Atomicity via quantum measure theory

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Abstract: In this paper, different results concerning (pseudo)-atomicity are obtained from the quantum measure theory perspective and several applications are provided.

Key–Words: Atom; Pseudo-atom; Null-additive; Fuzzy measure; Quantum measure.

1 Introduction

Measure theory concerns with assigning a notion of size to sets. In the last years, non-additive measures theory was given an increasing interest due to its various applications in a wide range of areas (such as, economics, social sciences, biology, philosophy etc.). It is used to describe situations concerning conflicts or cooperations among intelligent rational players, giving an appropriate mathematical framework to predict the outcome of the process. Precisely, theories dealing with (pseudo)atoms and monotonicity are used in statistics, game theory, probabilities, artificial intelligence. The notion of non-atomicity for set (multi)functions plays a key role in measure theory and its applications and extensions. For classical measures taking values in finite dimensional Banach spaces, it guarantees the connectedness of range. Even just replacing σ-additivity with finite additivity for measures requires some stronger nonatomicity property for the same conclusion to hold. Because of their multiple applications, in game theory or mathematical economics, the study concerning atoms and non-atomicity for additive, respectively, non-additive set functions has developed. Particularly, (non)atomic measures and purely atomic measures have been investigated (in different variants) due to their special form and their special properties (see Chiţescu [1,2], Cavaliere and Ventriglia [5], Gavriluţ and Agop [4], Gavriluţ and Croitoru [6,8,10,11], Gavriluţ [7,9], Gavriluţ, Iosif and Croitoru [12], Khare and Singh [18], Li *et al*. [19,20], Pap [22-24], Pap *et al*. [25], Rao and Rao [26], Suzuki [32], Wu and Bo [33] etc.).

Thus, one important application of measure theory is in probability, where a measurable set is interpreted as an event and its measure as the probability that the event will occur. Since probability is an important notion in quantum mechanics, measure theory's techniques could be used to study quantum phenomena. Unfortunately, one of the foundational axioms of measure theory do not remain valid in its intuitive application to quantum mechanics. Although classical measure theory imposes strict additivity conditions, a rich theory of non-additive measures developed. Precisely, modifications of traditional measure theory [23,24] led to quantum measure theory (Gudder [13-17], Salgado [27], Schmitz [28], Sorkin [29,30]). Practically, an extended notion of a measure has been introduced and its applications to the study of interference, probability, and spacetime histories in quantum mechanics have been discussed.

Introduced by Sorkin in [29,30], quantum measures help us to describe quantum mechanics and its applications to quantum gravity and cosmology. Quantum measure theory indicates a wide variety of applications, its mathematical structure being used in the standard quantum formalism.

The present paper is organized as follows. After Introduction, in Sections 2 and 3, different results concerning (pseudo)-atomicity and decoherent functions are provided from the quantum measure theory perspective.

2 Atomicity and pseudo-atomicity from quantum measure theory perspective

Let T be an abstract nonvoid set, $\mathcal{C}$ a ring of subsets of T and suppose $(V, +, \cdot)$ is a real linear space with the origin 0.

Definition 2.1. Let $m : \mathcal{C} \to V$ be a set function, with $m(\emptyset) = 0$.

I) m is said to be:

i) *finitely additive* (or, *grade-1-additive*) if $m(\overset{n}{\underset{i=1}{\cup}} A_i) = \sum_{i=1}^{n} m(A_i)$, for any arbitrary pairwise disjoint sets $(A_i)_{i\in\{1,2,...,n\}} \subset \mathcal{C}, n \in \mathbb{N}^*$;

ii) *a grade-2-measure* if

$$m(A \cup B \cup C) + m(A) + m(B) + m(C)$$
$$= m(A \cup B) + m(B \cup C) + m(A \cup C) (**)$$

for any pairwise disjoint sets $A, B, C \in \mathcal{C}$;

II) Two sets $A, B \in \mathcal{C}$ are called m-*compatible* (denoted by AmB) if

$$m(A \cup B) + m(A \cap B) = m(A) + m(B) (*).$$

(i.e., m-compatible sets are those two sets for which the set function m behaves like a grade-1-measure);

III) An arbitrary fixed set $A \in \mathcal{C}$ that is m-compatible with any set $B \in \mathcal{C}$ is said to be a *macroscopic set*.

Remark 2.2. (Schmitz [28]) i) Since some quantum objects interfere with each other, but others do not, one can justify the name of a "macroscopic set" by the fact that it does not interfere with any set and thus it behaves like a non-quantum object in the macroscopic world.

ii) One can immediately verify that the relation given by m-compatibility is reflexive, symmetric but it is not transitive.

iii) Evidently, if m is grade-1-additive, then it is also a grade-2-measure, but the converse is not valid.

iv) If $A \in \mathcal{C}$ is arbitrary, then A and $\emptyset$ are m-compatible.

v) If $V = \mathbb{R}$, one obtains the notions from [28].

vi) Suppose t_i, where $i \in \{1, 2, ..., n\}, n \in \mathbb{N}^*$ represent quantum objects or quantum events and let be their collection $T = \{t_1, t_2, ..., t_n\}$. One can need an interpretation of a "measure" on T, in situations when the additivity condition $(**)$ is not fulfilled:

Examples 2.3. (Schmitz [28]) Suppose $(T = \{t_1, t_2, ..., t_n\}, m : \mathcal{P}(T) \to \mathbb{R}_+)$ is a finite measure space, where for every $i \in \{1, 2, ..., n\}$, t_i represents the particle, and m the mass. The real valued set function associated to the mass is additive in the macroscopic world. On the quantum scale, these statements do not remain valid due to the annihilation and binding energy effects. For instance, if t_1 and t_2 represent an electron and a positron respectively, then $m(\{t_1\}) = m(\{t_2\}) = 9,11\times\ 10^{-31}\ kg$, but $m(\{t_1, t_2\}) = m(\{t_1\} \cup \{t_2\}) = 0$.

These are some reasons for in what follows we introduce several notions, weaker than classical additivity and also than those from Definition 2.1-i), ii):

Definition 2.4. A set function $m : \mathcal{C} \to V$, with $m(\emptyset) = 0$, is said to be:

i) *null-additive*^{1} if $m(A \cup B) = m(A)$, for every disjoint $A, B \in \mathcal{C}$, with $m(B) = 0$;

ii) *null-additive*^{2} if $m(A \cup B) = m(A)$, for every $A, B \in \mathcal{C}$, with $m(B) = 0$;

iii) *null-null-additive* if $m(A \cup B) = 0$, for every $A, B \in \mathcal{C}$, with $m(A) = m(B) = 0$;

iv) *null-equal* if $m(A) = m(B)$, for every $A, B \in \mathcal{C}$, with $m(A \cup B) = 0$;

v) a *quantum measure* (*q-measure*, for short) if it is a null-additive^1 and null-equal grade-2-measure;

vi) *diffused* if $m(\{t\}) = 0$, whenever $\{t\} \in \mathcal{C}$.

Definition 2.5. (Gavriluţ, Iosif and Croitoru [12]) If V is, moreover, a Banach lattice, a set function $m : \mathcal{C} \to V$, with $m(\emptyset) = 0$, is said to be:

i) *null-monotone* if for every $A, B \in \mathcal{C}$, with $A \subseteq B$, if $m(B) = 0$, then $m(A) = 0$;

ii) *monotone* (or, *fuzzy*) if $m(A) \leq m(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$;

iii) a *submeasure* if m is monotone and *subadditive*, i.e., $m(A \cup B) \leq m(A) + \nu(B)$, for every (disjoint) $A, B \in \mathcal{C}$;

iv) σ-*additive* (or, *a (vector) measure)* if $m(\overset{\infty}{\underset{n=1}{\cup}} A_n) = \lim_{n\to\infty} \sum_{k=1}^{n} m(A_k)$, for every pairwise disjoint sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $\overset{\infty}{\underset{n=1}{\cup}} A_n \in \mathcal{C}$.

The notion of an algebra of sets is significant in measure theory since it contains conditions on which sets are measurable:

Definition 2.6. If $\mathcal{A}$ is an arbitrary σ-algebra of T and if $m : \mathcal{A} \to \mathbb{R}_+$ is a measure on $\mathcal{A}$, with $m(T) = 1$, then:

i) The space $(T, \mathcal{A}, m)$ is said to be a *sample space* and m is said to be a *probability measure*;

ii) The elements of T are called *sample points* or *outcomes* and the elements of $\mathcal{A}$ are called *events*.

In this case, for every $A \in \mathcal{A}, m(A)$ is interpreted as the probability of the event A to occur.

Remark 2.7. i) The notion of a null-equal-measure has the following physical interpretation [28]: in the situation involving destructive interference, in order for two waves to produce complete destructive interference, thereby "cancelling out" each other, their original amplitudes must have been equal.

ii) Any positive real valued finitely additive set function $m : \mathcal{C} \to \mathbb{R}_+$ is a q-measure.

iii) If $m(T) > 0$, then one can immediately generate a probability measure by means of a normalization process.

Remark 2.8. I) i) One observes that a set function $m : \mathcal{C} \to V$ is diffused if the measure of any singleton of the space is null. This means in the construction of a physical theory, the vacuum condition of the matter should be considered as its complement.

ii) If V is a Banach lattice, $T = \{t_1, t_2, ..., t_n\}, n \in \mathbb{N}^*$ is an arbitrary finite metric space and $m : \mathcal{P}(T) \to V$ (or, more general, if T is a T_1-separated topological space, $\mathcal{B}$ is the Borel σ-algebra of T generated by the lattice of all compact subsets of T and $m : \mathcal{B} \to V$) is null-additive and diffused, then $m(T) = 0$ (i.e., the space T is composed of particles which annihilate one each other).

II) If $m : \mathcal{C} \to V$ is null-additive^1, then any two disjoint sets $A, B \in \mathcal{C}$, with $m(B) = 0$, are m-compatible.

III) If $m : \mathcal{C} \to V$ is null-monotone, then:

i) m is null-additive^1 if and only if it is null-additive^2. In this case, m will be simply called *null-additive*.

ii) If m is null-null-additive, then it is null-equal.

Definition 2.9. (Gavriluţ and Croitoru [6,8,10,11]) Let $m : \mathcal{C} \to \mathbb{R}_+$ be a set function, with $m(\emptyset) = 0$.

i) A set $A \in \mathcal{C}$ is said to be an *atom* of ν if $m(A) > 0$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have $m(B) = 0$ or $m(A \backslash B) = 0$;

ii) m is said to be *non-atomic* (NA, for short) if it has no atoms (i.e., for every $A \in \mathcal{C}$ with $m(A) > 0$, there exists $B \in \mathcal{C}, B \subseteq A$, such that $m(B) > 0$ and $m(A \backslash B) > 0$);

iii) *finitely purely atomic* if there is a finite family $(A_i)_{i \in \{1,2,\ldots,n\}}$ of pairwise disjoint atoms of m so that $T = \bigcup\limits_{i=1}^{n} A_i$.

Definition 2.10. (Gavriluţ and Croitoru [6,8,10,11]) Let $m : \mathcal{C} \to \mathbb{R}_+$ be a set function, with $m(\emptyset) = 0$.

i) A set $A \in \mathcal{C}$ is called a *pseudo-atom* of ν if $m(A) > 0$ and $B \in \mathcal{C}$, $B \subseteq A$ implies $m(B) = 0$ or $m(B) = m(A)$;

ii) m is said to be *non-pseudo-atomic* (NPA, for short) if it has no pseudo-atoms (i.e., for every $A \in \mathcal{C}$ with $m(A) > 0$, there exists $B \in \mathcal{C}, B \subseteq A$, such that $m(B) > 0$ and $m(A) \neq m(B)$.

We now recall the following properties involving operations with atoms/pseudo-atoms:

Remark 2.11. (Gavriluţ and Croitoru [6,8,10,11]) Let be $m : \mathcal{C} \to \mathbb{R}_+$, with $m(\emptyset) = 0$.

i) If m is null-monotone, $A \in \mathcal{C}$ is an atom of m and $B \in \mathcal{C}$, $B \subseteq A$ is such that $m(B) > 0$, then B is an atom of m and $m(A \backslash B) = 0$.

ii) If $A \in \mathcal{C}$ is a pseudo-atom of and $B \in \mathcal{C}$, $B \subseteq A$ is such that $m(B) > 0$, then B is a pseudo-atom of and $m(B) = m(A)$.

iii) If $A, B \in \mathcal{C}$ are pseudo-atoms of and $m(A \cap B) > 0$, then $A \cap B$ is a pseudo-atom of m and $m(A \cap B) = m(A) = m(B)$.

iv) Let $m : \mathcal{C} \to \mathbb{R}_+$ be null-additive and let $A, B \in \mathcal{C}$ be pseudo-atoms of m.

1. If $m(A \cap B) = 0$, then $A \backslash B$ and $B \backslash A$ are pseudo-atoms of m and $m(A \backslash B) = m(A), m(B \backslash A) = m(B)$.
2. If $m(A) \neq m(B)$, then $m(A \cap B) = 0, m(A \backslash B) = m(A)$ and $m(B \backslash A) = m(B)$.

v) Let $m : \mathcal{C} \to \mathbb{R}_+$ be null-additive and let $A, B \in \mathcal{C}$ be pseudo-atoms of m. If $m(A \cap B) > 0$ and $m(A \backslash B) = m(B \backslash A) = 0$, then $A \cap B$ is a pseudo-atom of and $m(A \Delta B) = 0$.

Remark 2.12. Suppose $m : \mathcal{C} \to \mathbb{R}_+$ is so that $m(\emptyset) = 0$.

i) If m is finitely additive, then $A \in \mathcal{C}$ is an atom of m if and only if A is a pseudo-atom of m.

ii) Any $\{t\} \subseteq T$, provided $\{t\} \in \mathcal{C}$ and $m(\{t\}) > 0$, is an atom of m.

ii) If m is null-additive^1, then every atom of m is also a pseudo-atom. The converse is not generally valid.

Examples 2.13. Let $T = \{t_1, t_2\}$ be a finite abstract space composed of two elements.

i) We consider the set function $m : \mathcal{P}(T) \to \mathbb{R}_+$ defined for every $A \subset T$ by $m(A) = \begin{cases} 2, A = T \\ 1, A = \{t_1\} \\ 0, A = \{t_2\} \text{ or } A = \emptyset. \end{cases}$.

Then T is an atom and it is not a pseudo-atom of m.

ii) We define $m : \mathcal{P}(T) \to \mathbb{R}_+$ by $m(A) = \begin{cases} 1, A \neq \emptyset \\ 0, A = \emptyset \end{cases}$, for every $A \subset T$. Then m is null-additive and $T = \{t_1, t_2\}$ is a pseudo-atom of m, but it is not an atom.

Proposition 2.14. If $m : \mathcal{C} \to \mathbb{R}_+$ is null-monotone and null-additive and if $A \cup B$ is an atom of m, then A, B are m-compatible.

Proof. 1. If $m(A) = 0$, then $m(A \cap B) = 0$ and $m(A \cup B) = m(B)$, so the conclusion follows.

2. If $m(A) > 0$, then by Remark 2.11-i) A is an atom, too and $m((A \cup B) \backslash A) = m(B \backslash A) = 0$. Since $B = (B \backslash A) \cup (B \cap A)$, then $m(B) = m(A \cap B)$ and since $A \cup B = A \cup (B \backslash A)$, we get $m(A \cup B) = m(A)$.

Definition 2.15. If $m : \mathcal{C} \to \mathbb{R}_+$ is so that $m(\emptyset) = 0$, we consider *the variation of* m, $\overline{m} : \mathcal{P}(T) \to [0, \infty]$, defined for every $A \in \mathcal{P}(T)$ by:

$$\overline{m}(A) = \sup \left\{ \sum_{i=1}^{n} m(A_i); A = \bigcup_{i=1}^{n} A_i, A_i \in \mathcal{C}, \right.$$
$$\left. \forall i = \overline{1, n}, A_i \cap A_j = \emptyset,\ i \neq j \right\}.$$

We say that m is *of finite variation* if $\overline{m}(T) < \infty$.

In what follows, we give some examples of q-measures:

Proposition 2.16. If $m : \mathcal{C} \to \mathbb{R}_+$ is a submeasure of finite variation, then $\overline{m}$ is a q-measure.

Proof. Since m is a submeasure of finite variation, then, according to [3], $\overline{m} : \mathcal{C} \to [0, \infty)$ is finitely additive, so it is a q-measure.

In what follows, let $\mathcal{K}$ be the lattice of all compact subsets of a locally compact Hausdorff space T and $\mathcal{B}$ be the Borel σ-algebra generated by $\mathcal{K}$. The following definition is then consistent:

Definition 2.17. [22-25] $m : \mathcal{B} \to \mathbb{R}_+$ is said to be *regular* if for every $A \in \mathcal{B}$ and every $\varepsilon > 0$, there exist $K \in \mathcal{K}$ and an open set $D \in \mathcal{B}$ such that $K \subset A \subset D$ and $m(D \backslash K) < \varepsilon$.

Theorem 2.18. [22-25] *Suppose* $m : \mathcal{B} \to \mathbb{R}_+$ *is a monotone null-additive regular set function. If* $A \in \mathcal{B}$ *is an atom of* m*, then there exists a unique point* $a \in A$ *so that* $m(A \backslash \{a\}) = 0$ *(and so,* $m(A) = m(\{a\})$.

Theorem 2.19. *Suppose* $T = \{t_1, ..., t_n\}$ *is a Hausdorff topological space and it is also an atom of a monotone, null-additive regular set function* $m : \mathcal{P}(T) \to \mathbb{R}_+$*. Then* m *is a q-measure.*

Proof. Obviously, T is a compact space, so it is locally compact. Also, $\mathcal{B} = \mathcal{P}(T)$.

Since T is an atom, by the previous theorem there exists $t_1 \in T$ so that $m(\{t_2, ..., t_n\}) = m(T \backslash \{t_1\}) = 0$, whence $m(\{t_2\}) = ... = m(\{t_n\}) = 0$.

In consequence, for every $A \subset T$, if $t_1 \notin A$, then $m(A) = 0$ and if $t_1 \in A$, then $m(A) = m(\{t_1\}) = m(T)$.

Now, consider arbitrary pairwise disjoint $A, B, C \in \mathcal{P}(T)$.

If $t_1 \notin A \cup B \cup C$, then $t_1 \notin A, t_1 \notin B, t_1 \notin C$, so the conclusion follows.

If $t_1 \in A \cup B \cup C$, suppose without any lack of generality that $t_1 \in A$, $t_1 \notin B, t_1 \notin C$. Then $m(B \cup C) = m(B) = m(C) = 0, m(A \cup B \cup C) = m(T) = m(A \cup B) = m(A \cup C) = m(A)$ and the proof finishes.

3 Decoherence functions

In quantum mechanics, when a wavefunction becomes coupled to its environment, the objects involved interacting with the surroundings, the decoherence phenomenon occurs. It is also known as the ”wavefunction collapse” and it allows the classical limit to emerge on the macroscopic scale from a set of quantum events. After decoherence has occurred, the system's components can no longer interfere, so one could assign a well-defined probability to each possible decoherent outcome.

Using decoherence functions, one could define the probabilities of all decoherent outcomes for a particular event by quantifying the amount of interference among system's various components (see Pap, Gavriluţ and Agop [25] for details). So, interference has an important role in the mathematical formulation of quantum mechanics. One can define functions related to interference, that can be used in order to obtain q-measures:

Definition 3.1. (Schmitz [28]) Suppose T is an abstract space and $\mathcal{A}$ is an algebra of subsets of T. A function $D : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is said to be a *decoherence function* if the following conditions hold:

i) $D(A, B) = \overline{D(B, A)}$, for every $A, B \in \mathcal{A}$;

ii) $D(A, A) \geq 0$, for every $A \in \mathcal{A}$;

iii) $|D(A, B)| \leq D(A, A) \cdot D(B, B)$, for every $A, B \in \mathcal{A}$;

iv) $D(A \cup B, C) = D(A, C) + D(B, C)$, for every disjoint $A, B \in \mathcal{A}$ and every $C \in \mathcal{A}$.

Remark 3.2. (Schmitz [28]) i) Since $D(A, A) \in \mathbb{R}$, the conditions ii) and iii) are justified.

ii) By i), for arbitrary $A, B \in \mathcal{A}$ representing quantum objects, $Re[D(A, B)]$ can be interpreted as the interference between A and B, as we shall remark in what follows:

Proposition 3.3. (Schmitz [28]) If $D : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a decoherence function, then $m : \mathcal{A} \to \mathbb{C}$, $m(A) = D(A, A)$ is a q-measure.

Example 3.4. If V is a pre-Hilbert space and if $m : \mathcal{A} \to V$ is finitely additive, then $D : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$, $D(A, B) =< m(A), m(B) >$, for every $A, B \in \mathcal{A}$ is a decoherence function.

Particularly, if $m : \mathcal{A} \to \mathbb{C}$ is finitely additive (often interpreted as a quantum amplitude), then one can define the decoherence function defined for every $A, B \in \mathcal{A}$ by $D(A, B) = m(A) \cdot \overline{m(B)}$.

The corresponding q-measure is $\widetilde{m} : \mathcal{A} \to \mathbb{C}$, $\widetilde{m}(A) = D(A, A) = m(A) \cdot \overline{m(A)} = |m(A)|^2$, for every $A \in \mathcal{A}$.

Remark 3.5. i) (Schmitz [28]) If $A, B \in \mathcal{A}$ are disjoint, then $\widetilde{m}$ is not grade-1-additive. Indeed,

$$\begin{aligned}
&\widetilde{m}(A \cup B) = |m(A \cup B)|^2 = |m(A) + m(B)|^2 = \\
&= |m(A)|^2 + |m(B)|^2 + 2\Re\mathrm{e}[m(A)\overline{m(B)}] = \\
&= \widetilde{m}(A) + \widetilde{m}(B) + 2\Re\mathrm{e}D(A, B).
\end{aligned}$$

Also, $\widetilde{m}(A \cup B) = \widetilde{m}(A) + \widetilde{m}(B)$ iff $\Re\mathrm{e}D(A, B) = 0$, i.e., interference is represented by the real part of a decoherence function.

ii) If $m : \mathcal{A} \to \mathbb{R}$ is a real valued submeasure of finite variation, then $D : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$, $D(A, B) =< \overline{m}(A), \overline{m}(B) >$ is a decoherence function, where $\overline{m}$ is the variation of m.

4 Conclusion

In this paper, certain (pseudo)-atomicity and decoherent functions problems are treated from the quantum measure theory perspective. Several applications are also provided.

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