On the computational analysis of damage of quasi-brittle materials using integral-type nonlocal models

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Abstract: Reliable prediction of quasi-brittle behaviour of a large class of advanced building materials and structures under mechanical loads needs the implementation of some integral-type nonlocal constitutive strain-stress relation. This paper pays attention namely to the Eringen model for the generation of the multiplicative damage factor, to the related quasi-static analysis, to the existence of a weak solution of the corresponding boundary and initial value problem with a parabolic system of partial differential equation and to the convergence of an algorithm based on 3 types of Rothe sequences.

Key–Words: Nonlocal elasticity, quasi-brittle fracture, partial differential equations, computational analysis, Rothe sequences.

1 Introduction

Engineering structures subjected to loading may result in stresses in the body exceeding the material strength and thus results in the progressive failure. Such failures are often initiated by surface or near surface cracks, reducing the strength of the material. In quasi-brittle materials like rocks or concrete this is manifested by fracture process zones, in brittle materials like glass or welds in metal structures by discrete crack discontinuities, in elasto-plastic ductile metal or similar materials by shear (localization) bands; for much more details and references see [32]. Advanced building structures frequently use silicate composites reinforced by metal, plastic or other fibres, preventing undesirable micro- an macro-cracking effects.

In this paper we shall pay attention namely to the quasi-brittle damage, realistic for a large class of building materials and composites, with a primarily elastic behaviour. The presence of above sketched effects forces the implementation of some nonlocal strain-stress constitutive relation. Following [4], [11], [12], [14], [25], etc., we shall come out from the well-known Eringen model [7] and [8], although numerical results referring to its pure version are not quite satisfactory, as observed by [29], and its ill-possedness has been recently discovered by [9], except the homogeneous Dirichlet problems and certain simplified 1-dimensional formulations – cf. [35], contesting the incomplete existence results by [1]. The remedy suggested by [7] relies on the additive linear combination of the classical local and the nonlocal Eringen model; unlike this, we shall utilize the Eringen approach to the setting of the multiplicative damage factor only, as demonstrated by [14] heuristically.

2 A model problem

For simplicity, to avoid technical difficulties, let us suppose that Ω is a domain with Lipschitz boundary $\partial\Omega$ in $\mathcal{R}^3$ in the 3-dimensional Euclidean space $\mathcal{R}^3$, compound from 2 disjoint parts Θ (for homogeneous Dirichlet boundary conditions) and Γ (for non-homogeneous Neumann boundary conditions), Let us consider the Cartesian coordinate system $x = (x_1, x_2, x_3)$ in $\mathcal{R}^3$ and the time $t \in I$ where $I = [0, T]$ for some final time T; the limit passage $T \to \infty$ will be allowed, too. We shall also use the Hamilton operator $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, the local outward unit normal vector $n(\tilde{x}) = (n_1(\tilde{x}), n_2(\tilde{x}), n_3(\tilde{x}))$, introduced for $\tilde{x} \in \partial\Omega$, and the dot symbols instead of $\partial/\partial t$ for brevity. As the reference variable, let us choose the displacement $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ for any x from Ω or $\partial\Omega$ (in the sense of traces), and arbitrary time $t \in I$. For a fixed time t and any admissible displacement $v(x) = (v_1(x), v_2(x), v_3(x))$ let us introduce the small strain tensor $\varepsilon(v)$ in the form $\varepsilon_{ij}(v(x)) = (v_{i,j}(x) + v_{j,i}(x))/2$ where $i, j \in \{1, 2, 3\}$ and the stress tensor σ of the same type; any comma followed by an index $k \in \{1, 2, 3\}$ (i or j here) must be understood as $\partial/\partial x_k$ applied to the preceding variable.

We shall start with the heuristic formulation of a model problem, using the standard notations of the linearized theory of elasticity. Let us set the Cauchy initial condition $u(.,0) = 0$ on Ω. Moreover, let us consider some volume loads $f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t))$ and some surface loads $g(x,t) = (g_1(\tilde{x},t), g_2(\tilde{x},t), g_3(\tilde{x},t))$ where $x \in \Omega$, $\tilde{x} \in \Gamma$ and $t \in I$. The physical principle of energy conservation can be then reduced to the Cauchy equilibrium condition on $\Omega \times I$

$$\sigma_{ij,j} + f_i = 0\,, \tag{1}$$

for any $i \in \{1,2,3\}$ and an Einstein summation index $j \in \{1,2,3\}$; the Neumann boundary condition on $\Omega \times I$ is

$$\sigma_{ij} n_j = g_i \tag{2}$$

with i and j as in (1), whereas the Dirichlet one degenerates to $u = 0$ on $\Theta \times I$. Here $\sigma_{ik} = \sigma_{ki}$ everywhere for arbitrary $i,k \in \{1,2,3\}$, as usual in the theory of Boltzmann continuum. The constitutive relation between σ and $\varepsilon(u)$, motivated by the classical Kelvin viscoelastic model (containing parallel Hooke and Newton components), on $\Omega \times I$ reads

$$\sigma_{ij} = \alpha C_{ijkl}\varepsilon_{kl}(\dot{u}) + (1-\mathcal{D})C_{ijkl}\varepsilon_{kl}(u) \tag{3}$$

for any $i,j \in \{1,2,3\}$ and Einstein summation indices $i,j \in \{1,2,3\}$; here $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$ refer to 21 independent material characteristics in the empirical Hooke law (expressible using 2 Lamé coefficients for isotropic materials, or using the Young modulus and the Poisson ratio alternatively), α forces some energy dissipation in such non-closed physical process and $\mathcal{D}$ refers to the above announced damage factor, whose reasonable design determines both the well-possedness and the practical validity of the computational model.

Following [14] (slightly generalized, to enable the comparison with some other approaches), $\mathcal{D}$ in (3) can be set as the maximal value of $\omega(|\mathcal{A}(\sigma(.,\tilde{t})|_3) = \omega(|\mathcal{A}(C(.)\varepsilon(u(.,\tilde{t})|_3)$ with $\tilde{t} \in [0,t]$; here $t \in I$, $|\,.\,|_3$ refers to the norm in $\mathcal{R}^3$ and

$$\mathcal{A}(w(x)) = \int_\Omega \mathcal{K}(x,\tilde{x})\,w(\tilde{x})\,\mathrm{d}\tilde{x} \tag{4}$$

for any $x \in \Omega$ and integrable functions $\mathcal{K}$ (real-valued) and w ($\mathcal{R}^3$-valued) in the needed sense. In particular, $\mathcal{K}$ can be taken as a radial basis function (RBF) (or its interpolation or approximation), as reviewed by [31]. Let us also notice that [9] needs to have $\mathcal{K}$ as a symmetric positive kernel, whereas [27] formulates 5 requirements to $\mathcal{K}$, satisfied by the Gaussian error-like distributions by [10]

$$\mathcal{K}(x,\tilde{x}) = \exp\left(-|x-\tilde{x}|_3^2)/(4\eta)\right)/(8\pi\eta)^{3/2}$$

automatically where the parameter η should be either set by appropriate experiments or evaluated from the theory of atomic lattice.

The approach of [17] and [18] relies on the theory of dislocations, Burgers vectors, etc., and constructs $\mathcal{K}(x,\tilde{x})$ using the Green functions of certain bi-Helmholtz equation, i. e. the generalized Helmholtz equation of the 4th order, exploiting the Bessel function depending on appropriate real constants, whose number can be reduced to 2 in the isotropic case; this can be identified with the higher-order strain-gradient formulation in the thermodynamic framework by [20]. However, such considerations may be not realistic especially for concrete-like composites with a complicated non-deterministic structure in general. Unlike this, [14] introduces

$$\mathcal{K}(x,\tilde{x}) = \exp\left(-|x-\tilde{x}|_3)/\rho(x)\right)$$

(supplied by an additional normalization step); here $\rho(x)$ (constant in the first guess) scales the internal length depending on $|x-\tilde{x}|_3$ with the closest $\tilde{x} \in \partial\Omega$. The details of evaluation of $\rho(x)$ are are not unified: namely [14] takes $\gamma(x)$ as a piecewise linear function, unlike its exponential improvement by [13]. As a quite different example, [25] refers to the (rather complicated) Wendland RBF of the 5th order; for some classes of still other choices (as bell- or conical-shape) cf. [26].

To see the damage progress utilizing $\mathcal{D}$, we need now to introduce ω properly. Following [14], to force the objectivity, let us set w in (4) as a vector of principle stress values σ_k with $k \in \{1,2,3\}$, i. e. $\sigma_{ij}v_{jk} = \sigma_k v_{ik}$, using arbitrary indices $i,k \in \{1,2,3\}$ and $j \in \{1,2,3\}$ in the role of an Einstein summation index where v_{ik} generate an orthonormal matrix from $\mathcal{R}^{3\times 3}$ (on Ω locally). Consequently we can take w in (4) as $(\sigma_1,\sigma_2,\sigma_3)$, to obtain certain $\sigma_\star = \mathcal{K}(w)$ on Ω, An appropriate formula for the evaluation of ω on Ω is then

$$\omega(\sigma_\star) = 1 - \exp\left(-\frac{\sigma_\star/E - \varepsilon_0}{\varepsilon_f - \varepsilon_0}\right) \tag{5}$$

for $\sigma_\star/E \geq \varepsilon_0$, zero otherwise; here E is the (always positive) Young modulus on Ω and ε_0 and ε_f are 2 dimensionless parameters (also positive in practice) controlling the peak stress and the slope of the softening part of the strain-stress dependencies. More generally (in the brief notation), we have $\mathcal{D} = \mathfrak{E}(u)$ finally, with a rather complicated mapping $\mathfrak{E}$. Clearly $\mathcal{D} \in [\varsigma, 1]$ in all cases, with some real non-negative $\varsigma \leq 1$.

3 Existence and convergence properties

To precise the above sketched considerations, we shall use the standard notation of Lebesque, Sobolev, Bochner, etc. (abstract) function spaces, as introduced by [30]. Namely we shall need the Hilbert spaces $H = L^2(\Omega)^3$, $V = \{v \in W^{1,2}(\Omega)^3 : v = (0,0,0) \text{ on } \Theta\}$ and $Z = L^2(\Gamma)^3$ and the corresponding scalar products: $(.\,,.)$ both in H and $H \times H$ and $\langle .\,,.\rangle$ in Z. Later we shall need also some symbols for standard norms, namely $|\,.\,|$ both in H and $H \times H$, $\|\,.\,\|$ in V and $|\,.\,|_\Gamma$ in Z. We shall use the upper star symbols for dual spaces, $\subset$ for continuous embeddings, $\Subset$ for compact embeddings and $\cong$ for the identification of a space with its dual (following the Riesz representation theorem). Then in the Gelfand triple $V \subset H \cong H^* \subset V^*$ both inclusions are dense, with the guaranteed embedding $W^{1,2,2}(I,V,V^*) \subset C(I,H)$. Moreover $L^2(I,V)^* \cong L^2(I,V^*)$ holds (thus $L^2(I,V)$ is reflexive), together with other useful relations $H \Subset V$ (the Sobolev embedding theorem), $Z \Subset V$ (the trace theorem), forcing $|v|_\Gamma \leq \mathfrak{T}\|v\|$ for any $v \in V$ with a positive $\mathfrak{T}$ independent of v, and $W^{1,2,2}(I,V,V^*) \Subset L^2(I,X)$ with $X \in \{H,Z\}$ (the Aubin - Lions lemma). Clearly $\|v\|^2 = |v|^2 + |\nabla v|^2$ for any $v \in V$; an alternatively norm in V is generated by $|\varepsilon(v)|^2$ because $|\varepsilon(v)|^2 \leq |\nabla v|^2 \leq \|v\|^2$ and $|\varepsilon(v)|^2 \geq \mathfrak{K}\|v\|^2$ with a positive $\mathfrak{K}$ independent of v (the Korn inequality).

Combining (1) with (2) and (3), applying the Green - Ostrogradskiĭ theorem (on integration by parts), we obtain

$$\alpha(\varepsilon(v), C\dot{\varepsilon}(u)) + (\varepsilon(v), (1-\mathfrak{E}(u))\, C\varepsilon(u)) \\ = (v,f) + \langle v,g\rangle \tag{6}$$

on I, which can be understood as the weak formulation of our problem, using any virtual displacement $v \in V$. It is natural to assume $u \in L^2(I,V,V^*)$ (thus $u \in L^2(I,V)$, $\dot{u} \in L^2(I,V^*)$, $\nabla u \in L^2(I,H^2)$, $\varepsilon(u) \in L^2(I,H^2_{\text{sym}})$, etc.), $\sigma \in L^2(I,H^2_{\text{sym}})$, together with the volume forces $f \in L^2(I,H)$ and surface forces $g \in L^2(I,Z)$. One more assumption $C_{ijkl}a_{ij}a_{kl} \geq c a_{ij}a_{ij}$ for all $a \in \mathcal{R}^{3\times 3}_{\text{sym}}$ with i,j,k,l taken as Einstein summation indices and with a positive c independent of the choice of $x \in \Omega$ will be needed later; the existence of a positive $\bar{c}$ satisfying $C_{ijkl}a_{ij}b_{kl} \leq \bar{c} a_{ij}b_{ij}$ for all $a,b \in \mathcal{R}^{3\times 3}_{\text{sym}}$ is evident. Thus $\|v\|^2_C = |v|^2 + |\varepsilon(v)|^2_C$ with $|\varepsilon(v)|_C = |C^{1/2}\varepsilon(v)|$ for each $v \in V$ generates still another norm in V. The introduction of $\mathcal{D} \in [0, 1-\varsigma]$ with some prescribed real non-negative $\varsigma < 1$ will follow.

Let us remark that the hypothetical reverse approach (to obtain the strong formulation) would be more delicate: it must be understood in the distributive sense, or needs some non-trivial additional regularity assumptions. However, we shall need the following regularization (compactness) property of $\mathcal{K}$, taken from $L^2(\Omega \times \Omega)$, following (4): if $\{w^k\}_{k=1}^\infty$ is some sequence converging weakly to w in H then, taking $\tilde{w} = \mathcal{A}(w)$ and $\tilde{w}^k = \mathcal{A}(w^k)$, up to a subsequence, $\{\tilde{w}^k\}_{k=1}^\infty$ converges strongly to $\tilde{w}$ in H. Indeed, $\tilde{w}^k(x)_{k=1}^\infty$ converges locally to $\tilde{w}(x)$ for almost every $x \in \Omega$; by the Lebesgue dominated convergence theorem is then sufficient to verify the boundedness of $\{\tilde{w}^k\}_{k=1}^\infty$ in H, which is guaranteed by the weak convergence (thus also the boundedness) of $\{w^k\}_{k=1}^\infty$, by the Fubini theorem (on multiple integrals) and by the Cauchy - Schwarz inequality; for all details see [5], p. 81. An important consequence is that for a continuous ω (not just for the special one by (5)) and for any fixed $t \in I$ we are able to guarantee the strong convergence of $\{\mathfrak{E}(u(.,t))\}_{k=1}^\infty$ to $\mathfrak{E}(u(.,t))$ provided that $\{u^k(.,t)\}_{k=1}^\infty$ converges weakly to some $u(.,t)$ in V.

We shall now continue with the sketch of the existence proof for u by (6), applying the method of Rothe sequences. Let us divide I into a finite number m of subsets $I^m_s = \{t \in I : (s-1)\tau < t \leq s\tau\}$ with $s \in \{1,\ldots,m\}$, with the final aim $m \to \infty$; $\tau(m) = T/m$ is considered here, omitting its argument m. Let us consider the Clément quasi-interpolation f^m of f in $L^2(I,H)$ and g^m of g in $L^2(I,Z_\Lambda)$, defined as

$$f^m(t) = \tau^{-1}\int_{(s-1)\tau}^{s\tau} f(\tilde{t})\,\mathrm{d}\tilde{t}\,,$$

$$g^m(t) = \tau^{-1}\int_{(s-1)\tau}^{s\tau} g(\tilde{t})\,\mathrm{d}\tilde{t}$$

for $t \in I^m_s$, $s \in \{1,\ldots,m\}$. For an unknown u we can also introduce some u^m, $\bar{u}^m$ and $\breve{u}^m$ as $u^m(t) = u^m_{s-1} + (t-(s-1)\tau)(u^m_s - u^m_{s-1})$ (linear Lagrange splines), $\bar{u}^m(t) = u^m_s$ (standard simple functions) and $\breve{u}^m(t) = u^m_{s-1}$ (retarded simple functions), which generates 3 different types of the Rothe sequences; $u^m_0 = (0,0,0)$. The discrete variant of (6) for a fixed m then reads

$$\alpha(\varepsilon(v), C\varepsilon(\dot{u}^m)) + (\varepsilon(v), (1-\mathfrak{E}(\breve{u}^m))\, C\varepsilon(\bar{u}^m)) \\ = (v,f^m) + \langle v,g^m\rangle \tag{7}$$

on I; its linearity is evident from its form, rewritten step-by-step for $s \in \{1,\ldots,m\}$ on I^m_s,

$$\alpha(\varepsilon(v), C\varepsilon(u^m_s - u^m_{s-1})) \\ +\ \tau(\varepsilon(v), (1-\mathfrak{E}(u^m_{s-1}))\, C\varepsilon(u^m_{s-1})) \\ =\ \tau(v,f^m_s) + \tau\langle v,g^m_s\rangle\,. \tag{8}$$

Let us set $v = u^m_s$ in (8). Using the Cauchy - Schwarz inequality, taking an arbitrary positive ϖ, we have

$$\begin{aligned}&(\varepsilon(u_s^m),C\varepsilon(u_s^m))=|\varepsilon(u_s^m)|_C^2\geq c\mathfrak{K}\|u_s^m\|^2\,,\\&2(\varepsilon(u_s^m),C\varepsilon(u_s^m-u_{s-1}^m))\\&=\ |\varepsilon(u_s^m)|_C^2+|\varepsilon(u_s^m-u_{s-1}^m)|_C^2-|\varepsilon(u_s^m)|_C^2\,,\\&|\varepsilon(u_s^m-u_{s-1}^m)|_C^2\geq c\mathfrak{K}\|u_s^m-u_{s-1}^m\|^2\,,\\&2(u_s^m,f_s^m)\leq 2|u_s^m|\,|f_s^m|\leq\varpi|u_s^m|^2+\varpi^{-1}|f_s^m|^2\,,\\&2\langle u_s^m,g_s^m\rangle\leq 2|u_s^m|_\Gamma\,|g_s^m|_\Gamma\leq 2\mathfrak{T}\|u_s^m\|\,|g_s^m|_\Gamma\\&\leq\varpi\mathfrak{T}^2\|u_s^m\|^2+\varpi^{-1}|g_s^m|_\Gamma^2\,.\end{aligned}$$

Thus for any fixed $r \in \{1,\dots,m\}$, summing over $s \in \{1,\dots,r\}$, we receive

$$\begin{aligned}&\alpha c\,\mathfrak{K}\|u_r^m\|^2+\alpha c\,\mathfrak{K}\sum_{s=1}^{r}\|u_s^m-u_{s-1}^m\|^2\\&\qquad+\tau M(\varpi)\sum_{s=1}^{r}\|u_s^m\|^2\\&\leq\tau\varpi^{-1}\sum_{s=1}^{r}|f_s^m|^2+\tau\varpi^{-1}\sum_{s=1}^{r}|g_s^m|_\Gamma^2\end{aligned}\tag{9}$$

where $M(\varpi) = 2(1-\varsigma)c\,\mathfrak{K} - (1+\mathfrak{T}^2)\varpi$. Since, up to some multiplicative constants, the right-hand-side additive terms of (9) are just the squares of norms of f^m in $L^2(I,H)$ and of g^m in $L^2(I,Z)$ and those left-hand-side ones correspond to the squares of norms of $\bar{u}^m$ in $L^\infty(I,V)$ and of $\sqrt{\tau}\dot{u}^m = \sqrt{\tau}(\bar{u}^m - \breve{u}^m)$ in $L^2(I,V)$, we have $\{\bar{u}^m\}_{m=1}^\infty$ and $\{\breve{u}^m\}_{m=1}^\infty$ bounded in $L^2(I,V)$ for $M(\varpi) > 0$ directly; otherwise this follows from the discrete Gronwall lemma – cf. [30], p. 26, and [5], p. 99. Moreover, using the notation $[\,.\,]$ for the integration over I here, reformulating (7) with $w \in L^2(I,V)$ as

$$\begin{aligned}&\alpha[(\varepsilon(w),C\varepsilon(\dot{u}^m))]\\&=\ -\ [\,(\varepsilon(w),(1-\mathfrak{E}(\breve{u}^m))C\varepsilon(\bar{u}^m))]\\&\quad-\ [(w,f^m)]-[\langle w,g^m\rangle]\,,\end{aligned}$$

using the same estimates as above, we can see that the upper bound for $[(w,\dot{u}^m)]$ is just the norm of w in $L^2(I,V)$, multiplied by a positive constant; thus we have also $\{\dot{u}^m\}_{m=1}^\infty$ bounded in $L^2(I,V^*)$. Consequently the Eberlein - Shmul'yan theorem by [5], p. 67, implies, up to subsequences, the weak convergence of $\{\bar{u}^m\}_{m=1}^\infty$ and $\{\breve{u}^m\}_{m=1}^\infty$ to some $\bar{u}$ and $\breve{u}$ in $L^\infty(I,V)$ and of $\{\bar{u}^m\}_{m=1}^\infty$ to some $u^\times$ in $L^\infty(I,V^*)$.

To verify that both $\bar{u}$ and $\breve{u}$ can be identified with u, as well as $u^\times$ with $\dot{u}$, let us start with the obvious estimate

$$\begin{aligned}\|u-\bar{u}\|&\leq\|u-u^m\|+\|u^m-\bar{u}^m\|+\|\bar{u}^m-\bar{u}\|\\&=\|u-u^m\|+\tau\|\dot{u}^m\|+\|\bar{u}^m-\bar{u}\|\end{aligned}$$

on I. Passing $m \to \infty$, the 2nd additive term vanishes due to the boundedness of $\sqrt{\tau}\|\dot{u}^m\|$ and the 1st and 3rd ones tend to zero due to the convergence properties in $L^2(I,H) \subset L^\infty(I,H)$; this identifies u with $\bar{u}$. The same arguments can be repeated to identify $\bar{u}$ with $\breve{u}$, too. Moreover, the integration by parts

$$\begin{aligned}[(w,u^\times)]&=\lim_{m\to\infty}[(w,\dot{u}^m)]\\&=-\lim_{m\to\infty}[(\dot{w},u^m)]=-\,[(\dot{w},u)]\end{aligned}$$

is valid for any w from the space of distributions $C_0^\infty(I)$; this is sufficient to identify $u^\times$ with $\dot{u}$, as derived by [3], p. 49. Thanks to the compact embeddings we obtain the convergence properties for $m \to \infty$: $\{\bar{u}\}_{m=1}^\infty$ has its weak limit u in $L^\infty(I,V)$, and its strong limit u in $L^2(I,X)$ for $X \in \{H,Z\}$, the same holds for $\{\breve{u}^m\}_{m=1}^\infty$, whereas $\{\dot{u}^m\}_{m=1}^\infty$ has its weak limit $\dot{u}$ in $L^2(I,V^*)$. Therefore, taking the continuity of ω and ϕ into account, we are allowed to come from (7) to (6).

Let us remark that the presence of $\mathfrak{E}(\,.\,)$ in (8) and (7) brings significant difficulties to most consideration on the uniqueness, regularity, etc., of the solution of (6), including its quasi-static character, as usual in the (both physically and geometrically) linearized elasticity; one can expect some disturbing effects namely in the case $\varsigma = 0$, corresponding to the total loss of stiffness of some part of Ω. For example, let us hint (omitting technical details) how the artificial evolutionary term with $\alpha > 0$ in (8), thus in (7) and (6), too, could vanish. Setting $v = u_s^m - u_{s-1}^m$, the difference of (8) and the same equation with $s-1$ instead of s, can be handled as above by (9), provided that $s > 1$: if f and g are still (nearly) unchanging in time, then the decisive right-hand-side additive term, stemming from (8), reads $\tau(\varepsilon(u_s^m - u_{s-1}^m), (\mathfrak{E}(u_{s-2}^m) - \mathfrak{E}(u_{s-1}^m))C\varepsilon(u_{s-1}^m))$. Thanks to some more continuity arguments, related to a rather complicated $\mathfrak{E}(\,.\,)$, one should obtain $|\varepsilon(\dot{u}^m)|$ in (7) and $|\varepsilon(\dot{u})|$ in (6) decreasing to zero.

4 Some generalizations and applications

Let us start with the remark that, being ready to overcome some technical difficulties in proofs, numerous assumptions in our model problem could be weakened, e. g. the choice of f and g would be able to use the embeddings $L^{6-\epsilon}(\Omega) \Subset V$ (the Sobolev embedding theorem) and $L^{4-\tilde{\epsilon}}(\Gamma) \Subset V$ (the trace theorem) for any positive ϵ and $\tilde{\epsilon}$ instead of those with $\epsilon = 4$ for H and $\tilde{\epsilon} = 2$ for Z; even the boundary of Ω need not to be just the Lipschitz one. Another generalization has been prepared for [33] recently: Ω consists from a finite number of domains, Γ and Θ form the exterior boundary of the union of such domains, whose material properties can be different and the initiation and propagation of macroscopic cracks on their interfaces due to certain cohesive law, compatible with [25], is possible. An instructive numerical example of this type for a metal fibre reinforced cementitious composite (with all micro-fractured zones created only in the cementitious matrix) is presented in [34]; its computational discretization on Ω and required interfaces relies on the eXtended Finite Element Method (XFEM) by [22], referenced in [25], too.

In the last mentioned access most extensive calculations concentrate on the matrix / particle interfaces and in their vicinity. To avoid this phenomenon, various continuum "smeared crack" approaches have been developed; the concepts and history of such research activities from the late 1960s to recent achievements can be assessed by [2], [24] and [35]. In some very simplified formulations more numerical stable exact or fundamental solutions can be implemented then those generated by Green functions, e. g. in the special problem of [19], coupling the Timoshenko beam on elastic medium with the Eringen model of nonlocal Euler - Bernoulli nanobeams. Alternative research directions rely on the computational peridynamics, avoiding all gradient evaluations – cf. [6], [15] and [21], or on statistical physics, handling extremal dynamics in random threshold systems – see [28].. Numerous open questions still occur in the case of multiple scale bridging, i. e. of the computational homogenization at several (typically macro- and micro-) levels, namely for the non-periodic problems.

5 Conclusion

The extensive use of brittle matrix materials requires appropriate computational models to describe, with adequate accuracy, their mechanical behaviour. A possible choice of such model has been introduced in this paper, including numerous references to potential generalizations and alternative methods, whose mathematical analysis is frequently not closed. This can be seen as a) certain basis for the discussion on the advantages and drawbacks of various approaches. b) the research challenge for the near future.

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