About the Maxwell Equations for Electromagnetic Field and Peculiarity Analysis of the Wave Spreading

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Abstract: - Methodology of derivation and analysis of the Maxwell equations in total derivatives by time is considered. In continuation of the earlier published paper, this paper is devoted to the methodology of derivation of the Maxwell equations for the electromagnetic field - with total time derivatives - in contrast to the partial derivatives in the "classical" Maxwell equations. Analysis of the electromagnetic wave spreading is performed in detail and the parameters of the waves are derived and analyzed. It is shown that the modified Maxwell equations contain a description of the Doppler Effect, which takes place when the waves of any nature (not only electromagnetic) propagate in a homogeneous and isotropic continuous medium. Also some novel teaching methodologies are mentioned, where these ideas may be spread for the students to learn the Maxwell equations and to train in their analysis and comprehend the wave spreading phenomena.

Key-Words: - Maxwell Equations, Total Derivatives, Electromagnetic Waves, Wavelength Reduction, Expansion, Green Lagging Function, Fresnel Equation, Doppler Effect

1 Introduction

The presented paper has been prepared in continuation of the [1], with the research and educational purposes. The equation of spherical wave was obtained from the wave equation, which is derived from the Maxwell equations as shown below:

$$\omega^2 - \boldsymbol{k}^2 = 0.$$

Dividing this equation by $k^2 \neq 0$, we get:

$$\left(\frac{\omega}{k}\frac{\boldsymbol{k}}{k}\right)^2 = 1.$$

This is equation of the sphere of unit radius in the velocity space (square of the phase speed of the wave is equal to unity). The Akimov's formula [2], for the speed of wave, the source of which is moving with a speed $\boldsymbol{v}$ is as follows:

$$\left(\frac{\omega}{k}\frac{\boldsymbol{k}}{k} - \boldsymbol{v}\right)^2 = 1. \qquad (1)$$

Then from (1) yields:

$$\left(\frac{\omega}{k}\frac{\boldsymbol{k}}{k}\right)^2 - 2\left(\frac{\omega}{k}\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{k}\right) + v^2 = 1,$$

or

$$\omega^2 - 2\omega\boldsymbol{k}\cdot\boldsymbol{v} + k^2v^2 = k^2,$$

$$\omega^2 - 2\omega\boldsymbol{k}\cdot\boldsymbol{v} + (\boldsymbol{k}\cdot\boldsymbol{v})^2 + [\boldsymbol{k}\times\boldsymbol{v}]^2 = k^2,$$

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2 + [\boldsymbol{k}\times\boldsymbol{v}]^2 - k^2 = 0.$$

From the last equation, the solution is as follows:

$$\omega = \boldsymbol{k}\cdot\boldsymbol{v} \pm \sqrt{k^2 - [\boldsymbol{k}\times\boldsymbol{v}]^2},$$

$$\omega = k\cdot v\cdot cos\theta \pm k\cdot\sqrt{1 - v^2\cdot sin\theta^2},$$

$$\frac{\omega}{k} = v\cdot cos\theta \pm \sqrt{1 - v^2\cdot sin\theta^2},$$

where are: θ – the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{v}$, $\frac{\omega}{k}$ – the amplitude of phase speed of the wave. In this form, the last formula was given in [2].

In the right hand of (1) the unity is put just for simplicity. Actually, it is the following equation with the right part the speed of light:

$$\left(\frac{\omega}{k}\frac{\boldsymbol{k}}{k} - \boldsymbol{v}\right)^2 = c^2. \qquad (2)$$

In fact, the equation (2) is the **law of cosines** for the difference of the vectors.

If the receiver is moving too, with a speed $\boldsymbol{w}$, then according to the rule of vectors' adding yields:

$$\left(\frac{\omega}{k}\frac{\boldsymbol{k}}{k} - \boldsymbol{v} + \boldsymbol{w}\right)^2 = c^2.$$

Plus is chosen because if $\boldsymbol{v} = \boldsymbol{w}$ (the receiver and transmitter move in the same direction), then the Doppler Effect is absent.

2 Modification of Maxwell Equations

Now the Maxwell equation array (see Appendix) is considered to reveal, which modification is needed to obtain of it the Akimov's formula [2] (the speed of light is put $c = 1$ just for simplicity) as in (1).

2.1 The Linear Equation Array for the Modified Maxwell Equations

In case of linear differential equations with constant coefficients, it is easier to work with the equivalent their algebraic equations for the Fourier amplitudes. Thus, the linear equation array corresponding to the Maxwell equations in their modified, more general form (see Appendix) is as follows:

$$s\boldsymbol{\beta}_1 + \mathbf{p}_2\boldsymbol{\varepsilon}_3 - \mathbf{p}_3\boldsymbol{\varepsilon}_2 = \mathbf{0},\ s\boldsymbol{\beta}_2 + \mathbf{p}_3\boldsymbol{\varepsilon}_1 - \mathbf{p}_1\boldsymbol{\varepsilon}_3 = \mathbf{0},$$

$$s\boldsymbol{\beta}_3 + \mathbf{p}_1\boldsymbol{\varepsilon}_2 - \mathbf{p}_2\boldsymbol{\varepsilon}_1 = \mathbf{0},$$

$$-\mathbf{q}_2\boldsymbol{\beta}_3 + \mathbf{q}_3\boldsymbol{\beta}_2 + s\boldsymbol{\varepsilon}_1 = \mathbf{I}\boldsymbol{\gamma}_1, \qquad (3)$$

$$-\mathbf{q}_3\boldsymbol{\beta}_1 + \mathbf{q}_1\boldsymbol{\beta}_3 + s\boldsymbol{\varepsilon}_2 = \mathbf{I}\boldsymbol{\gamma}_2,$$

$$-\mathbf{q}_1\boldsymbol{\beta}_2 + \mathbf{q}_2\boldsymbol{\beta}_1 + s\boldsymbol{\varepsilon}_3 = \mathbf{I}\boldsymbol{\gamma}_3,$$

where: $s = -\omega + \boldsymbol{v}\cdot\boldsymbol{k}$. The determinant of this system is: $(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 \cdot ((\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - \boldsymbol{p}\cdot\boldsymbol{q})^2$, so that there are two longitudinal (plane) waves and four «spherical» transversal waves.

Then the coefficients $\boldsymbol{p}$ and $\boldsymbol{q}$ are selected such ones, which allow getting the expression

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2 + [\boldsymbol{k}\times\boldsymbol{v}]^2 - k^2$$

from $(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - \boldsymbol{p}\cdot\boldsymbol{q}$. Two cases are available for this:

$$\boldsymbol{p} = \boldsymbol{k} + [\boldsymbol{v}\times\boldsymbol{k}], \qquad \boldsymbol{q} = \boldsymbol{k} - [\boldsymbol{v}\times\boldsymbol{k}], \qquad (a)$$

Or

$$\boldsymbol{p} = \boldsymbol{k}, \qquad \boldsymbol{q} = \boldsymbol{k} + \big[\boldsymbol{v}\times[\boldsymbol{v}\times\boldsymbol{k}]\big]. \qquad (b)$$

The choice (a) yields:

$$(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - \boldsymbol{p}\cdot\boldsymbol{q} = (\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - (\boldsymbol{k} + [\boldsymbol{v}\times\boldsymbol{k}])\cdot(\boldsymbol{k} - [\boldsymbol{v}\times\boldsymbol{k}]) = (\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - k^2 + [\boldsymbol{v}\times\boldsymbol{k}]^2.$$

The choice (b) yields:

$$\begin{aligned}(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - \boldsymbol{p}\cdot\boldsymbol{q} &= (\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - k^2 \\ &- \big(\boldsymbol{k}\cdot\big[\boldsymbol{v}\times[\boldsymbol{v}\times\boldsymbol{k}]\big]\big) = \\ &= (\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - k^2 - ([\boldsymbol{k}\times\boldsymbol{v}]\cdot[\boldsymbol{v}\times\boldsymbol{k}]) = \\ &(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - k^2 + [\boldsymbol{v}\times\boldsymbol{k}]^2.\end{aligned}$$

Here the vector $\boldsymbol{v}$ is constant.

2.2 The Fourier Transform of the Equations

Having a Fourier transform of the system of equations, one can restore its coordinate representation, and two options are possible:

$$\begin{aligned}&\frac{d\boldsymbol{B}}{dt} + (\boldsymbol{\nabla} + [\boldsymbol{v}\times\boldsymbol{\nabla}])\times\boldsymbol{E} = 0,\\ &\frac{d\boldsymbol{E}}{dt} - (\boldsymbol{\nabla} - [\boldsymbol{v}\times\boldsymbol{\nabla}])\times\boldsymbol{B} + \boldsymbol{j} = 0.\end{aligned} \qquad (a)$$

or

$$\begin{aligned}&\frac{d\boldsymbol{B}}{dt} + \boldsymbol{\nabla}\times\boldsymbol{E} = 0,\\ &\frac{d\boldsymbol{E}}{dt} - \big(\boldsymbol{\nabla} + \big[\boldsymbol{v}\times[\boldsymbol{v}\times\boldsymbol{\nabla}]\big]\big)\times\boldsymbol{B} + \boldsymbol{j} = 0.\end{aligned} \qquad (b)$$

Here the conventional form of the total derivative by time was used:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla}).$$

3 Solution of the Equations

3.1 The System in Case (b)

3.1.1 Transformation of the Equations

For the analysis of the equation array (b), the substitution is implemented, which turns the first equation into an identity:

$$\boldsymbol{B} = [\boldsymbol{\nabla}\times\boldsymbol{A}], \quad \boldsymbol{E} = -\frac{d\boldsymbol{A}}{dt} - \boldsymbol{\nabla}\Phi.$$

The second equation of the system is transformed as

$$-\frac{d}{dt}\left(\frac{d\boldsymbol{A}}{dt}\right)-\frac{d}{dt}\boldsymbol{\nabla}\Phi-(1-v^2)\big[\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times\boldsymbol{A}]\big]-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\big[\boldsymbol{v}\times[\boldsymbol{\nabla}\times\boldsymbol{A}]\big]=-\boldsymbol{j},$$

$$\frac{d^2\boldsymbol{A}}{dt^2}-(\boldsymbol{\nabla}^2-[\boldsymbol{v}\times\boldsymbol{\nabla}]^2)\boldsymbol{A}+\boldsymbol{\nabla}\left(\frac{d}{dt}\Phi+(1-v^2)(\boldsymbol{\nabla}\cdot\boldsymbol{A})+(\boldsymbol{v}\cdot\boldsymbol{\nabla})(\boldsymbol{v}\cdot\boldsymbol{A})\right)=\boldsymbol{j},$$

$$\frac{d^2\boldsymbol{A}}{dt^2}-(\boldsymbol{\nabla}^2-[\boldsymbol{v}\times\boldsymbol{\nabla}]^2)\boldsymbol{A}+\boldsymbol{\nabla}\left(\frac{d}{dt}\Phi+\left(\boldsymbol{\nabla}_q\cdot\boldsymbol{A}\right)\right)=\boldsymbol{j},$$

where: $\boldsymbol{\nabla}_q\equiv(1-v^2)\boldsymbol{\nabla}+\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{\nabla})$.

3.1.2 the Lorentz's Calibration

The analogy of the Lorentz's calibration gives

$$\frac{d}{dt}\Phi+\left(\boldsymbol{\nabla}_q\cdot\boldsymbol{A}\right)=0,$$

while the second equation results in

$$-\frac{d^2\boldsymbol{A}}{dt^2}+(\boldsymbol{\nabla}^2-[\boldsymbol{v}\times\boldsymbol{\nabla}]^2)\boldsymbol{A}=-\boldsymbol{j}.$$

The equation

$$\left(\boldsymbol{\nabla}_q\cdot\boldsymbol{E}\right)=\rho$$

transforms into

$$-\frac{d}{dt}\left(\boldsymbol{\nabla}_q\cdot\boldsymbol{A}\right)-\left(\boldsymbol{\nabla}_q\cdot\boldsymbol{\nabla}\right)\Phi=\rho.$$

With the Lorentz's calibration follows

$$\frac{d}{dt}\frac{d}{dt}\Phi-\left(\boldsymbol{\nabla}_q\cdot\boldsymbol{\nabla}\right)\Phi=\rho,$$

$$-\frac{d^2}{dt^2}\Phi+(\boldsymbol{\nabla}^2-[\boldsymbol{v}\times\boldsymbol{\nabla}]^2)\Phi=-\rho.$$

3.2 Parameters of the Spreading Electromagnetic Waves

3.2.1 The phase speeds of the waves

The scalar Φ and vector $\boldsymbol{A}$ potentials, in this mathematical model considered, satisfy the modified wave equation:

$$\left(-\frac{d^2}{dt^2}+\boldsymbol{\nabla}^2-[\boldsymbol{v}\times\boldsymbol{\nabla}]^2\right)f=-g. \qquad (4)$$

The waves described by the equation (4) have different phase speeds for their spreading (along the vector $\boldsymbol{v}$ and perpendicular to it):

$$(\omega-\boldsymbol{v}\cdot\boldsymbol{k})^2-k^2+[\boldsymbol{v}\times\boldsymbol{k}]^2=0, \qquad (5)$$

where from

$$\omega=\boldsymbol{k}\cdot\boldsymbol{v}\pm\sqrt{k^2-[\boldsymbol{k}\times\boldsymbol{v}]^2},$$

$$\frac{\omega}{k}=v\cdot cos\theta\pm\sqrt{1-v^2\cdot sin\theta^2}.$$

One could see analogy in (5) with a spreading of the waves in anisotropic medium, e.g. in the single-axis crystal.

In the anisotropic medium the group speed $\boldsymbol{V}\equiv\frac{\partial\omega}{\partial\boldsymbol{k}}$ does not coincide with the phase speed $\equiv\frac{\omega}{k}\frac{\boldsymbol{k}}{k}$. The phase speed is the speed of the moving wave front (surface of constant phase), while the group speed is a characteristic of the energy transfer by the wave packet.

3.2.2 Choosing the Coordinate System

The vector of the group speed is

$$\boldsymbol{V}=\frac{\partial\omega}{\partial\boldsymbol{k}}=\boldsymbol{v}\pm\frac{(1-v^2)\boldsymbol{k}+(\boldsymbol{v}\cdot\boldsymbol{k})\boldsymbol{v}}{\sqrt{k^2-[\boldsymbol{k}\times\boldsymbol{v}]^2}}. \qquad (6)$$

Let us choose the coordinate system so that the vector $\boldsymbol{v}$ would be directed by the axis z: $\boldsymbol{v}=[0,0,u]$. Then from (6) follows

$$\omega=k_3\cdot u\pm\sqrt{(1-u^2)\left({k_1}^2+{k_2}^2\right)+{k_3}^2},$$

$$V_1=\frac{\partial\omega}{\partial k_1}=\pm\frac{(1-u^2)k_1}{\sqrt{(1-u^2)\left({k_1}^2+{k_2}^2\right)+{k_3}^2}}, \qquad (7)$$

$$V_2=\frac{\partial\omega}{\partial k_2}=\pm\frac{(1-u^2)k_2}{\sqrt{(1-u^2)\left({k_1}^2+{k_2}^2\right)+{k_3}^2}},$$

$$V_3=\frac{\partial\omega}{\partial k_3}=u\pm\frac{k_3}{\sqrt{(1-u^2)\left({k_1}^2+{k_2}^2\right)+{k_3}^2}}.$$

3.2.3 The Wave Fronts

It is easily proved with (7) that

$$\frac{{V_1}^2+{V_2}^2}{(1-u^2)}+(V_3-u)^2=\mathbf{1}. \qquad (8)$$

The equation (8) represents the equation of:

- One-axis ellipsoid by $u^2<1$;

- Single-cavity hyperboloid by $u^2 > 1$.

The case $u^2 = 1$ is degenerate (plane wave).

3.3 The Fresnel Equation

Rewriting the equation (5):

$$(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 + [\boldsymbol{v}\times\boldsymbol{k}]^2 = k^2$$

through the conventional in the crystal optics vector $\boldsymbol{n} = \boldsymbol{k}/_{\omega}$ ([3], p. 307), and divide by $\omega \neq 0$:

$$(1 - \boldsymbol{v}\cdot\boldsymbol{n})^2 + [\boldsymbol{v}\times\boldsymbol{n}]^2 = n^2. \qquad (9)$$

This equation (9) is called in optics the Fresnel equation and defines the surface of wave vectors - the optical indicatrix. It is written in the spherical coordinates as follows

$$\left(\boldsymbol{n}\cdot\boldsymbol{cos\theta} + \frac{v}{1-v^2}\right)^{\boldsymbol{2}} + (\boldsymbol{n}\cdot\boldsymbol{sin\theta})^{\boldsymbol{2}} = \frac{\boldsymbol{1}}{(\boldsymbol{1}-\boldsymbol{v^2})^{\boldsymbol{2}}}, \quad (10)$$

where $\boldsymbol{n}$, $\boldsymbol{v}$ are the amplitudes of the vectors $\boldsymbol{n}$, $\boldsymbol{v}$ respectively.

The equation (10) describes the sphere with the radius $\frac{\boldsymbol{1}}{\boldsymbol{1}-\boldsymbol{v^2}}$, the center of which is shifted by $\frac{\boldsymbol{v}}{\boldsymbol{1}-\boldsymbol{v^2}}$ along the vector $\boldsymbol{v}$. Thus, the vector $\boldsymbol{v}$ of relative motion determines the chosen direction as the optical axis of space. It defines the anisotropy of the space for the propagation of electromagnetic waves.

The equation of the equal phases is as follows

$$\left(\frac{\omega}{k}\frac{\boldsymbol{k}}{k} - \boldsymbol{v}\right)^2 = 1,$$

the sphere with a center shifted by the vector $\boldsymbol{v}$.

The group speed surface is

$$\frac{\boldsymbol{V_1}^{\boldsymbol{2}} + \boldsymbol{V_2}^{\boldsymbol{2}}}{(1-u^2)} + (V_3 - u)^{\boldsymbol{2}} = \boldsymbol{1},$$

the surface of stretched along $\boldsymbol{v}$ rotation ellipsoid.

4 The Green Lagging Function

4.1 The Equation with Right Hand

Solution of the equation with right hand side

$$\left(-\frac{d^2}{dt^2} + \boldsymbol{\nabla}^2 - [\boldsymbol{v}\times\boldsymbol{\nabla}]^2\right) f = -g \qquad (11)$$

can be presented as the integral of right hand side with the Green function:

$$f(t,x,y,z) = \int dt'dx'dy'dz' G(t-t', x-x', y-y', z-z')\cdot g(t',x',y',z'),$$

where the Green function satisfies the equation (11):

$$\left(\frac{d^2}{dt^2} - \boldsymbol{\nabla}^2 + [\boldsymbol{v}\times\boldsymbol{\nabla}]^2\right) G(t-t', x-x', y-y', z-z') = -\delta(t-t', x-x', y-y', z-z').$$

Using the presentation of the functions through their Fourier integral yields:

$$G(t,\boldsymbol{r}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \frac{e^{i(-\omega t + \boldsymbol{k}\cdot\boldsymbol{r})}}{(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - (1-v^2)k^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2}.$$

4.2 Integral by Frequency

The integral by frequency

$$\boldsymbol{I_k}(t,\boldsymbol{r}) = \int \frac{d\omega}{2\pi} \frac{e^{i(-\omega t + \boldsymbol{k}\cdot\boldsymbol{r})}}{(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - (1-v^2)k^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2}$$

is computed using the series expansion of the denominator of fraction by the factors

$$\frac{1}{(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 - (1-v^2)k^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2} = \frac{1}{(\omega - \Omega_-)\cdot(\omega - \Omega_+)} = \frac{1}{(\Omega_+ - \Omega_-)}\left(\frac{1}{(\omega - \Omega_+)} - \frac{1}{(\omega - \Omega_-)}\right),$$

where are

$$\Omega_- \equiv \boldsymbol{v}\cdot\boldsymbol{k} - \sqrt{(1-v^2)k^2 + (\boldsymbol{v}\cdot\boldsymbol{k})^2},$$

$$\Omega_+ \equiv \boldsymbol{v}\cdot\boldsymbol{k} + \sqrt{(1-v^2)k^2 + (\boldsymbol{v}\cdot\boldsymbol{k})^2}.$$

The integral

$$\boldsymbol{I_k}(t,\boldsymbol{r}) = \frac{e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}}{2\pi(\Omega_+ - \Omega_-)} \int d\omega \left(\frac{e^{-i\omega t}}{(\omega - \Omega_+)} - \frac{e^{-i\omega t}}{(\omega - \Omega_-)}\right)$$

is calculated by the theorem on the residues of the analytic function, the closure of the integration contour in the lower half-plane by $t > 0$:

$$\boldsymbol{I_k}(t,\boldsymbol{r}) = \frac{e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}}{2\pi(\Omega_+ - \Omega_-)}(2\pi i)\left(e^{-i\Omega_+ t} - e^{-i\Omega_- t}\right)\cdot \upsilon(t),$$

$$\boldsymbol{I_k}(t,\boldsymbol{r}) = \frac{e^{i(\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{v}t))}}{\sqrt{(1-v^2)k^2 + (\boldsymbol{v}\cdot\boldsymbol{k})^2}} \cdot \frac{\left(e^{it\sqrt{(1-v^2)k^2 + (\boldsymbol{v}\cdot\boldsymbol{k})^2}} - e^{-it\sqrt{(1-v^2)k^2 + (\boldsymbol{v}\cdot\boldsymbol{k})^2}}\right)}{2i}.$$

$$\upsilon(t),$$

$$\boldsymbol{I_k}(t,\boldsymbol{r}) = \frac{e^{i(\boldsymbol{k}\cdot(r-vt))}}{\sqrt{(1-v^2)k^2+(\boldsymbol{v}\cdot\boldsymbol{k})^2}} \cdot sin\left(t\sqrt{(1-v^2)k^2+(\boldsymbol{v}\cdot\boldsymbol{k})^2}\right)\cdot \upsilon(t),$$

where $\upsilon(t) = \{0\ (t<0)|1\ (0<t)\}$.

4.3 The Green Function

The integral

$$G(t,\boldsymbol{r}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{e^{i(\boldsymbol{k}\cdot(r-vt))}}{\sqrt{(1-v^2)k^2+(\boldsymbol{v}\cdot\boldsymbol{k})^2}}\cdot sin\left(t\sqrt{(1-v^2)k^2+(\boldsymbol{v}\cdot\boldsymbol{k})^2}\right)\cdot \upsilon(t)$$

is calculated by all space $\boldsymbol{k^3}$ in the special coordinate system with the vector $\boldsymbol{v} = [0,0,u]$:

$$G(t,\boldsymbol{r}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{e^{i(k_1x+k_2y+k_3(z-ut))}}{\sqrt{(1-u^2)\left({k_1}^2+{k_2}^2\right)+{k_3}^2}}\cdot sin\left(t\sqrt{(1-u^2)\left({k_1}^2+{k_2}^2\right)+{k_3}^2}\right)\cdot \upsilon(t),$$

$$G(t,\boldsymbol{r}) = \frac{1}{(1-u^2)}\int \frac{d^3\boldsymbol{\kappa}}{(2\pi)^3}\frac{e^{i(\kappa_1q_1+\kappa_2q_2+\kappa_3q_3)}}{\sqrt{{\kappa_1}^2+{\kappa_2}^2+{\kappa_3}^2}}\cdot sin\left(t\sqrt{{\kappa_1}^2+{\kappa_2}^2+{\kappa_3}^2}\right)\cdot \upsilon(t),$$

where are:

$$q_1 = \frac{x}{\sqrt{1-u^2}},\ q_2 = \frac{y}{\sqrt{1-u^2}},\ q_3 = z-ut.$$

In the spherical coordinates:

$$G(t,\boldsymbol{r}) = \frac{1}{(2\pi)^2(1-u^2)}\int_0^\infty \kappa d\kappa\cdot sin(\kappa t)\cdot \int_{-1}^{1} ds e^{i\kappa Qs}\cdot \upsilon(t),$$

$$G(t,\boldsymbol{r}) = \frac{2}{(2\pi)^2(1-u^2)Q}\int_0^\infty d\kappa\cdot sin(\kappa t)\cdot sin(\kappa Q)\cdot \upsilon(t),$$

where

$$Q = \sqrt{{q_1}^2+{q_2}^2+{q_3}^2} = \sqrt{\frac{x^2+y^2}{1-u^2}+(z-ut)^2}.$$

The use of identities

$$sin(\kappa t)\cdot sin(\kappa Q) = \frac{1}{2}\Big(cos\big(\kappa(t-Q)\big) - cos\big(\kappa(t+Q)\big)\Big),$$

$$\int_{\boldsymbol{0}}^{\infty} \boldsymbol{d}\kappa\cdot cos(\kappa\lambda) = \boldsymbol{\pi\delta}(\lambda),$$

gives

$$G(t,\boldsymbol{r}) = \frac{1}{4\pi(1-u^2)R}\big(\boldsymbol{\delta}(Q-t)-\boldsymbol{\delta}(Q+t)\big)\cdot \upsilon(t).$$

Here

$$Q = \sqrt{\frac{x^2+y^2}{1-u^2}+(z-ut)^2} > 0,$$

therefore, $\boldsymbol{\delta}(Q+t) = \boldsymbol{0}$ by $t>0$ and can be omitted. Then $\boldsymbol{\delta}(Q-t)\neq \boldsymbol{0}$ only for $t>0$, therefore, $\upsilon(t)$ is not needed. Finally,

$$G(t,\boldsymbol{r}) = \frac{\boldsymbol{\delta}(Q-t)}{4\pi(1-u^2)Q}. \tag{12}$$

The Green function (12) can be presented also in the following form:

$$G(t,\boldsymbol{r}) = \frac{2\boldsymbol{\delta}(Q-t)}{4\pi(1-u^2)(Q+t)} = \frac{\boldsymbol{\delta}\big((Q+t)(Q-t)\big)}{2\pi(1-u^2)} = \frac{\boldsymbol{\delta}\big((1-u^2)(Q^2-t^2)\big)}{2\pi},$$

$$G(t,\boldsymbol{r}) = \frac{\boldsymbol{\delta}\Big(x^2+y^2+(1-u^2)\big((z-ut)^2-t^2\big)\Big)}{2\pi} = \frac{\boldsymbol{\delta}\Big(x^2+y^2+z^2-\big((1-u^2)t+uz\big)^2\Big)}{2\pi}.$$

Then, in the spherical coordinates
x= $R\cdot$ sinθ $\cdot$ cosφ, y= $R\cdot$ sinθ $\cdot$ sinφ, z= $R\cdot cos$θ,

$$G(t,\boldsymbol{r}) = \frac{\boldsymbol{\delta}\Big(R^2-\big((1-u^2)t+u\cdot R\cdot cos\theta\big)^2\Big)}{2\pi}.$$

The condition of zero argument for the $\boldsymbol{\delta}$-function is the equation of cone of causality

$$R^2 = \big((1-u^2)t+u\cdot R\cdot cos\theta\big)^{\boldsymbol{2}}.$$

Thus,

$$\boldsymbol{V} = \frac{R}{t} = -\frac{(1-u^2)}{(u\cdot cos\theta\pm 1)}. \tag{13}$$

4.4 The Graphs of Speed for Spreading the Wave Fronts Depending on Angle

The graphs of function (13) of the speed of spreading for the wave front against the latitudinal angle θ are given in Fig. 1 for $u = 0.618$ and Fig. 2 for $u = 1.618$, respectively:

4.4.1 Ellipse $(u < 1)$ Dependence of Speed of Wave Front versus Latitudinal Angle θ

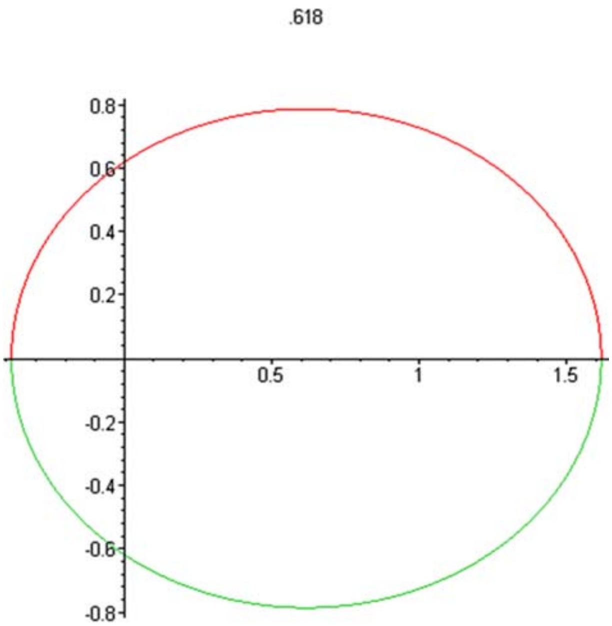

Fig. 1 The ellipse $(u < 1)$ dependence (13) of the speed of spreading for the wave front against the latitudinal angle θ

4.4.2 Hyperbole $(u > 1)$ dependence of Speed of Wave Front versus Latitudinal Angle θ

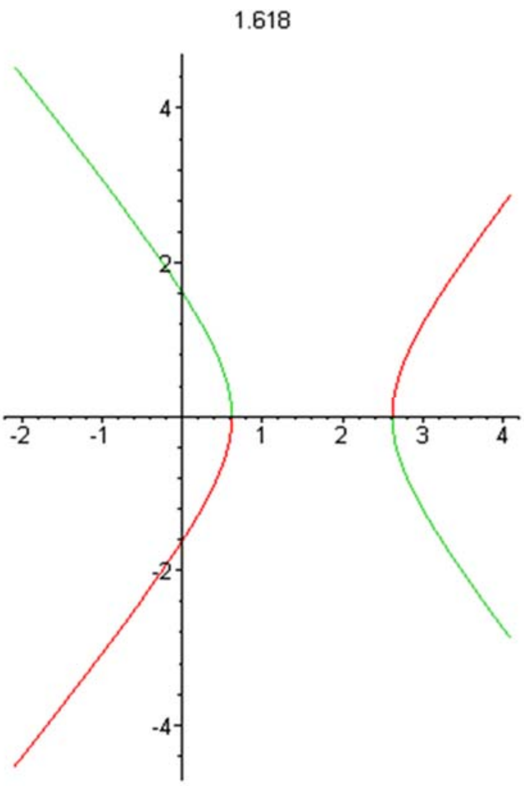

Fig. 2 The hyperbole $(u > 1)$ dependence (13) of the speed of spreading for the wave front against the latitudinal angle θ

5 Equations in the Canonical Form

5.1 Reduction of Equations to Canonical Form

5.1.1 The Maxwell Equations in Local System

In the local system with the vector $\boldsymbol{v} = [0,0,u]$ the Maxwell equations are the following:

$$\frac{dB_1}{dt} + \partial_2 E_3 - \partial_3 E_2 = 0, \frac{dB_2}{dt} + \partial_3 E_1 - \partial_1 E_3 = 0\,,$$

$$\frac{dB_3}{dt} + \partial_1 E_2 - \partial_2 E_1 = 0\,,$$

$$\frac{dE_1}{dt} - (1-u^2)\partial_2 B_3 + \partial_3 B_2 = -j_1, \quad (14)$$

$$\frac{dE_2}{dt} - \partial_3 B_1 + (1-u^2)\partial_1 B_3 = -j_2,$$

$$\frac{dE_3}{dt} - (1-u^2)(\partial_1 B_2 - \partial_2 B_1) = -j_3,$$

where:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial z}\,.$$

Transforming to the new independent variables:

$$\partial_x = \sqrt{1-u^2}\partial_1,\ \ \partial_y = \sqrt{1-u^2}\partial_2,\ \ \partial_z = \partial_3.$$

yields (14) as follows

$$\frac{dB_1}{dt} + \partial_y\left(\frac{E_3}{\sqrt{1-u^2}}\right) - \partial_z E_2 = 0,$$

$$\frac{dB_2}{dt} + \partial_z E_1 - \partial_x\left(\frac{E_3}{\sqrt{1-u^2}}\right) = 0\,, \quad (15)$$

$$\frac{d}{dt}\left(\sqrt{1-u^2}B_3\right) + \partial_x E_2 - \partial_y E_1 = 0\,,$$

$$\frac{dE_1}{dt} - \partial_y\left(\sqrt{1-u^2}B_3\right) + \partial_z B_2 = -j_1,$$

$$\frac{dE_2}{dt} - \partial_z B_1 + \partial_x\left(\sqrt{1-u^2}B_3\right) = -j_2,$$

$$\frac{d}{dt}\left(\frac{E_3}{\sqrt{1-u^2}}\right) - \partial_x B_2 + \partial_y B_1 = -\left(\frac{j_3}{\sqrt{1-u^2}}\right).$$

5.1.2 The Conventional Maxwell Equations

Now introduce the new variables:

$$\beta_1 = B_1, \qquad \varepsilon_1 = E_1, \qquad \iota_1 = j_1,$$
$$\beta_2 = B_2, \qquad \varepsilon_2 = E_2, \qquad \iota_2 = j_2,$$
$$\beta_3 = \sqrt{1-u^2}B_3,\ \varepsilon_3 = \frac{E_3}{\sqrt{1-u^2}},\ \iota_3 = \frac{j_3}{\sqrt{1-u^2}}.$$

In these assignments the equations (15) look like the conventional Maxwell equations

$$\frac{d\beta_1}{dt} + \partial_y\varepsilon_3 - \partial_z\varepsilon_2 = 0,\ \frac{d\beta_2}{dt} + \partial_z\varepsilon_1 - \partial_x\varepsilon_3 = 0\ ,$$

$$\frac{d\beta_3}{dt} + \partial_x\varepsilon_2 - \partial_y\varepsilon_1 = 0\ ,$$

$$\frac{d\varepsilon_1}{dt} - \partial_y\beta_3 + \partial_z\beta_2 = -\iota_1, \qquad (16)$$

$$\frac{d\varepsilon_2}{dt} - \partial_z\beta_1 + \partial_x\beta_3 = -\iota_2,$$

$$\frac{d\varepsilon_3}{dt} - \partial_x\beta_2 + \partial_y\beta_1 = -\iota_3.$$

Thus, for each value of speed $\boldsymbol{v}$, the local transformation of the coordinates exists, which leads to the “canonical” form of the equations:

$$\frac{d\boldsymbol{\beta}}{dt} + \boldsymbol{\nabla}\times\boldsymbol{\varepsilon} = 0, \qquad \frac{d\boldsymbol{\varepsilon}}{dt} - \boldsymbol{\nabla}\times\boldsymbol{\beta} + \boldsymbol{\iota} = 0,$$

where all derivatives by time are total derivatives:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla}).$$

When $u > 1$, the local transformation of the coordinates requires the multiplier $\sqrt{u^2-1}$ instead of $\sqrt{1-u^2}$.

5.2 Energy Conservation Law

Normally in derivation of the law for conservation of energy-momentum the following is applied. The first Maxwell equation (see Appendix) is got in scalar product by vector $\boldsymbol{B}$, and the second one – by vector $\boldsymbol{E}$. The resulted equations are added. It yields

$$\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{2}\right) + \boldsymbol{\nabla}\cdot[\boldsymbol{E}\times\boldsymbol{B}] = -\boldsymbol{j}\cdot\boldsymbol{E}. \qquad (17)$$

Similar to (17), for the canonical form is got:

$$\frac{d}{dt}\left(\frac{\varepsilon^2+\beta^2}{2}\right) + \boldsymbol{\nabla}\cdot[\boldsymbol{\varepsilon}\times\boldsymbol{\beta}] = -\boldsymbol{\iota}\cdot\boldsymbol{\varepsilon}. \qquad (18)$$

6 Appendix

6.1 The Maxwell Equations

The Maxwell equations have the form:

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0},\ \ \frac{\partial \boldsymbol{E}}{\partial t} - \boldsymbol{\nabla}\times\boldsymbol{B} = -\boldsymbol{j}.$$

In the Cartesian coordinate system is:

$$\frac{\partial b_1}{\partial t} + \frac{\partial e_3}{\partial y} - \frac{\partial e_2}{\partial z} = \boldsymbol{0},$$

$$\frac{\partial b_2}{\partial t} + \frac{\partial e_1}{\partial z} - \frac{\partial e_3}{\partial x} = \boldsymbol{0},$$

$$\frac{\partial b_3}{\partial t} + \frac{\partial e_2}{\partial x} - \frac{\partial e_1}{\partial y} = \boldsymbol{0},$$

$$\frac{\partial e_1}{\partial t} - \frac{\partial b_3}{\partial y} + \frac{\partial b_2}{\partial z} = -\boldsymbol{j_1},$$

$$\frac{\partial e_2}{\partial t} - \frac{\partial b_1}{\partial z} + \frac{\partial b_3}{\partial x} = -\boldsymbol{j_2},$$

$$\frac{\partial e_3}{\partial t} - \frac{\partial b_2}{\partial x} + \frac{\partial b_1}{\partial y} = -\boldsymbol{j_3}.$$

The linear system of the partial differential equations with constant coefficients can be solved using the Fourier transformation:

$$\boldsymbol{b_i} = \boldsymbol{\beta_i exp}\big(\boldsymbol{i}\cdot(-\boldsymbol{\omega}\cdot\boldsymbol{t} + \boldsymbol{k_1 x} + \boldsymbol{k_2 y} + \boldsymbol{k_3 z})\big),$$

$$\boldsymbol{e_i} = \boldsymbol{\varepsilon_i exp}\big(\boldsymbol{i}\cdot(-\boldsymbol{\omega}\cdot\boldsymbol{t} + \boldsymbol{k_1 x} + \boldsymbol{k_2 y} + \boldsymbol{k_3 z})\big),$$

$$\boldsymbol{j_i} = \boldsymbol{\gamma_i exp}\big(\boldsymbol{i}\cdot(-\boldsymbol{\omega}\cdot\boldsymbol{t} + \boldsymbol{k_1 x} + \boldsymbol{k_2 y} + \boldsymbol{k_3 z})\big).$$

After substitution, omitting by exponential (non-zero) results in the system of algebraic equations for the Fourier amplitudes of the fields $\boldsymbol{\beta_i}$ and $\boldsymbol{\varepsilon_i}$:

$$-\omega\boldsymbol{\beta_1} - \boldsymbol{k_3\varepsilon_2} + \boldsymbol{k_2\varepsilon_3} = \boldsymbol{0},$$

$$-\omega\boldsymbol{\beta_2} + \boldsymbol{k_3\varepsilon_1} - \boldsymbol{k_1\varepsilon_3} = \boldsymbol{0},$$

$$-\omega\boldsymbol{\beta_3} - \boldsymbol{k_2\varepsilon_1} + \boldsymbol{k_1\varepsilon_2} = \boldsymbol{0},$$

$$+\boldsymbol{k_3\beta_2} - \boldsymbol{k_2\beta_3} - \omega\boldsymbol{\varepsilon_1} = \boldsymbol{i\gamma_1},$$

$$-\boldsymbol{k_3\beta_1} + \boldsymbol{k_1\beta_3} - \omega\boldsymbol{\varepsilon_2} = \boldsymbol{i\gamma_2},$$

$$+\boldsymbol{k_2\beta_1} - \boldsymbol{k_1\beta_2} - \omega\boldsymbol{\varepsilon_3} = \boldsymbol{i\gamma_3}.$$

Now the system has «classic» view of the linear algebraic equations: $\widehat{M}\cdot\bar{u} = \bar{w}$, where are:
$\bar{u} = [\beta_1, \beta_2, \beta_3, \varepsilon_1, \varepsilon_2, \varepsilon_3], \overline{w} = [0,0,0, i\boldsymbol{\gamma_1}, \boldsymbol{i\gamma_2}, \boldsymbol{i\gamma_3}]$
And the determinant of the matrix $\widehat{M}$ is:

$$det(M) = \omega^2 \cdot (\omega^2 - \boldsymbol{k}^2)^2.$$

6.2 Solution of the Equations

Solution of the system is as follows:

$$\beta_1 = ik_2A_3 - ik_3A_2, \quad \beta_2 = ik_3A_1 - ik_1A_3,$$

$$\beta_3 = ik_1A_2 - ik_2A_1, \quad \varepsilon_1 = i\omega A_1 - ik_1\Phi,$$

$$\varepsilon_2 = i\omega A_2 - ik_2\Phi, \quad \varepsilon_3 = i\omega A_3 - ik_3\Phi,$$

where are:

$$A_i = -\frac{\gamma_i}{\omega^2 - {k_1}^2 - {k_2}^2 - {k_3}^2},$$

$$\Phi = (k_1A_1 + k_2A_2 + k_3A_3)/\omega.$$

The solution is written in the variables $\boldsymbol{B}, \boldsymbol{E}, \boldsymbol{j}$ as

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \boldsymbol{\nabla}\,\Phi,$$

where $\boldsymbol{A}$ satisfies the wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right)\boldsymbol{A} = \boldsymbol{j},$$

and Φ satisfies the Lorentz condition:

$$\frac{\partial}{\partial t}\,\Phi + \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0.$$

Solution of the wave equation may be done using for example the Green lagging function. The Lorentz condition can be also reduced to the wave equation computing the divergence of the vector $\boldsymbol{E}$:

$$div\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \boldsymbol{A} - \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\,\Phi = \left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right)\Phi.$$

Thus, for Φ we get also the wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right)\Phi = \rho,$$

Where $\rho = div\boldsymbol{E}$ is the charge density satisfying the continuity equation

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0,$$

which is assumed to be a known function.

7 Conclusion

Derivation of the Doppler Effect from the modified Maxwell equations with total derivatives by time was considered and analyzed in this paper. It was shown that the modified Maxwell equations contain a description of the Doppler Effect in the form of O.E. Akimov [2]. The Doppler Effect takes place when waves of any nature (not only electromagnetic) propagate in a homogeneous and isotropic continuous medium. The problem raised the new attention of scientists during the last time, e.g. [4].

Researchers in the field and the students including the application of the Computerized Educational Platform (CompEdu) [5-7] may use the presented materials by the spreading of the electromagnetic waves and analysis of the Maxwell equations. It can also be useful for studying the Doppler effect of electromagnetic wave propagation.

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