An Asymptoticity Based Solution to Parametric Recurrences: Formulation and the Confirmation via Quantum Hydrogen-like Systems' Expectation Value Evolutions

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Abstract: The main subject of this work comes from the applications of Probabilistic Evolution Theory (PREVTH) proposed and then developed in few recent years. Basic aim is to use a set of time-independent expectation value equations based on a given univariate function of the position operator. By choosing an appropriate basis set it is possible to construct a recurrence which contains the unknown energy parameter of the system. By finding the asymptotic solution of the recursion and then using it in the very first finite number of equations we have shown that an approximation scheme can be established to get the energy of the system and a set of expectation values constructed from the basis set of the recurrence. The confirmative applications to the stationary solutions of the hydrogen-like systems are also given.

Key–Words: Parametric recurrences, Asymptoticity, Quantum spectral problems, Autonomous systems, Hydrogen-like systems

1 Introduction and Positional Function Image Expectation Value Equation

The Hamilton operator of a one-degree-of-freedom system which evolves under a potential function of position operator $\widehat{q}$, $V\left(\widehat{q}\right)$, can be defined as follows

$$\widehat{H} \equiv \frac{1}{2\mu}\widehat{p}^2 + V\left(\widehat{q}\right). \tag{1}$$

Herein $\widehat{p}$ and $\widehat{q}$ symbolize the momentum and position operators whose definitions are given below as $g(x)$ denotes an arbitrary function in the space where the wave function of system lies.

$$\widehat{p}g(x) \equiv -i\hbar g'(x), \qquad \widehat{q}g(x) \equiv xg(x), \quad x \in (-\infty, \infty) \tag{2}$$

where $\hbar$ denotes the reduced Planck constant defined as the ratio of the Planck constant to 2π while x stands for a scalar variable which can be called "Position Variable". In other words, it symbolizes the positional variable of Schrödinger equation. Even though x position variable is considered to be located anywhere on the real axis, it can also be considered as taking values from semi infinite or finite intervals on the real axis.

After this preformatting, we can focus on the Poisson Bracket evaluation between the system Hamiltonian and a function operator depending on the position operator defined through an analytic univariate function. We can immediately write the following equality by using the distributive property of the addition in Poisson bracket.

$$\begin{aligned}\left\{\widehat{H}, f\left(\widehat{q}\right)\right\} &= \frac{1}{2\mu}\left\{\widehat{p}^2, f\left(\widehat{q}\right)\right\} + \left\{V\left(\widehat{q}\right), f\left(\widehat{q}\right)\right\} \\ &= \frac{1}{2\mu}\left\{\widehat{p}^2, f\left(\widehat{q}\right)\right\}\end{aligned} \tag{3}$$

In this formula, we have used the fact that the Poisson bracket between $V\left(\widehat{q}\right)$ and $f\left(\widehat{q}\right)$, each of which has an algebraic multiplication operator structure, vanishes.

When one of the operands of a Poisson bracket is a binary product, the correspondant of the Leibnitz rule used in first order differentiation can be applied on that Poisson bracket. This fact allows us to write the following equality

$$\begin{aligned}\left\{\widehat{H}, f\left(\widehat{q}\right)\right\} &= \frac{1}{2\mu}\left\{\widehat{p}^2, f\left(\widehat{q}\right)\right\} \\ &= \frac{1}{2\mu}\left\{\widehat{p}, f\left(\widehat{q}\right)\right\}\widehat{p} + \frac{1}{2\mu}\widehat{p}\left\{\widehat{p}, f\left(\widehat{q}\right)\right\}\end{aligned} \tag{4}$$

The following manipulation stages can be realized for the rightmost Poisson bracket by using just differentiation

$$\{\widehat{p}, f(\widehat{q})\} g(x) = \frac{i}{\hbar}\widehat{p} f(\widehat{q}) g(x) - \frac{i}{\hbar} f(\widehat{q}) \widehat{p} g(x) \quad (5)$$

$$\widehat{q} g(x) = x g(x) \implies f(\widehat{q}) g(x) = f(x) g(x) \quad (6)$$

$$\begin{aligned} \widehat{p} f(\widehat{q}) g(x) &= \widehat{p}(f(\widehat{q}) g(x)) = \widehat{p}(f(x) g(x)) \\ &= -i\hbar (f(x) g(x))' \\ &= -i\hbar f'(x) g(x) - i\hbar f(x) g'(x) \end{aligned} \quad (7)$$

$$\begin{aligned} f(\widehat{q}) \widehat{p} g(x) &= f(\widehat{q})(\widehat{p} g(x)) = f(\widehat{q})(-i\hbar g'(x)) \\ &= -i\hbar f(\widehat{q}) g'(x) \\ &= -i\hbar f(x) g'(x) \end{aligned} \quad (8)$$

$$\implies$$
$$\{\widehat{p}, f(\widehat{q})\} g(x) = f'(x) g(x) = f'(\widehat{q}) g(x) \quad (9)$$

Thus, we can arrive at

$$\{\widehat{p}, f(\widehat{q})\} = f'(\widehat{q}) \quad (10)$$

and therefore

$$\left\{\widehat{H}, f(\widehat{q})\right\} = \frac{1}{2\mu}\left[\widehat{p} f'(\widehat{q}) + f'(\widehat{q}) \widehat{p}\right]. \quad (11)$$

If the expectation value of both sides under the system's wavefunction is taken and the following fact from Ehrenfest Theorem is used

$$\left\langle \left\{\widehat{H}, f(\widehat{q})\right\}\right\rangle (t) = \frac{d}{dt}\langle f(\widehat{q})\rangle (t) \quad (12)$$

then the below dynamical equation can be obtained.

$$\frac{d}{dt}\langle f(\widehat{q})\rangle (t) = \frac{1}{2\mu}\left\langle \widehat{p} f'(\widehat{q}) + f'(\widehat{q}) \widehat{p}\right\rangle (t) \quad (13)$$

This equality means that the temporal variation of the expectation value of $f(\widehat{q})$ is described by the expectation value of an operator depending on both momentum and position operators and appearing at the right hand side. Whereas an expression which contains no momentum operator but Hamiltonian would be more preferable. However, this situation is not possible for the first temporal derivative. This urges us to focus on higher temporal derivative of the expectation value. The first thing to do this end appears to be concerning with the second temporal derivative. If the following equality, which can be derived from (12) by temporal differentiation, is considered For this purpose. Then it becomes clear that the analytic expression of two nested Poisson brackets should be evaluated.

$$\frac{d^2}{dt^2}\langle f(\widehat{q})\rangle (t) = \left\langle \left\{\widehat{H}, \left\{\widehat{H}, f(\widehat{q})\right\}\right\}\right\rangle (t) \quad (14)$$

If the Poisson bracket of both sides of (11) has been taken then we can immediately write

$$\begin{aligned} &\left\{\widehat{H}, \left\{\widehat{H}, f(\widehat{q})\right\}\right\} = \\ &\qquad \left\{\widehat{H}, \frac{1}{2\mu}\left[\widehat{p} f'(\widehat{q}) + f'(\widehat{q}) \widehat{p}\right]\right\} \end{aligned} \quad (15)$$

which the distributive aspect of Poisson bracket on the binary products at the right hand side enables us to write the following equality

$$\begin{aligned} &\left\{\widehat{H}, \left\{\widehat{H}, f(\widehat{q})\right\}\right\} = \frac{1}{2\mu}\left[\left\{\widehat{H}, \widehat{p}\right\} f'(\widehat{q}) + \right. \\ &+ f'(\widehat{q}) \left\{\widehat{H}, \widehat{p}\right\}\left] + \frac{1}{2\mu}\right[\left\{\widehat{H}, f'(\widehat{q})\right\} \widehat{p} + \\ &\left. + \widehat{p}\left\{\widehat{H}, f'(\widehat{q})\right\}\right] \end{aligned} \quad (16)$$

In this formula, the Poisson brackets at the right hand side can be evaluated by tracing the routes mentioned above, and then, the following equalities can be obtained

$$\left\{\widehat{H}, \widehat{p}\right\} = \{V(\widehat{q}), \widehat{p}\} = -V'(\widehat{q}) \quad (17)$$

$$\left\{\widehat{H}, f'(\widehat{q})\right\} = \frac{1}{2\mu}\left[\widehat{p} f''(\widehat{q}) + f''(\widehat{q}) \widehat{p}\right] \quad (18)$$

The following intermediate result can be obtained instead of (16) with the aid of these equalities

$$\begin{aligned} &\left\{\widehat{H}, \left\{\widehat{H}, f(\widehat{q})\right\}\right\} = -\frac{1}{\mu} V'(\widehat{q}) f'(\widehat{q}) + \\ &+ \frac{1}{2\mu^2}\widehat{p} f''(\widehat{q}) \widehat{p} + \frac{1}{4\mu^2}\left[\widehat{p}^2 f''(\widehat{q}) + f''(\widehat{q}) \widehat{p}^2\right] \end{aligned} \quad (19)$$

where the first term at the right hand side is compatible with what we desire since it does not contain momentum, while the last term contains no momentum but its square. This permits us to replace the momentum square with the expression in terms of the Hamiltonian and the position operator. On the other hand, the second term outside these two terms contains momentum but not in its square. This pushes us to investigate the possibility of converting this term into an expression such that it contains nothing involving the

momentum but its square. Towards this end we can write

$$\begin{aligned}(\widehat{p}f''(\widehat{q})-f''(\widehat{q})\widehat{p})\,g(x) &= -i\hbar\,(f''(x)g(x))' \\ &\quad +i\hbar f''(x)g'(x) \\ &= -i\hbar f'''(x)g(x) \\ &\Longrightarrow \\ \widehat{p}\,f''(\widehat{q})-f''(\widehat{q})\,\widehat{p} &= -i\hbar f'''(\widehat{q}) \\ &\Longrightarrow \\ \widehat{p}\,f''(\widehat{q}) &= f''(\widehat{q})\,\widehat{p}-i\hbar f'''(\widehat{q}) \\ f''(\widehat{q})\,\widehat{p} &= \widehat{p}\,f''(\widehat{q})+i\hbar f'''(\widehat{q})\,. \end{aligned} \quad (20)$$

Here, if the first and second of the last two equalities are multiplied by momentum from the right and the left respectively and the resulting equalities are added side by side and then the resulting equality's both sides are divided by 2 then

$$\begin{aligned}\widehat{p}\,f''(\widehat{q})\,\widehat{p} &= \frac{1}{2}\left[\widehat{p}^2 f''(\widehat{q})+f''(\widehat{q})\,\widehat{p}^2\right] \\ &\quad +\frac{i\hbar}{2}\left[\widehat{p}f'''(\widehat{q})-f'''(\widehat{q})\,\widehat{p}\right]\end{aligned} \quad (21)$$

is obtained. For the second term at the right hand side of this equality it is not hard to produce the following equality

$$\begin{aligned}i\hbar\left[\widehat{p}f'''(\widehat{q})-f'''(\widehat{q})\,\widehat{p}\right] &= \hbar^2\left\{\widehat{p},f'''(\widehat{q})\right\} \\ &= \hbar^2 f^{(4)}(\widehat{q})\end{aligned} \quad (22)$$

Thus the following equality, which contains momentum only in its square, is obtained

$$\begin{aligned}\widehat{p}f''(\widehat{q})\,\widehat{p} &= \frac{1}{2}\left[\widehat{p}^2 f''(\widehat{q})+f''[\,\widehat{q}\,)\,\widehat{p}^2\right] \\ &\quad +\frac{\hbar^2}{2}f^{(4)}(\widehat{q})\end{aligned} \quad (23)$$

After all these investigations we can write the following equality instead of (19)

$$\begin{aligned}&\left\{\widehat{H},\left\{\widehat{H},f(\widehat{q})\right\}\right\}=-\frac{1}{\mu}V'(\widehat{q})\,f'(\widehat{q})+ \\ &+\frac{\hbar^2}{4\mu^2}f^{(4)}(\widehat{q})+\frac{1}{2\mu^2}\left[\widehat{p}^2 f''(\widehat{q})+f''(\widehat{q})\,\widehat{p}^2\right]\end{aligned} \quad (24)$$

and from this, by using the Hamiltonian instead of the momentum square, we can get

$$\begin{aligned}&\left\{\widehat{H},\left\{\widehat{H},f(\widehat{q})\right\}\right\}=-\frac{1}{\mu}V'(\widehat{q})\,f'(\widehat{q}) \\ &-\frac{2}{\mu}V(\widehat{q})\,f''(\widehat{q})+\frac{\hbar^2}{4\mu^2}f^{(4)}(\widehat{q}) \\ &+\frac{1}{\mu}\left[\widehat{H}f''(\widehat{q})+f''(\widehat{q})\,\widehat{H}\right]\end{aligned} \quad (25)$$

If we take the expectation value of both sides of this equality under the system's wave function then we can arrive at the following equality

$$\begin{aligned}\frac{d^2\langle f(\widehat{q})\rangle(t)}{dt^2} &= -\frac{1}{\mu}\langle V'(\widehat{q})\,f'(\widehat{q})\rangle(t) \\ &\quad -\frac{2}{\mu}\langle V(\widehat{q})\,f''(\widehat{q})\rangle(t) \\ &\quad +\frac{\hbar^2}{4\mu^2}\left\langle f^{(4)}(\widehat{q})\right\rangle(t) \\ &\quad +\frac{1}{\mu}\left\langle\widehat{H}f''(\widehat{q})+f''(\widehat{q})\,\widehat{H}\right\rangle(t)\end{aligned} \quad (26)$$

(26) has a quite flexible structure since the function f has no other constraint except it analyticity. This also gives a high level universality. However, this flexibility and universality is not only due to f. Even though we have not explicitly emphasized on there is also a relation to the spectral properties of the system. The expectation value is taken under the system's wave function which is evolved from a given initial state wave function. Hence it is strongly depending on how the initial state wave function is given. In other words there is a flexibility in the selection of the initial state.

In the systems whose Hamilton operator is temporally autonomous, in toher words, not changing in time, it is possible to mention the overall (remaining constant in time) spectral properties of the Hamiltonian. The eigenvalues of this operator correspond to the stationary energy values of the system.

The wave function of a state whose energy is denoted by E temporally changes in accordance with the following equality.

$$\psi(x,t)=\mathrm{e}^{-\frac{i}{\hbar}Et}\psi_0(x) \quad (27)$$

where ψ_0 stands for the system Hamiltonian's eigenfunction corresponding to E. It can also be taken as the initial wave function for a motion where system energy remains unchanged.

(27) enables us to write the following equalities for any given linear operator corresponding to an observable of the system under consideration.

$$\begin{aligned}\langle\,\widehat{o}\,\rangle(t) &\equiv \int_{-\infty}^{\infty}dx\psi(x,t)^*\widehat{o}\psi(x,t) \\ &= \int_{-\infty}^{\infty}dx\mathrm{e}^{\frac{i}{\hbar}Et}\psi_0(x)^*\widehat{o}\mathrm{e}^{-\frac{i}{\hbar}Et}\psi_0(x) \\ &= \int_{-\infty}^{\infty}dx\psi_0(x)^*\widehat{o}\psi_0(x).\end{aligned} \quad (28)$$

which states that the expectation value of a temporally autonomous operator does not vary in time if

the system under consideration stands in a specific energy state. The stationary nature of autonomous states arises from this reality in fact.

(26) takes the following form for a stationary state with energy E.

$$\begin{aligned}&-\frac{1}{\mu}\left\langle V'\left(\widehat{q}\right)f'\left(\widehat{q}\right)\right\rangle-\frac{2}{\mu}\left\langle V\left(\widehat{q}\right)f''\left(\widehat{q}\right)\right\rangle\\&+\frac{\hbar^2}{4\mu^2}\left\langle f^{(4)}\left(\widehat{q}\right)\right\rangle+\frac{1}{\mu}\left\langle \widehat{H}f''\left(\widehat{q}\right)\right.\\&\left.+f''\left(\widehat{q}\right)\widehat{H}\right\rangle=0\end{aligned}\tag{29}$$

All expressions whose expectation values are appearing at the right hand side of this equality are autonomous (unchanging in time) operators. Hence their expectation values are also unchanging in time. For this reason, we have removed the time dependence originally appearing in (26).

E and $\psi_0(x)$ in (27) are not independent from each other. They have a relation through the eigenequation of the Hamilton operator. Hence,

$$\widehat{H}\psi_0(x)=E\psi_0(x)\tag{30}$$

can be written. We can also write the following equalities for the system state whose energy is represented by E.

$$\begin{aligned}\left\langle \widehat{H}f''\left(\widehat{q}\right)\right\rangle &= \int_{-\infty}^{\infty}dx\psi_0(x)^*\widehat{H}f''\left(\widehat{q}\right)\psi_0(x)\\&= \int_{-\infty}^{\infty}dx\widehat{H}\psi_0(x)^*f''\left(\widehat{q}\right)\psi_0(x)\\&= E\left\langle f''\left(\widehat{q}\right)\right\rangle\end{aligned}\tag{31}$$

$$\begin{aligned}\left\langle f''\left(\widehat{q}\right)\widehat{H}\right\rangle &= \int_{-\infty}^{\infty}dx\psi_0(x)^*f''\left(\widehat{q}\right)\widehat{H}\psi_0(x)\\&= E\left\langle f''\left(\widehat{q}\right)\right\rangle\end{aligned}\tag{32}$$

where we have used the self-adjointness (or Hermiticity) of the Hamilton operator. The use of these identities in (29) gives

$$\begin{aligned}&-\frac{1}{\mu}\left\langle V'\left(\widehat{q}\right)f'\left(\widehat{q}\right)\right\rangle-\frac{2}{\mu}\left\langle V\left(\widehat{q}\right)f''\left(\widehat{q}\right)\right\rangle\\&+\frac{\hbar^2}{4\mu^2}\left\langle f^{(4)}\left(\widehat{q}\right)\right\rangle+2\frac{E}{\mu}\left\langle f''\left(\widehat{q}\right)\right\rangle=0\end{aligned}\tag{33}$$

The entities appearing in this equality have values with physical units. Certain appropriate scalings in the relevant entities may change the form to a new one wthout physical units. However, for this the potential parameters may need to be treated in the same way as well. It is also possible to show that these changes can be actualized by directly taking $\hbar=1$ and $\mu=1$ together with certain similar standardizations in the potential. If we assume that all these changes have been made and the energy parameter has been changed with its physical unit free correspondant then we can write

$$\begin{aligned}&-\left\langle V'\left(\widehat{q}\right)f'\left(\widehat{q}\right)\right\rangle-2\left\langle V\left(\widehat{q}\right)f''\left(\widehat{q}\right)\right\rangle\\&+\frac{1}{4}\left\langle f^{(4)}\left(\widehat{q}\right)\right\rangle+2E\left\langle f''\left(\widehat{q}\right)\right\rangle=0\end{aligned}\tag{34}$$

We call f "Positional Function" and the equality in (34) "Positional Function Image Expectation Value (POFIMEV) equation. This is the most important equation we are going to use in our investigations in this work.

2 Constructing Singular Recursions via Positive Function Image Expectation Value Equation

To understand the singular recursion structure produced by using Positive Function Image Expectation Value Equation (POFIMEV), first we are going to focus on the quantum systems composed of two particles interacting with a potential depending on the interparticular distance only then we can separate out the mass center motion and the relative motion of the particles. The latter can be reformulated in spherical coordinates and then the potential becomes depending just on the radial variable and angular behavior appears only in the kinetic energy terms of the Hamiltonian, and in such a way that, the angular and the radial motion can also be separated. When this is realized the potential term gains an extra additive term proportional to the reciprocal square of interparticle distance. Hence, we can mathematically consider the potential functions having a second degree polar term located at the origin. Beyond this, the potential is defined to have a Coulomb interaction term. So to proceed, we can write

$$V\left(\widehat{q}\right)\equiv\sum_{j=0}^{\infty}V_{k-2}\widehat{q}^{k-2}\tag{35}$$

which takes us to the following recursion when we set f equal to jth power of the position operator.

$$\begin{aligned}&\frac{1}{2(j+1)}\sum_{k=0}^{\infty}(2j+k)V_{k-2}\left\langle \widehat{q}^{j+k-2}\right\rangle\\&-\frac{j(j-1)}{8}\left\langle \widehat{q}^{j-2}\right\rangle=E\left\langle \widehat{q}^{j}\right\rangle,\\&j=0,1,2,...\end{aligned}\tag{36}$$

This recursion contains the expectation value of the position operator reciprocal in its first two equations. Since we have considered the natural number power expectation values only, this is a one dimensional overflow from the operator space spanned by the natural number powers of the position operator. This overflow changes the character of the spectral problem we have previously focused on and impels us to work on the extended space by adding the expectation value of the position operator's reciprocal an additional component. So we focus on the phase space spanned by the expectation values of the natural number powers and reciprocal of the position operator. This urges us to define a vector, having denumerable infinite number of components, whose elements are defined as follows

$$h_j = \left\langle \widehat{q}^{j-2} \right\rangle, \qquad j = 1, 2, 3, ... \tag{37}$$

We also define the following matrices via their elements

$$\begin{aligned} H_{j,j+k} &\equiv \frac{2j+k-1}{2j} V_{k-1}, \\ &\quad j = 1, 2, 3, ...; \qquad k = 0, 1, 2, ... \\ H_{j,j-1} &\equiv \frac{j-1}{j} V_{-2} - \frac{(j-1)(j-2)}{8}, \\ &\quad j = 2, 3, 4, ... \end{aligned} \tag{38}$$

$$S_{j,j+1} \equiv 1, \qquad j = 1, 2, 3, ... \tag{39}$$

where all other elements (which do not show up in the above formulae) of the matrices $\mathbf{H}$ and $\mathbf{S}$ vanish.

These definitions allow us to write the following concise equation instead of (36)

$$\mathbf{Hh} = E\mathbf{Sh} \tag{40}$$

(40) is an algebraic weighted eigenvalue equation. In its present form $\mathbf{S}$ can be considered as a weight matrix at the first glance. However, it is a singular matrix and therefore can not be considered as a true weight matrix. Then we can divide this equation by E at its both sides and have E^{-1} as the eigenvalue parameter. This urges us to consider $\mathbf{H}$ as a weight matrix, even though it may have zero eigenvalue depending on the potential structure. Weight matrix must be positive definite by definition. This may not be encountered in the structure of $\mathbf{H}$. If this happens then we do not have a weight matrix at the both sides of the eigenequation, neither at the left nor the right. This is apparently a singularity. In fact the singularity of $\mathbf{S}$ is sufficient to make the problem singular, since the reciprocal of E will have zero value if $\mathbf{H}$ is singular or its nullspaces intersects with the nullspaces of $\mathbf{S}$.

The singular nature of (40) reflects to the relevant recursion as singularity. Hence, the ordinary truncations we have mentioned in the previous section should not be expected to give good quality results. Convergence slow downs must be anticipated. This urges us to develop a little bit different approaches for the solution of the recursion.

We have used the power basis operator in position, therefore the analiticity is preserved in f. However, overflows from the operator space spanned by power set happen because of the appearence of the first and second reciprocal powers of the position operator. Even though the expectation value of the reciprocal square of the position operator does not show up the expectation value of the position operator reciprocal exists in the first two equations of the recursion. This produces element shifts to the right in the coefficient matrix and has the basic responsibility for the singularity.

3 An Experimentative Application: Hydrogen-like Quantum Systems

Let us now consider the hydrogen-like quantum systems which are composed of two electrically charged particles having opposite signs and interacting via a Coulomb potential. In the physically unitless representation we can write the following formula for the potential

$$V(\widehat{q}) \equiv -\widehat{q}^{-1} \tag{41}$$

which takes us from (38) to the following matrix structure

$$\begin{aligned} H_{j,j} &\equiv -\frac{2j-1}{2j}, \quad j = 1, 2, 3, ...; \\ &\qquad\qquad k = 0, 1, 2, ... \\ H_{j,j-1} &\equiv -\frac{(j-1)(j-2)}{8}, \\ &\qquad\qquad j = 2, 3, 4, ... \end{aligned} \tag{42}$$

where only nonzero elements are given. The previously defined matrix $\mathbf{S}$ remains unchanged. The recursion for this system can be explicitly given as follows

$$-\frac{(j-1)(j-2)}{8} h_{j-1} - \frac{2j-1}{2j} h_j - E h_{j+1} = 0, \qquad j = 1, 2, 3, ... \tag{43}$$

This is a second order, linear, homogeneous recursion with varying coefficients. Since its first term grows unboundedly while the others remain constant,

its asymptotic counterpart when j climbs to infinity gives inconsistency. Whereas we prefer to have its asymptotic counterpart as a recursion which is same kind but having constant coefficients. To get this situation we may use the transformation $h_j \equiv (j+1)!\widetilde{h}_j$. This gives

$$-\frac{(j-1)(j-2)}{8(j+2)(j+1)}\widetilde{h}_{j-1} - \frac{2j-1}{2j(j+2)}\widetilde{h}_j - E\widetilde{h}_{j+1} = 0, \qquad j = 1,2,3,... \tag{44}$$

whose asymptotic form for unboundedly increasing j values is as follows

$$\frac{1}{8}\widetilde{h}^{(\infty)}_{j-1} + E\widetilde{h}^{(\infty)}_{j+1} = 0, \qquad j = 1,2,3,... \tag{45}$$

which has the following general solution

$$\widetilde{h}^{(\infty)}_j = c_1\left(\frac{1}{2\sqrt{-2E}}\right)^{j-1} + c_2\left(-\frac{1}{2\sqrt{-2E}}\right)^{j-1}, \quad j = 1,2,3,... \tag{46}$$

where c_1 and c_2 stand for the constants which are momentarily arbitrary.

Now we can rewrite the following approximation for the recursion in (44) by preserving the first n equations as they are while the remaining equations are replaced by their asymptotic counterparts

$$\begin{aligned} &-\frac{(j-1)(j-2)}{8(j+2)(j+1)}\widetilde{h}_{j-1} - \frac{2j-1}{2j(j+2)}\widetilde{h}_j \\ &\qquad -E\widetilde{h}_{j+1} = 0, \qquad j = 1,2,3,...,n \\ &\frac{1}{8}\widetilde{h}_{j-1} + E\widetilde{h}_{j+1} = 0, \\ &\qquad j = n+1, n+2, n+3,... \end{aligned} \tag{47}$$

whose second part inspires us to assume the following form for the solution

$$\begin{aligned} \widetilde{h}_j(c_1,c_2,E) &= c_1\left(\frac{1}{2\sqrt{-2E}}\right)^{j-1} \\ &\quad + c_2\left(-\frac{1}{2\sqrt{-2E}}\right)^{j-1}, \\ &\quad j = 1,2,3,... \end{aligned} \tag{48}$$

Even though this form does not satisfy the first part of (47) we may expect that the satisfaction can be asymptotically achieved as n grows unboundedly. This urges us to minimize the square sum of the lefthand side expressions in (47) with respect to c_1 and c_2. To present an explicit formulation for the minimization we can define

$$\begin{aligned} \mathbf{h}_1(E) &\equiv \left[\widetilde{h}_1(1,0,E)\ \widetilde{h}_2(1,0,E)\ ...\ \right]^T, \\ \mathbf{h}_2(E) &\equiv \left[\widetilde{h}_1(0,1,E)\ \widetilde{h}_2(0,1,E)\ ...\ \right]^T \end{aligned} \tag{49}$$

which gives the following cost functional to minimize the square sum of the left hand side terms of the first part in (47).

$$\mathcal{J}(c_1,c_2;E)\,\mathbf{c}^T\mathbf{G}(\mathbf{E})\mathbf{c} \equiv [\,c_1\ \ c_2\,]\begin{bmatrix} G_{1,1}(E) & G_{1,2}(E) \\ G_{2,1}(E) & G_{2,2}(E) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{50}$$

where

$$G_{i,j}(E) \equiv \left(\sum_{k=1}^{n}\mathbf{H}_k\mathbf{h}_i(E)\right)\left(\sum_{k=1}^{n}\mathbf{H}_k\mathbf{h}_j(E)\right)^T \tag{51}$$

and $\mathbf{H}_k$ stands for the transpose of the kth row of the matrix $\mathbf{H}$.

The cost functional given through (51) is in a quadratic form format. Its minimum with respect to the vector $\mathbf{c}$ corresponds to the least eigenvalue of the $\mathbf{G}$ matrix which is positive definite. This eigenvalue depends on the system energy paramater which is unknown. Hence, it can be minimized with respect to this parameter. However there is no warranty to get a real value for the energy parameter which makes this eigenvalue minimum. Beyond that the expressions are not so manageable as expected from the quite simple structure of the matrix $\mathbf{G}(E)$. Hence the optimization of this cost functional may be avoided.

The abovementioned optimization is unconstrained. We can also launch a constrained optimization by taking the condition $c_1 + c_2 = 1$ which can be produced from the requirement $h_2 = 1$. This can be formulated either by eliminating c_2 in terms of c_1 in the above cost functional and then minimizing the resulting entity with respect to c_1 or we can use the following new cost functional

$$\begin{aligned} \mathcal{J}(c_1,c_2,\lambda;E) &\equiv \mathbf{c}^T\mathbf{G}(\mathbf{E})\mathbf{c} \\ &\equiv [\,c_1\ \ c_2\,]\begin{bmatrix} G_{1,1}(E) & G_{1,2}(E) \\ G_{2,1}(E) & G_{2,2}(E) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &\quad +\lambda(c_1+c_2-1) \end{aligned} \tag{52}$$

where λ stands for the Lagrange parameter. This optimization problem can be analytically and uniquely solved and all unknowns except E can be determined. This takes us to the following equality

$$\begin{aligned} \mathcal{J}(c_1,c_2,\lambda;E) &\equiv \mathbf{c}^T\mathbf{G}(\mathbf{E})\mathbf{c} \\ &= \frac{\det(\mathbf{G}(\mathbf{E}))}{G_{1,1}(E)+G_{2,2}(E)-2G_{1,2}(E)} \end{aligned} \tag{53}$$

which depends on the energy parameter and therefore can be further optimized with respect to E. However,

it is better to deal with the expressions depending on not E but $\alpha \equiv 1/2\sqrt{-2E}$ for brevity.

The rightmost ratio of (53) has a double root corresponding to $\alpha = 1/2$ and therefore $E = -1/2$ for all truncations except the very beginning one term one where this ratio spontaneously vanishes.

This portion of this section is not completely new. We have announced it in a very concise manner [14]. Beyond that all intermediate but conceptually important details of the issue have been given here very first time and belong to the research group of the authors.

4 Concluding Remarks

This work has a two sided goal. First of all to establish a new solution method for parametric recurrence. Today's computer technologies do not almost leave a gap to develeop a novel approach for the solution of non-parametric recurrences because of the concrete structure. However, parametric recurrences contain one or more undetermined parameters and direct computations via computers become quite difficult. They enforces us to develop different efficient algorithms. This is the one side of the goal of this work.

The second side of the goal is to test a recently developed quantum dynamical relation based recurrences and their solutions [1–13]. In this work we take the hydrogen-like systems as foci for testing and show that all expected conceptuality appear to verify the method proposed here.

We now are at such a point that more complicated systems' eigenvalue problems can be treated by the method we have developed here.

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