Validity and Failure of the Autonomy Imposition on the Eigenfunctions in Zero Interval Limit Perturbation Expansion for Hilbert-Schmidt Integral Operators

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Abstract: In this paper, we focus on the autonomy issue in the perturbation expansions of eigenfunctions. We consider the zero interval limit perturbation expansion of a Hilbert-Schmidt Integral Operator here and the autonomy means that the eigenfunction depends on the perturbation parameter not only through the independent variable argument but via an additional argument which just the perturbation parameter. Our purpose is to show that the autonomy puts an important restriction on the kernel of the operator and the resulting perturbation series fails to exist unless a specific and appropriate kernel is used. The proof is also supported by the given illustrative implementations.

Key–Words: Integral operators, eigenfunctions, perturbation expansion, autonomy

1 Introduction

Integral equations and integral operators are one of the fundamental study areas in science and engineering. They can be encountered in chemical and physical applications[1, 2, 3] as well as in multivariate function approximation problems such as Enhanced Multivariance Products Representation (EMPR)[4, 5]. In EMPR, an analytic multivariate (bivariate case is considered in [4]) function under consideration is exerted to be approximated with the help of the univariate support functions[4]. Through the optimization process of these support functions, an eigenvalue problem of an integral operator involving symmetric, thus self-adjoint, and bivariate kernel is constructed. If this eigenvalue problem is solved, the eigenfunction accompanied by eigenvalue with the greatest absolute value can be utilized as the optimized univariate support function. The other support function can be evaluated using the same process by solving another integral operator's eigenvalue problem having, again, a symmetric kernel. By the utilization of these support functions, an efficient approximation to the bivariate function under consideration is obtained.

The content of the present paper is as follows. First, a brief information about integral operators is revived in the second section. Then, the details of the zero integral limit perturbation expansion for the eigenpairs of the considered operator are explained in the third section while the error analysis of this expansion is given in the fourth one. The efficiency of the developed method in order to calculate the eigenfunction which is related to the greatest eigenvalue is shown via numerical implementations in the fifth section. The paper is finalized with the concluding remarks and discussions in the last section. The third section contains the proof of the failure of the autonomy imposition on the eigenfunction whereas the fifth section implementations confirms the proof.

2 Integral Operators and Their Spectral Properties

Consider the following operator

$$\mathcal{I}\, g(x) \equiv \int_a^b d\xi K\,(x,\xi)\, g\,(\xi) \tag{1}$$

where $K(x,\xi)$ is an analytic and bounded bivariate function whose domain is $[\,a,b\,]^2$ and named "the kernel of the operator in (1)". On the other hand g is any analytic univariate function defined on the interval $[\,a,b\,]$. It is obvious that the operator $\mathcal{I}$ in (1) is a linear mathematical object and, in particular, is called "*an integral operator*" whose kernel is $K(x,\xi)$[6, 7, 8].

If an integral operator having its kernel as $K(x,\xi)$ satisfies the the below equation

$$\int_a^b d\xi K\,(x,\xi)\,\psi\,(\xi) = \lambda\psi\,(x)\,; \quad x,\xi \in [\,a,b\,] \tag{2}$$

where ψ is an unknown univariate function while scalar λ is again an unknown, then the problem in (2) is called "the eigenvalue or the spectral problem of the corresponding integral operator"[6, 7, 8]. Thus, the scalar λ is named "the eigenvalue" and ψ is called "the relevant eigenfunction" of the integral operator under consideration. If the kernel of the integral operator is symmetric, that is, $K(x,\xi) = K(\xi,x)$ for all x's and ξ's lying in the domain of the integral operator, then the corresponding integral operator is stated as a *self-adjoint* operator[6]. As an analogy to the classical linear algebra, all eigenvalues of a self-adjoint integral operator are real by implying that the spectrum of the relevant operator is located on the real axis[6]. Moreover, in many methods based on the spectral properties of the linear operators such as Spectral Decomposition[9] and Principal Component Analysis[10], the most dominant and therefore the most important component of the spectrum is the greatest eigenvalue and thus its accompanying eigenfunction (or eigenvector). For this reason, obtaining these entities even using analytical or numerical methods becomes an important issue to this end.

3 Perturbation Scheme at the Zero-Interval-Limit

Consider the following equation

$$\frac{1}{b-a}\int_a^b d\xi K(x,\xi)\,\psi(\xi) = \lambda\psi(x)\,; \quad x,\xi \in [\,a,b\,]\,. \tag{3}$$

By the brief knowledge given in the previous section, it is possible to say that the equation in (3) defines an eigenvalue problem of an integral operator whose kernel is $K(x,\xi)$. Then, λ and $\psi(x)$ can be named "the eigenvalue and its corresponding eigenfunction" of the relevant integral operator respectively from this point of view. In order to develop a general numerical solution method for the eigenproblem in (3), a universal interval should be used instead of the existing one for the problem under consideration. For this purpose, the following affine transformations can be brought forward

$$x \equiv \frac{a+b}{2} + \frac{b-a}{2}\,y, \quad \xi \equiv \frac{a+b}{2} + \frac{b-a}{2}\,\eta. \tag{4}$$

It is obvious that the transformations in (4) take any closed and bounded $[\,a,b\,]$ interval to $[\,-1,1\,]$. By making the following shorthand notations

$$x_{mp} \equiv \frac{a+b}{2}, \quad \varepsilon \equiv \frac{b-a}{2} \tag{5}$$

and applying the transformations in (4) to the integral equation in (3)

$$\frac{1}{2}\int_{-1}^{1} d\eta K\left(x_{mp}+\varepsilon y, x_{mp}+\varepsilon\eta\right)\psi\left(x_{mp}+\varepsilon\eta\right) = \lambda(\varepsilon)\psi\left(x_{mp}+\varepsilon y\right) \tag{6}$$

is obtained. In the above equation, if ε is thought as a small positive value which is close to zero, then it becomes convenient to consider it as a perturbation parameter[11, 12, 13]. Beyond that, x_{mp} is the midpoint of the interval to be worked on and is an important issue which will be discussed a bit later. Thus, by assuming that the unknowns λ and ψ are analytic in the vicinity of x_{mp}, a perturbation equation can be achieved with the help of two relevant power series. On the other hand, by assuming that the bivariate kernel in (6) is analytic in the vicinity of (x_{mp}, x_{mp}), it can be expanded to a series in terms of the nonnegative powers of the small parameter ε as follows

$$K\left(x_{mp}+\varepsilon y, x_{mp}+\varepsilon\eta\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{j} K_{k,j-k}\,y^k\eta^{j-k}\varepsilon^j \tag{7}$$

and the corresponding linear combination coefficients can be calculated as

$$K_{k,j-k} \equiv \frac{1}{k!(j-k)!}\frac{\partial^j K}{\partial x^k\,\partial\xi^{j-k}}\left(x_{mp}, x_{mp}\right) \tag{8}$$

In other respects, the eigenvalue λ and the relevant eigenfunction ψ can be expanded into nonnegative power series of the perturbation parameter ϵ as follows

$$\lambda(\varepsilon) = \sum_{j=0}^{\infty}\lambda_j\varepsilon^j \tag{9}$$

$$\psi\left(x_{mp}+\varepsilon\eta\right) = \sum_{j=0}^{\infty}\psi_j\eta^j\varepsilon^j,$$
$$\psi\left(x_{mp}+\varepsilon y\right) = \sum_{j=0}^{\infty}\psi_j y^j\varepsilon^j, \tag{10}$$

If the infinite series in (7), (9) and (10) are embedded into their places in the transformed equation (6) then we can get

$$\frac{1}{2}\int_{-1}^{1} d\eta\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j} K_{k,j-k}\,y^k\eta^{j-k}\varepsilon^j\right)\left(\sum_{j=0}^{\infty}\psi_j\eta^j\varepsilon^j\right) = \left(\sum_{j=0}^{\infty}\lambda_j\varepsilon^j\right)\left(\sum_{j=0}^{\infty}\psi_j y^j\varepsilon^j\right) \tag{11}$$

from which the following equality can be obtained by using Cauchy product and rearranging the terms

$$\frac{1}{2}\int_{-1}^{1} d\eta \sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{k=0}^{n} K_{k,n-k}y^k\eta^{n-k}\psi_{m-n}\eta^{m-n}\varepsilon^m = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\lambda_n\psi_{m-n}y^{m-n}\varepsilon^m. \quad (12)$$

The above entity can be considered as the perturbation equation of the problem in (3). To solve this equation at zero interval limit by taking ε as the perturbation parameter, one should take the definite integral of the both side along the interval of the integral operator with respect to η first. To this end, the following equation can be obtained

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{k=0}^{n} K_{k,n-k}\psi_{m-n}\frac{\widehat{I}_{m-k}}{2}y^k\varepsilon^m = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\lambda_n\psi_{m-n}y^{m-n}\varepsilon^m \quad (13)$$

where

$$\widehat{I}_n \equiv \frac{1+(-1)^n}{n+1}, \quad n=0,1,... \quad (14)$$

Thus, the equation in (13) can be identified as the final form of the relevant perturbation recursion amongst the eigenpair components for the eigenvalue problem in (3). By employing this equation, unknown ψ_j and λ_j $(j = 0, 1, ...)$ coefficients are expected to be determined uniquely in order to obtain a unique approximation to the eigenfunction $\psi(x)$ corresponding to the greatest eigenvalue which is also expected to be determined uniquely. Due to this aim, if ε is taken to zero limit in both sides of the equation in (13) then

$$\lambda_0 = K_{0,0} \quad (15)$$

is obtained. This result denotes that the zeroth coefficient of the expansion for the eigenvalue λ is equal to the value of the relevant kernel at the point (x_{mp}, x_{mp}).

It is important to remember that the equation in (13) holds for any ε. Thus, it can be assessable as an identity with respect to ε, instead of an equation. Due to this reason, any number of consecutive differentiation of this equation with respect to ε does not annihilate its validity. Hence, the r-times (r is a nonnegative integer) consecutive differentiation with respect to ε and then division to $r!$ for both sides of the equation in (13) gives the following equation

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{k=0}^{n} K_{k,n-k}\psi_{m-n}\frac{\widehat{I}_{m-k}}{2}y^k\left(\varepsilon^m\right)^{(r)} = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\lambda_n\psi_{m-n}y^{m-n}\left(\varepsilon^m\right)^{(r)}. \quad (16)$$

The following equality can be easily written

$$\left\{\left(\varepsilon^m\right)^{(r)}\right\}_{\varepsilon\to 0} = \delta_{m,r} \quad (17)$$

Now if ε is taken to zero limit in (16) and (17) is used to get the limit then the following result is achieved

$$\sum_{n=0}^{r}\sum_{k=0}^{n} K_{k,n-k}\psi_{r-n}\frac{\widehat{I}_{r-k}}{2}y^k = \sum_{n=0}^{r}\lambda_n\psi_{r-n}y^{r-n} \quad (18)$$

which can be simplified via summation index transformation. We give just the result by skipping the intermediate steps

$$\lambda_k\psi_{r-k} = \sum_{n=r-k}^{r} K_{r-k,n-r+k}\frac{\widehat{I}_k}{2}\psi_{r-n}, \quad r=0,1,..., \quad k=0,1,...,r \quad (19)$$

This equation implies $\lambda_0 = K_{0,0}$ for $r = k = 0$ as long as ψ_0 does not vanish. The vanishing ψ_0 can be considered as another option by leaving λ_0 arbitrary. However, it corresponds to an inconsistency removing the normalizability of the eigenfunction, and hence, should be discarded. The case where $r = 1$ produces the following results as long as ψ_0 does not vanish

$$\lambda_1 = 0, \qquad \psi_1 = \frac{K_{1,0}}{K_{0,0}}\psi_0 \quad (20)$$

while the case where $r = 2$ takes us to the following results

$$\lambda_2 = \frac{K_{2,0}}{3} + \frac{K_{0,2}}{3} + \frac{K_{1,0}K_{0,1}}{3K_{0,0}}, \quad \psi_2 = \frac{K_{2,0}}{K_{0,0}}\psi_0 \quad (21)$$

Thus we do not encounter any other inconsistency in these cases. On the other hand, the case where $r = 3$ produces the following results through its forms for $k = 3$ and $k = 0$ respectively as long as ψ_0 does not vanish

$$\lambda_3 = 0, \qquad \psi_3 = \frac{K_{3,0}}{K_{0,0}} \quad (22)$$

whereas the case where $r = 3$ and $k = 1$ gives an equation which is spontaneously satisfied because of

the vanishing value of λ_1. The remaining subcase for $r = 3$ corresponds to $k = 2$ and produces the following requirement on the kernel coefficients

$$\left(K_{0,2} + \frac{K_{1,0}K_{0,1}}{K_{0,0}} - K_{1,1}\right)\frac{K_{1,0}}{K_{0,0}} - K_{1,2} = 0 \quad (23)$$

which does not determine anything unknown but puts a limitation on the kernel coefficients. This requirement is apparently an inconsistency. As we proceed by ascending the values of r we encounter more and more requirements on the kernel coefficients and therefore more and more inconsistencies. These constraints bring a denumerable infinite number of structural limitations on the kernel and makes it possible to use this perturbation expansion only for very specific kernels. We do not intend to get the family of kernels which permit the zero integral limit perturbation expansion in this conference proceedings.

Even though infinite number of constraints are needed to be satisfied we can prove that the following results hold for this approach

$$\psi_r = \frac{K_{r,0}}{K_{0,0}}\,\psi_0, \qquad r = 1, 2, ... \qquad (24)$$

by assuming that $K_{0,0}$, which is the value of the corresponding kernel at the point (x_{mp}, x_{mp}), does not vanish. By embedding the general structure for ψ_r obtained above into the expansion of the eigenfunction in (10) and utilizing the inverse of the first affine transformation in (4)

$$\psi(x) = \psi_0 \sum_{r=0}^{\infty} \frac{K_{r,0}}{K_{0,0}}\,(x - x_{mp})^r \qquad (25)$$

is generated as the structure of the corresponding eigenfunction to be dealt with. In other words, by using a perturbation method, the univariate eigenfunction $\psi(x)$ is written as an infinite series. Beside all these, the zeroth coefficient, that is ψ_0, is another important issue about the structure in (25) and can be perceived as an *arbitrary constant* at this moment. But, as it can be remembered, no assumption has been made about the normalization condition that nothing is imposed on the eigenfunction to be calculated as normalization during the perturbation procedure because the nonexistence of any requirement to this end. There is only one common flexibility ψ_0 to be determined in this fashion. This freedom brings a common arbitrariness on the eigenfunction coefficients. This undesired status in fact is originated from the autonomy imposition on the eigenfunction and can be globally removed only by normalizing the corresponding eigenfunction. Thus, ψ_0 is computed as the normalization factor after truncating the series (25) at a certain level. Hereby, the truncated and normalized series can be used as approximations for the relevant eigenfunction $\psi(x)$.

Before closing the section we need to emphasize on the limitations or constraints told above. This perturbation expansion can be valid only for a very specific family of kernels for which all constraints are satisfied.

4 Error Analysis

Error estimation is one of the important issues in approximation problems. Hence the eigenfunction under consideration is determined by utilizing an approximation procedure through a perturbation expansion in this work, this section is devoted to the concern on the error bound of the relevant method. Concordantly, one can easily verify that the series which will be utilized for approximation in (25) is a power series with the coefficients involving the derivatives of the relevant kernel. Although the kernel under consideration is bivariate and has differentiation possibilities with respect to both of its independent variables, the related coefficients in (24) include only the derivatives with respect to its first variable, x (but of course this is valid under the satisfied constraint mentioned above). This feature reveals the similarity between the approximation series in (25) and the well known Taylor series. To this end, it is reasonable to make an error estimation for the developed method by utilizing the error bound theorem for the Taylor series.

As it is obvious from the univariate Taylor theorem, assuming that $g(x)$ has the derivatives up to $(n+1)$th order, which are continous on a closed interval and the last of them, that is, the $(n+1)$th one, is bounded from above such as

$$\left|g^{(n+1)}\right| \leq G \qquad (26)$$

where G is an n-independent positive constant, then the remainder function $R(x)$ corresponding to $(n+1)$th degree Taylor polynomial for the function $g(x)$ can be bounded from above[14] as

$$|\,R(x)\,| \leq \frac{G\,(x-a)^{n+1}}{(n+1)!}, \quad n = 0, 1, 2, ... \qquad (27)$$

In the light of the brief information given above, we can assume that the bounds for the bivariate function $K(x, \xi)$ and its $(n+1)$th order derivative with respect to x can be written as follows

$$m \leq |\,K\,| \leq M, \qquad \left|\frac{\partial^{n+1}K}{\partial x^{n+1}}\right| \leq M_{n+1}, \quad n = 0, 1, 2, ... \qquad (28)$$

where m, M are the lower and upper bound of the kernel K respectively and M_{n+1} is the upper bound for the $(n+1)$th derivative of K with respect to its first independent variable x.

On the other hand, the zeroth coefficient of the expansion for the kernel in (7) is denoted as $K_{0,0}$ which does not vanish by assumption. By using the lower bound for K in (28)

$$|K_{0,0}| \geq m \tag{29}$$

is arrived at as the lower bound for $K_{0,0}$. By adapting the series in (25) and rewrite the kernel coefficients in accordance with the definition in (8) explicitly,

$$\psi(x) = \sum_{r=0}^{\infty} \frac{\psi_0}{K_{0,0}} \frac{1}{r!} \frac{\partial^r K}{\partial x^r}(x_{mp}, x_{mp}) (x - x_{mp})^r \tag{30}$$

is acquired where $x_{mp} = (a+b)/2$. If the remainder term for the relevant eigenfunction $\psi(x)$ is symbolized by $R(x)$ then the upper and the lower bounds in (28) and (29) respectively are utilized and combined with the remainder bound in (27)

$$\begin{aligned} |R(x)| &\leq \frac{\psi_0 M}{|K_{0,0}|(n+1)!} |x - x_{mp}|^{n+1} \\ &\leq \frac{\psi_0 M}{m(n+1)!} |x - x_{mp}|^{n+1} \end{aligned} \tag{31}$$

is obtained. If it is recalled that $x - x_{mp} = \varepsilon y$ and $|y| \leq 1$ from (4)

$$|R(x)| \leq \frac{\psi_0 M}{m(n+1)!} \varepsilon^{n+1} \tag{32}$$

can be taken as the remainder bound for the approximation of the the relevant eigenfunction $\psi(x)$ where ψ_0 is a positive constant which normalizes the corresponding eigenfunction.

5 Numerical Implementations

In this section, four numerical implementations are given in order to show the efficiency of the method described in the third section. To this end, the exact solution of the eigenvalue problem in (25) and its perturbation based numerical solutions at different truncation levels for various kernels will be compared through the plots. The kernels which are utilized in the implementations in this section are as follows

$$\begin{aligned} K_1(x,\xi) &= \exp(2x) + \exp(2\xi) \\ K_2(x,\xi) &= \exp(x+\xi) + \exp(-x-\xi) \\ K_3(x,\xi) &= \cos(\pi(x+\xi)) \\ K_4(x,\xi) &= \tan x \sin 3\xi + \sin 3x \tan \xi \end{aligned}$$

where all of them are chosen as Pincherle-Goursat type of dimension two. The reason why to choose this type of kernels is to obtain their eigenvalue problem solutions easily hence the eigenvalue problems of the integral operators having Pincherle-Goursat type kernels can be reduced into a matrix algebraic eigenvalue problems without any appearable difficulties.

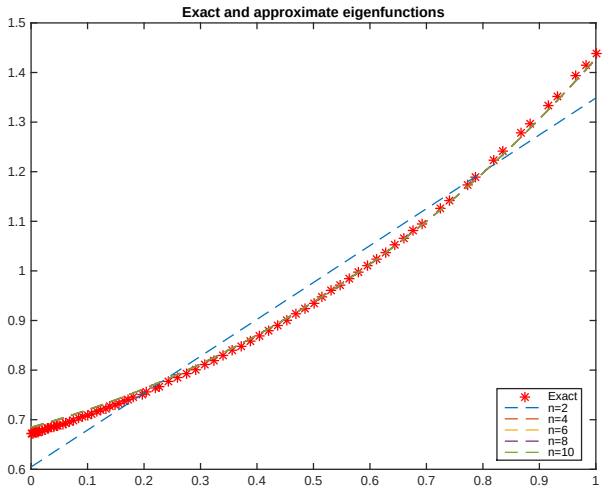

Figure 1: Exact and approximate eigenfunctions for the kernel $K(x,\xi) = \exp(2x) + \exp(2\xi)$ over $[0,1]^2$ at different truncation orders

On the other hand, each graphic is plotted in the interval $[0,1]$ since the corresponding eigenvalue problems are designed as having just the integration domain $[0,1]$. Beside this, it becomes useful to state that all the eigenfunctions plotted are the ones which are accompanied by the eigenvalue whose absolute value is the greatest and all are normalized over the interval $[0,1]$.

In each figure, the curves which are constructed using red asterisks implies the exact solution of the eigenvalue problem of the integral operator under consideration while the dashed colorful curves dictates the truncated approximations to the corresponding exact eigenfunction. The truncation orders start from $n = 2$ and go to $n = 10$ with an increment of 2 except Figure 4 which includes the truncated approximation for $n = 12$ and $n = 14$. In Figure 1 and Figure 3, one can easily verify that increment in n values, that is the truncation order in (25), seems to cause better approximation for the eigenfunction under consideration since the difference between the dashed curves and red asterisk curve in each figure starts to get smaller, even though the restriction equation in (23) is not satisfied, but the right hand side of the equation becomes relatively small and close to 0. On the other hand, in In Figure 2 and Figure 4, the approximation curves diverge from the related exact eigenfunction curves

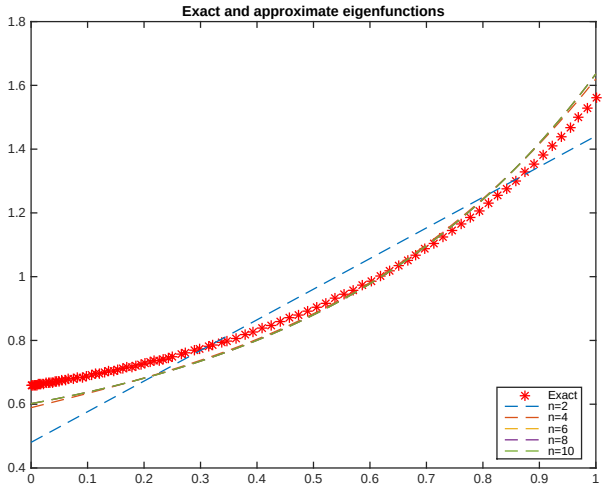

Figure 2: Exact and approximate eigenfunctions for the kernel $K(x,\xi) = \exp(x+\xi) + \exp(-x-\xi)$ over $[0,1]^2$ at different truncation orders

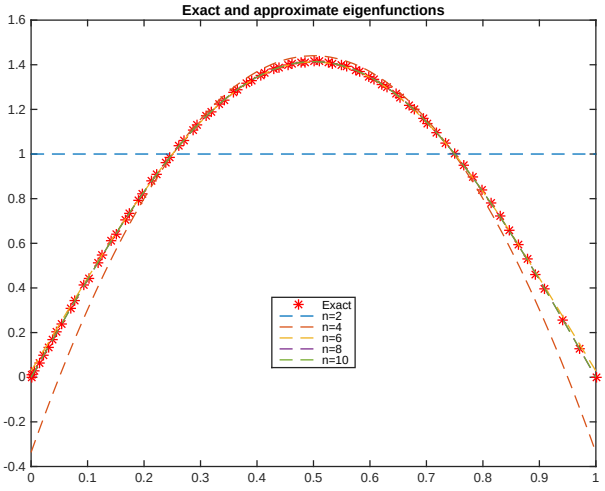

Figure 3: Exact and approximate eigenfunctions for the kernel $K(x,\xi) = \cos(\pi(x+\xi))$ over $[0,1]^2$ at different truncation orders

hence the restriction in (23) is not be able to satisfied. However, the chosen function need to satisfy the constraint equations to get an exact match.

6 Conclusion

In this work, a perturbation expansion based approximation method is proposed in order to approximate to the greatest eigenvalues' eigenfunction of an integral operator having symmetric kernel. Throughout the development of this method, the halfinterval length of the domain of the integral operator is considered as the perturbation parameter and all investigations are realized by assuming that this parameter is small enough, which means it approaches to zero. Thus, this process can be interpreted as a perturbation method at zero-interval-limit.

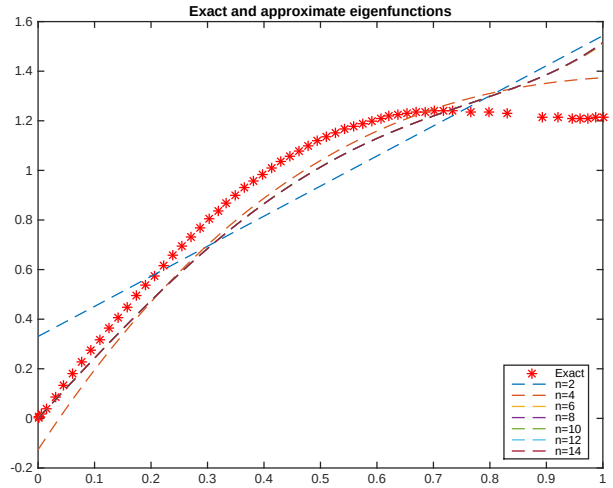

Figure 4: Exact and approximate eigenfunctions for the kernel $K(x,\xi) = \tan x \sin 3\xi + \sin 3x \tan \xi$ over $[0,1]^2$ at different truncation orders

With the help of the numerical implementations, the efficiency of the method is confirmed under the constraint mentioned in the text. Also the effects of the values of the lower and upper bounds, ψ_0 coefficient and truncation order to the approximation quality is observed by the error analysis.

As a final remark, if the restrictions amongst the kernel coefficients depicted in the third section are not satisfied, the generated perturbation series in (25) may not converge to the exact solution for the relevant eigenfunction. Thus, it becomes wiser to work with the kernels satisfying these restrictions, or to deal with the intervals which enables the kernel coefficients satisfy the mentioned relations. Even if the restriction equations are not exactly satisfied, their closeness to zero may result effective approximations.

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