1 system's seven-planet structure. Mutual gravitational interactions were taken into consideration when numerically integrating the equations of motion. For successive triplets of planets, a series of Laplace-type resonant angles was created. These angles' time history shows constrained oscillatory behavior, which suggests that the system is trapped in a resonant chain. The system's long-term dynamical stability is enhanced by this resonance network, which drastically narrows the accessible phase space.

The temporal evolution of the Laplace resonance angles created from successive triplets of planets is shown in Figure 9. The resonant angles show distinct libration, being contained to a small interval rather than moving across the entire angular range.

For the Figure 9 we have the results: $\sigma(\varphi_1) = 1.8066$, $\bar{\varphi}_1 = 0.0026$, $\sigma(\varphi_2) = 0.1290$, $\bar{\varphi}_2 = 1.6728$, $\sigma(\varphi_3) = 0.9548$, $\bar{\varphi}_3 = -2.6842$, $\sigma(\varphi_4) = 0.0464$, $\bar{\varphi}_4 = 0.6501$, $\sigma(\varphi_5) = 0.9776$, $\bar{\varphi}_5 = -2.7401$.

Observations show that multiple consecutive triplets of planets exhibit near Laplace-type

The trajectory's geometry offers qualitative proof of: quasi-periodic motion during certain intervals and shifts towards weakly chaotic behavior caused by multi-body perturbations.

resonances, crucial for ensuring long-term dynamical stability.

In this model, we simulate the movement of a massless test particle influenced by the gravity of the central star and all seven planets through a planar N-body framework. Initial conditions of the planets are selected to simulate near-resonant orbital arrangements that align with the observed period ratios. The objective is to examine phase-space configurations and detect indicators of resonant and chaotic dynamics.

Figure 10 depicts the two-dimensional path of the test particle moving within the gravitational influence of the central star and the seven planets in the TRAPPIST-1 system. The orbit depicted is not entirely Keplerian because of ongoing disturbances from the planetary system. Rather, the trajectory shows: non-closed orbital paths, denoting a departure from straightforward periodic motion, minor oscillations and distortions due to ongoing gravitational interactions with adjacent planets and potential temporary trapping near orbital areas linked to resonances. These characteristics indicate that the particle's movement is affected by the system's resonant chain configuration. Specifically, nearby encounters with planets like TRAPPIST-1e and TRAPPIST-1f can lead to significant changes in the path.

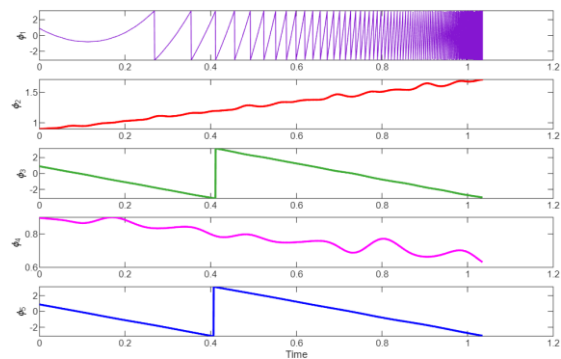


Fig. 9: Laplace-type resonant angle time evolution for successive planet triplets in the TRAPPIST-1 system. With λ being the instantaneous orbital longitude, each panel represents a resonant angle of the form $\varphi_i = \lambda_i - 3\lambda_{i+1} + 2\lambda_{i+2}$. These angles exhibit bounded oscillatory behavior (libration), which suggests that the system is trapped in a series of three-body resonances

Figure 11 shows a phase-space projection of the identical test particle, mapping its position x versus its velocity component v_x . In contrast to the spatial trajectory, the phase-space representation highlights the dynamic characteristics of the motion: closed or

smooth paths signify quasi-periodic (regular) behavior, while scattered or blurred formations suggest sensitivity to disturbances and potential chaotic dynamics, and layered structures or islands might indicate resonant areas associated with Laplace-type resonances within the planetary system. This figure is essential for determining: the level of stability of the orbit, shifts between regular and chaotic states, indications of fundamental resonant mechanisms within the TRAPPIST-1 system.

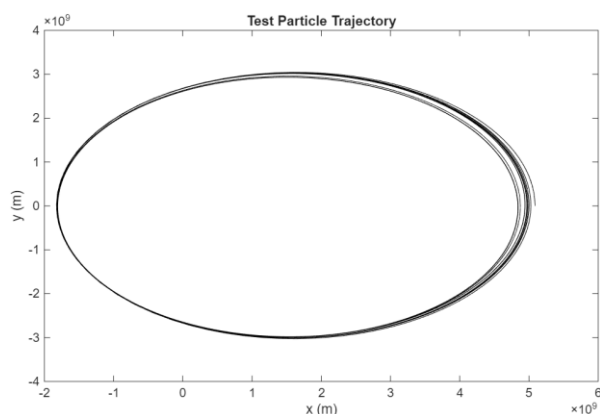


Fig. 10: Trajectory of the test particle in the TRAPPIST-1 system. The particle evolves under the gravitational influence of the central star and all seven planets. The complex orbital structure reflects perturbations induced by the near-resonant planetary chain

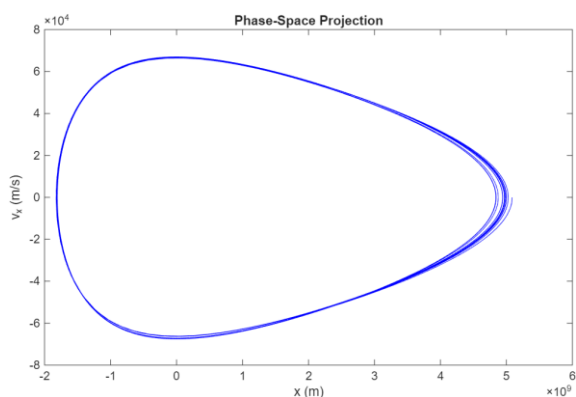


Fig. 11: Phase-space projection (x vs. v_x) of the test particle. The structure of the phase-space reveals regions of quasi-periodic motion interspersed with irregular trajectories, indicating the presence of resonant and weakly chaotic dynamics

Three-body resonance locking is strongly indicated by this phenomenon. The persistence of limited oscillations shows that nearby planets' orbital phases are constrained by approximate integral relations rather than being independent. Consequently, in phase space, the system evolves on a reduced-dimensional manifold.

Furthermore, the strength of the resonance can be inferred from the amplitude of libration. Long-term stability is closely related to tighter phase locking and improved dynamical coherence, which are shown by smaller amplitudes.

4 Discussions

The numerical results obtained in this study provide a detailed picture of the dynamical structures that arise in the CRTBP when applied to the TRAPPIST-1–TRAPPIST-1e configuration. The computed locations of the five Lagrange points and their associated Jacobi constants are consistent with the theoretical expectations for systems characterized by a very small mass ratio. In particular, the proximity of the triangular points L_4 and L_5 to the equilateral configuration reflects the weak perturbative influence of the secondary body, while the collinear points L_1 , L_2 , and L_3 define critical gateways that regulate the accessibility of different regions of phase space.

The simulated trajectories demonstrate a rich variety of dynamical regimes, including bounded motion around the primary star, resonant orbits influenced by the secondary body, and long-lived quasiperiodic trajectories. The existence of periodic and resonant orbits, identified both in configuration space and in phase-plane representations, indicates the presence of underlying invariant structures typical of Hamiltonian systems. These results emphasize that even in a simplified CRTBP framework, the phase space exhibits a complex organization, where small variations in initial conditions can lead to qualitatively different orbital outcomes.

Phase-plane analysis further reveals the nonlinear nature of the dynamics, showing closed curves associated with regular motion as well as more intricate structures linked to resonant behavior. Such representations are particularly useful for distinguishing stable configurations from those that are sensitive to perturbations. The emergence of resonant islands and quasiperiodic motion suggests that resonance mechanisms play an important role in sustaining long-term stability, even in compact planetary systems where gravitational interactions are strong.

From a broader perspective, the findings confirm that the CRTBP, despite its idealized assumptions, captures essential dynamical features relevant to tightly packed exoplanetary systems. The results provide qualitative insight into stability domains, resonance confinement, and transport mechanisms that may also operate in more realistic N-body models. Consequently, the combination of analytical

concepts—such as the Jacobi constant and Lagrange points—with systematic numerical integration proves to be an effective approach for exploring orbital dynamics in complex gravitational systems.

Strong evidence that the TRAPPIST-1 planetary system functions inside a highly ordered dynamical regime structured by a chain of three-body resonances is provided by the numerical studies mentioned above. The system evolves on a limited subset of phase space, which is consistent with near-integrable Hamiltonian dynamics, according to the observed libration of Laplace-type angles.

From a theoretical standpoint, KAM theory can be used to explain this phenomenon [24]. Phase space is dominated by invariant tori in the absence of severe perturbations, allowing for quasiperiodic motion. Bounded resonant angles' persistence indicates that the system is still close enough to integrability for a sizable portion of these invariant structures to endure in spite of mutual gravitational interactions.

However, because of its compact design, the TRAPPIST-1 system operates in a zone where resonant interactions are not insignificant. The applicability of KAM theory becomes nuanced in such circumstances. Resonance overlap, a technique known to produce broad chaotic behavior by deleting invariant tori, is made more likely by the proximity of several mean-motion resonances. Long-term instability can result from paths diffusing over phase space when nearby resonances meet, according to classical conditions.

It's interesting to note that despite this dense resonance structure, the numerical simulations indicate that the system escapes global chaos. By taking into account the stabilizing function of the resonant chain itself, this seeming contradiction can be reconciled. The resonances efficiently organize the dynamics into a coherent structure by being phase-correlated through Laplace-type relations rather than functioning as independent perturbations.

The system functions as a resonantly connected network in this way, with each resonance imposing limitations that strengthen the others.

One way to understand this phenomenon is as a type of dynamic self-organization. The motion is restricted to a lower-dimensional manifold by the resonant chain, which lowers the effective number of degrees of freedom. Consequently, large-scale chaotic diffusion is suppressed by the collective influence of resonances, notwithstanding the strength of individual resonances. This is in line with the lack of noticeable phase drift and the observed libration of several resonant angles. Furthermore, the system most likely exists close to the line between chaotic and regular dynamics. Tidal

dissipation, planetary migration, and external force are examples of small perturbations that could cause the system to move past this limit. Resonance overlap may become effective in such a situation, resulting in partial or global instability. As a result, dissipative processes and the exact values of orbital parameters may have a sensitive impact on the system's long-term stability.

Lastly, the findings lend credence to the more general theory that resonant locking can provide long-term stability for compact multiplanet systems. Strong gravitational interactions can result in highly ordered and stable dynamical structures rather than chaos, as demonstrated by the TRAPPIST-1 system.

5 Conclusions

In this paper, the Circular Restricted Three-Body Problem has been used to analyze the dynamical behavior of a test particle in the TRAPPIST-1–TRAPPIST-1e configuration. The equations of motion were obtained in both Lagrangian and Hamiltonian forms and stated in nondimensional rotating coordinates, beginning with the general N-body formulation.

All things considered, the study demonstrates that the Circular Restricted Three-Body Problem offers a solid theoretical and computational framework for examining resonance dynamics and orbital stability in compact exoplanetary systems. The approach used here can be expanded to more general N-body configurations, incorporate more perturbations, or use sophisticated numerical integrators to examine long-term stability. Thus, the TRAPPIST-1 system continues to be a valuable natural laboratory for investigating resonance-driven orbital structures and nonlinear gravitational dynamics.

All things considered, the TRAPPIST-1 system serves as an example of how intricate gravitational interactions can result in dynamically ordered states via resonance locking. These findings emphasize the significance of multi-body resonances in influencing the long-term evolution of compact planetary systems and advance our knowledge of stability in these systems.

These results pave the door for further investigation into the potential impacts of minor disturbances, like planetary migration or tidal effects, on the fragile equilibrium between chaos and order. Similar resonant mechanisms in other closely spaced exoplanetary systems could be investigated in future research, possibly revealing general principles of long-term stability in multiplanet structures.

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