

Mathematical Modelling for Electromagnetic Fields Equally Polarized in Four Directions

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Abstract: - There are neither longitudinal nor transverse (globally) waves, because the wave front can change its direction and what was longitudinal will become transverse after turning. Therefore, we pass from a globally transverse EM field to fields equally polarized in four directions: longitudinal, two transverse and "temporal". What is significantly new in the model: there is no scalar potential; instead of a light cone, there is an ellipsoid, which at $v \geq c$ turns into a two-cavity hyperboloid (as with the trajectories of celestial bodies in classical mechanics: closed elliptical orbits at the second cosmic velocity ~ 11.2 km/s turned into open hyperbolic ones).

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1 Statement of the Problem

1.1 Introduction to the problem

Feynman in his Nobel lecture gives interesting considerations regarding the four polarization states of light [1]. It has recently been shown [2] that a metamaterial supports longitudinal waves over an extremely wide frequency range from very low frequencies to the Bragg resonances of the structure, with waves of unprecedentedly short lengths, comparable to the period of the material. The observed effects emphasize the spatial-dispersive response of a twisted wire medium and provide a way to generate electromagnetic fields with strong spatial variations.

From Maxwell's equations [3] in their modern form, proposed by O. Heaviside and G. Hertz [4], this should not have happened. Maxwell himself, who predicted the possibility of electromagnetic waves, in theoretical discussions denied the possibility of a unidirectional vector field generated by the pulsation of the "density" of electrostatic fields, similar to the pressure pulsations in sound waves. Such pulsations would imply the presence of a variable concentration of electrostatic field lines propagating along the electrostatic field lines. Therefore, Maxwell's original equations did not contain solutions corresponding to these waves, and subsequent efforts

of experimenters were directed mainly to the detection of transverse electromagnetic waves, i.e. waves in which the directions of the electric and magnetic fields are perpendicular to the direction of their propagation. G. Hertz first reported the experimental detection of such waves in 1887 [5].

However, as N. Tesla showed two years later, the effects observed by Hertz could also be caused by longitudinal waves [6]. Such waves probably consisted of a sequence of unidirectional shock waves caused by the rupture of the electrostatic field, and were able to affect charges in the direction of their propagation. Some of them have been known for a long time, for example, Langmuir waves generated by collective oscillatory processes of volume charge in plasma.

1.2 The goal of the paper

Researchers have discovered other types of longitudinal electromagnetic waves in waveguides, resonators, piezoelectrics, semiconductors, liquid crystals, single-wire power transmission lines, as well as a special class of receiving and transmitting antennas (the so-called EN - antennas that emit longitudinal waves and provide communication through the water column and rocks [7], etc.).

LW generators, transverse wave converters into longitudinal waves, detectors, mixers and power meters have been created [8]. Such waves are

recorded by Schottky diodes, photo materials protected by a light-tight foil screen, liquid crystal indicators, phase-contrast microscopy of high-purity water, etc.

Along with the existence of longitudinal magnetic waves, experiments often also reveal the appearance of forces acting along a conductor with current. The presence of such forces acting on electrons moving along the axis of the toroid and the copper conductor was discovered in the experiments of A. Solunin and confirmed in the experiments of S. Grano and G. Nikolaev [9].

Taking into account all of the above, we move from a globally transverse electromagnetic field to fields equally polarized in four directions: longitudinal, two transverse, and “temporal” in contrast to our previous papers [11,12], where only transverse polarization was considered.

2 Basic Equations of the Electromagnetic Field

2.1 Two different values of the “speed of light”

Since the wave front can change, the division of waves into transversely polarized and longitudinally polarized can only be performed locally. In this regard, let us consider – starting from equations [10] – the following original basic equations

$$\begin{aligned} \frac{d}{dt}B_0 + c_2 \nabla \cdot \mathbf{B} - \mathbf{v} \cdot [\nabla \times \mathbf{E}] &= 0, \\ \frac{d}{dt}\mathbf{B} + [(\mathbf{v} \times \nabla) \times \mathbf{B}] + c_2 \nabla B_0 + c_1 [\nabla \times \mathbf{E}] - [\mathbf{v} \times \nabla]E_0 &= 0, \\ \frac{d}{dt}E_0 + c_2 \nabla \cdot \mathbf{E} + \mathbf{v} \cdot [\nabla \times \mathbf{B}] &= 0, \\ \frac{d}{dt}\mathbf{E} + [(\mathbf{v} \times \nabla) \times \mathbf{E}] + c_2 \nabla E_0 - c_1 [\nabla \times \mathbf{B}] + [\mathbf{v} \times \nabla]B_0 &= -\mathbf{J}, \end{aligned} \quad (1)$$

where B_0 , E_0 are the “scalar” components of the electromagnetic field, $\mathbf{v}$ is the velocity vector for the “source” of the field. Here the time derivatives are complete, so $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$.

The array of partial differential equations (1) contains two different values of the “speed of light”: c_1 and c_2 for the “longitudinal” and “transverse” waves, respectively. Next, we turn to the Fourier amplitudes of the fields according to the formula

$$A_j(t, \mathbf{x}) = \int_{-\infty}^{+\infty} \frac{d\omega d\mathbf{k}}{(2\pi)^4} e^{i(-\omega t + \mathbf{k} \cdot \mathbf{x})} a_j(\omega, \mathbf{k}), \quad (2)$$

we obtain the following array of linear algebraic equations

$$sb_0 + c_2 \mathbf{k} \cdot \mathbf{b} - \mathbf{v} \cdot [\mathbf{k} \times \mathbf{e}] = 0,$$

$$\begin{aligned} s\mathbf{b} + [(\mathbf{v} \times \mathbf{k}) \times \mathbf{b}] + c_2 \mathbf{k} b_0 + c_1 [\mathbf{k} \times \mathbf{e}] - \\ - [\mathbf{v} \times \mathbf{k}]e_0 &= 0, \\ se_0 + c_2 \mathbf{k} \cdot \mathbf{e} + \mathbf{v} \cdot [\mathbf{k} \times \mathbf{b}] &= 0, \\ s\mathbf{e} + [(\mathbf{v} \times \mathbf{k}) \times \mathbf{e}] + c_2 \mathbf{k} e_0 - c_1 [\mathbf{k} \times \mathbf{b}] + \\ + [\mathbf{v} \times \mathbf{k}]b_0 &= i\mathbf{j}. \end{aligned} \quad (3)$$

The determinant of this system is equal to

$$\begin{aligned} \det = ((s - c_1 k)(s + c_2 k) + [\mathbf{v} \times \mathbf{k}]^2) \cdot \\ \cdot ((s + c_1 k)(s - c_2 k) + [\mathbf{v} \times \mathbf{k}]^2)^2 \end{aligned} \quad (4)$$

where $s \equiv -\omega + \mathbf{v} \cdot \mathbf{k}$.

2.2 Two waves propagating at different speeds

It is obvious that the determinant of the system is equal to the square of the value, thus, system (3) is a “dual set” of equations describing the physics of the same wave process. In the case of a stationary source $\mathbf{v} = 0$, the wave front splits into two waves propagating with different velocities

$$\det = (s^2 - c_1^2 k^2)^2 (s^2 - c_2^2 k^2)^2.$$

The system of eight linear equations (3) can be broken down into two independent systems of four equations each using linear substitution of variables:

$$s\xi_0 + \mathbf{p}_2 \cdot \boldsymbol{\xi} = 0, \quad s\boldsymbol{\xi} + \mathbf{p}_2 \xi_0 + i\mathbf{p}_1 \times \boldsymbol{\xi} = i\mathbf{j}; \quad (5)$$

$$s\zeta_0 + \mathbf{q}_2 \cdot \boldsymbol{\zeta} = 0, \quad s\boldsymbol{\zeta} + \mathbf{q}_2 \zeta_0 - i\mathbf{q}_1 \times \boldsymbol{\zeta} = i\mathbf{j} \quad (6)$$

where $\xi_l \equiv E_l + iB_l$, $\zeta_l \equiv E_l - iB_l$, $l = 0, 1, 2, 3$

$$\mathbf{p}_1 \equiv c_1 \mathbf{k} - i[\mathbf{v} \times \mathbf{k}], \quad \mathbf{p}_2 \equiv c_2 \mathbf{k} - i[\mathbf{v} \times \mathbf{k}]; \quad (7)$$

$$\mathbf{q}_1 \equiv c_1 \mathbf{k} + i[\mathbf{v} \times \mathbf{k}], \quad \mathbf{q}_2 \equiv c_2 \mathbf{k} + i[\mathbf{v} \times \mathbf{k}].$$

The determinant of each of the systems (5), (6) is equal to

$$\begin{aligned} \det = ((s - c_1 k)(s + c_2 k) + [\mathbf{v} \times \mathbf{k}]^2) \cdot \\ \cdot ((s + c_1 k)(s - c_2 k) + [\mathbf{v} \times \mathbf{k}]^2). \end{aligned} \quad (8)$$

The last expression can be written as

$$\begin{aligned} \det = (s^2 - (\sqrt{c_1 c_2} k)^2 + [\mathbf{v} \times \mathbf{k}]^2)^2 - \\ - (c_2 - c_1)^2 s^2 k^2. \end{aligned}$$

2.3 The case of an isotropic system

Let us further consider the case of an isotropic medium in which $c_1 = c_2 = c$. In this case, the determinant (8) is the square of the number

$$\det = (s^2 - c^2 k^2 + [\mathbf{v} \times \mathbf{k}]^2)^2$$

or

$$\det = (s^2 - c^2 \kappa^2)^2, \quad (9)$$

where $\kappa \equiv \mathbf{k} + \frac{i}{c} \mathbf{v} \times \mathbf{k}$

are the «new» momentums.

Using a linear combination of equations, each of the systems (5), (6) can be transformed into a pair of independent systems with two unknowns. For example, the system of four coupled equations (6)

$$\begin{aligned} s\zeta_0 + c\kappa_1\zeta_1 + c\kappa_2\zeta_2 + c\kappa_3\zeta_3 &= 0, \\ c\kappa_1\zeta_0 + s\zeta_1 + i c\kappa_3\zeta_2 - i c\kappa_2\zeta_3 &= i j_1, \\ c\kappa_2\zeta_0 - i c\kappa_3\zeta_1 + s\zeta_2 + i c\kappa_1\zeta_3 &= i j_2, \\ c\kappa_3\zeta_0 + i c\kappa_2\zeta_1 - i c\kappa_1\zeta_2 + s\zeta_3 &= i j_3, \end{aligned} \quad (10)$$

with

$$\det = (s^2 - c^2 \kappa^2)^2, \quad (11)$$

from which we can write the following two independent systems:

$$(s - c\kappa_3)(\zeta_0 - \zeta_3) + c(\kappa_1 - i\kappa_2)(\zeta_1 + i\zeta_2) = -i j_3,$$

$$c(\kappa_1 + i\kappa_2)(\zeta_0 - \zeta_3) + (s + c\kappa_3)(\zeta_1 + i\zeta_2) = i(j_1 + i j_2)$$

and

$$\begin{aligned} (s + c\kappa_3)(\zeta_0 + \zeta_3) + c(\kappa_1 + i\kappa_2)(\zeta_1 - i\zeta_2) &= i j_3, \\ c(\kappa_1 - i\kappa_2)(\zeta_0 + \zeta_3) + (s - c\kappa_3)(\zeta_1 - i\zeta_2) &= i(j_1 - i j_2) \end{aligned}$$

$$\text{with } \det = s^2 - c^2 \kappa^2 \quad (12)$$

for each of the systems.

3 Solution of the Equations

3.1 Elongation of Derivatives and Coordinate Transformation

Let us calculate, based on formula (9) – the transformation of coordinate gradients – the formula for the transformation of the coordinates themselves.

Multiplying equation (9) by $\mathbf{v}$ scalarly, we obtain that $\mathbf{v} \cdot \kappa = \mathbf{v} \cdot \mathbf{k}$. Multiplying (9) on the left by $\mathbf{v}$ vectorly, we obtain

$$\mathbf{v} \times \kappa = \mathbf{v} \times \mathbf{k} + \frac{i}{c} \mathbf{v} \times [\mathbf{v} \times \mathbf{k}],$$

or

$$\frac{i}{c} \mathbf{v} \times \kappa + \frac{1}{c^2} \mathbf{v}(\mathbf{v} \cdot \kappa) = \frac{i}{c} \mathbf{v} \times \mathbf{k} + \frac{1}{c^2} v^2 \mathbf{k},$$

where

$$\frac{i}{c} \mathbf{v} \times \mathbf{k} = \kappa - \mathbf{k} \text{ according to (9), so that}$$

$$\left(1 - \frac{v^2}{c^2}\right) \mathbf{k} = \kappa - \frac{1}{c^2} \mathbf{v}(\mathbf{v} \cdot \kappa) - \frac{i}{c} \mathbf{v} \times \kappa. \quad (13)$$

Finally, the expression of the "old" impulses in terms of the "new" ones is as follows:

$$\mathbf{k} = \frac{\kappa - \frac{1}{c^2} \mathbf{v}(\mathbf{v} \cdot \kappa) - \frac{i}{c} \mathbf{v} \times \kappa}{\left(1 - \frac{v^2}{c^2}\right)}, \quad (14)$$

and the "old" coordinates are expressed in terms of the "new" coordinates by the formula

$$\mathbf{x} = \mathbf{y} + \frac{i}{c} \mathbf{v} \times \mathbf{y}. \quad (15)$$

Solution of the algebraic equation array (3) under condition $c_1 = c_2 = c$ is as follows:

$$b_0 = -i \left[\frac{\mathbf{v}}{c} \times \mathbf{k} \right] \cdot \mathbf{a}, \quad \mathbf{b} = i[\mathbf{k} \times \mathbf{a}], \quad (16)$$

$$e_0 = i \mathbf{k} \cdot \mathbf{a}, \quad \mathbf{e} = -i \frac{\mathbf{v}}{c} \cdot \mathbf{a} + i \left[\frac{\mathbf{v}}{c} \times \mathbf{k} \right] \times \mathbf{a},$$

where the Fourier amplitude of the vector potential is equal to

$$\mathbf{a} = -\frac{j}{s^2 - c^2 k^2 + [\mathbf{v} \times \mathbf{k}]^2}. \quad (17)$$

3.2 Solution of the Problem in Coordinates Time-Space

To write the solution in coordinates $t, \mathbf{x}$, it is necessary to perform integration for the vector potential according to formula (2), then the solution of system (1) can be written in the form

$$B_0 = -\left[\frac{\mathbf{v}}{c} \times \nabla \right] \cdot \mathbf{A}, \quad \mathbf{B} = [\nabla \times \mathbf{A}], \quad (18)$$

$$E_0 = \nabla \cdot \mathbf{A}, \quad \mathbf{E} = -\frac{1}{c} \frac{d}{dt} \mathbf{A} + \left[\frac{\mathbf{v}}{c} \times \nabla \right] \times \mathbf{A}, \quad (19)$$

where

$$A(t, \mathbf{x}) = -\int_{-\infty}^{+\infty} \frac{d\omega d^3 \mathbf{k}}{(2\pi)^4} e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})} \frac{J(\omega, \mathbf{k})}{(\omega - \mathbf{v} \cdot \mathbf{k})^2 - c^2 k^2 + [\mathbf{v} \times \mathbf{k}]^2},$$

because of

$$J(\omega, \mathbf{k}) = \int_{-\infty}^{+\infty} d\tau d^3 \mathbf{x} e^{i(\omega \tau - \mathbf{k} \cdot \mathbf{y})} J(\tau, \mathbf{y}).$$

Then we are calculating the Green's function

$$G(t - \tau, \mathbf{x} - \mathbf{y}) = -\int_{-\infty}^{+\infty} \frac{d\omega d^3 \mathbf{k}}{(2\pi)^4} \frac{e^{-i(\omega(t-\tau) - \mathbf{k} \cdot (\mathbf{x}-\mathbf{y}))}}{(\omega - \mathbf{v} \cdot \mathbf{k})^2 - c^2 k^2 + [\mathbf{v} \times \mathbf{k}]^2}. \quad (20)$$

According to formula (13), there are two fundamentally different cases:

1. $v^2 \neq c^2$ the source moves at a speed different from the speed of light;
2. $v^2 = c^2$ the source moves at the speed of light.

To clearly imagine the picture of wave front propagation according to the characteristic equation (12), let us introduce the phase velocity of the wave front in the cylindrical coordinate system $\omega = \theta k$, then $(\partial k - v k \cos(\theta))^2 - c^2 k^2 + v^2 k^2 \sin(\theta)^2 = 0$. (21)

Dividing equation (21) by $k \neq 0$ and $c \neq 0$, we obtain the equation of a circle of unit radius, the centre of which is shifted from the origin, following the source:

$$\left(\frac{\vartheta}{c} - \frac{v}{c} \cos(\theta)\right)^2 + \frac{v^2}{c^2} \sin^2(\theta) = 1. \quad (22)$$

It is obvious that the wave front has no features at $\frac{v^2}{c^2} = 1$, so no "light barrier" is observed in this model (see Fig. 1).

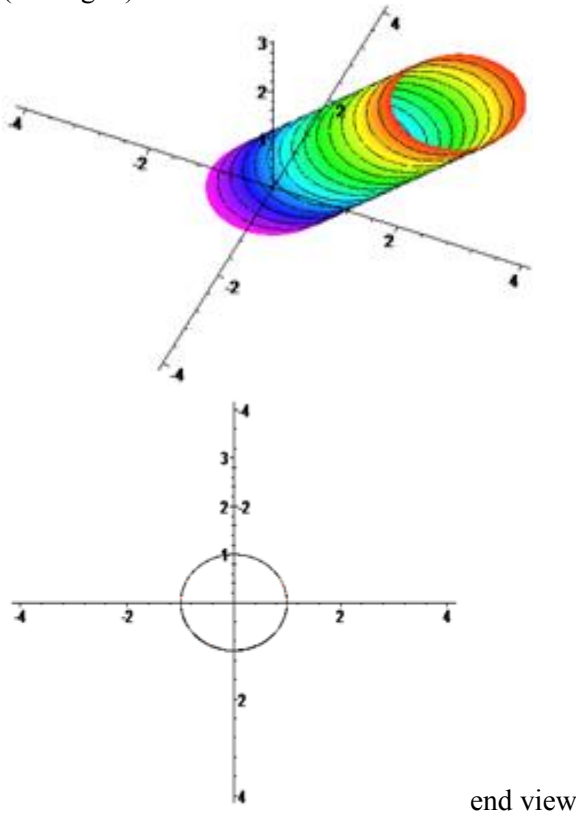


Fig. 1 Wave front diagram

3.3 Calculation of the Green's Function for $v^2 \neq c^2$

According to the formula (9):

$$s^2 - c^2 k^2 + [\mathbf{v} \times \mathbf{k}]^2 = s^2 - c^2 \kappa^2, \\ \frac{1}{s^2 - c^2 k^2 + [\mathbf{v} \times \mathbf{k}]^2} = \frac{1}{s^2 - c^2 \kappa^2} = \frac{1}{2c\kappa} \left(\frac{1}{s - c\kappa} - \frac{1}{s + c\kappa} \right) = \\ \frac{1}{2c\kappa} \left(\frac{1}{\omega - \mathbf{v} \cdot \mathbf{k} - c\kappa} - \frac{1}{\omega - \mathbf{v} \cdot \mathbf{k} + c\kappa} \right).$$

The integral over ω in formula (20) can be calculated using the residue theorem:

$$\int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{s^2 - c^2 \kappa^2} d\omega = \frac{1}{2c\kappa} \left(\int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{\omega - \mathbf{v} \cdot \mathbf{k} - c\kappa} d\omega - \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{\omega - \mathbf{v} \cdot \mathbf{k} + c\kappa} d\omega \right) = -2\pi e^{-i\mathbf{v} \cdot \mathbf{k} t} \frac{\sin(c\kappa t)}{c\kappa}.$$

Thus, the following integral must be calculated

$$\int_{-\infty}^{+\infty} \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}) - v(t - \tau)} \frac{\sin(c\kappa(t - \tau))}{c\kappa}$$

in the space of «new» impulses κ .

According to (14), we have a linear dependence (the notation $\mathbf{u} \equiv \frac{\mathbf{v}}{c}$ is introduced)

$$k_l = \frac{\kappa_l - u_l(u_m \kappa_m) - i\epsilon_{lmn} u_m \kappa_n}{(1 - u^2)},$$

or in matrix form

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \frac{1}{(1 - u^2)} \cdot \begin{pmatrix} 1 - u_1^2 & -u_1 u_2 + i u_3 & -u_1 u_3 - i u_2 \\ -u_2 u_1 - i u_3 & 1 - u_2^2 & -u_2 u_3 + i u_1 \\ -u_3 u_1 + i u_2 & -u_3 u_2 - i u_1 & 1 - u_3^2 \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix}$$

with determinant $\det = (1 - u^2)$, so that

$$d^3 \mathbf{k} = (1 - u^2) d^3 \kappa$$

and

$$\int_{-\infty}^{+\infty} \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y}) - v(t - \tau)} \frac{\sin(c\kappa(t - \tau))}{c\kappa} = (1 - u^2) \int \frac{d^3 \kappa}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{L}} \frac{\sin(c\kappa(t - \tau))}{c\kappa}, \quad (23)$$

where

$$\mathbf{L} \equiv \boldsymbol{\delta} - \mathbf{u}(\mathbf{u} \cdot \boldsymbol{\delta}) + i[\mathbf{u} \times \boldsymbol{\delta}], \quad (24)$$

$$\boldsymbol{\delta} \equiv \mathbf{x} - \mathbf{y} - \mathbf{u}c(t - \tau).$$

Recalling the expression for the Pauli function:

$$D(z) = \int \frac{d^3 \kappa}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{L}} \frac{\sin(c\kappa(t - \tau))}{\kappa}, \quad (25)$$

we obtain the final expression of the Green's function for $u^2 \neq 1$:

$$G(t - \tau, \mathbf{x} - \mathbf{y}) = (1 - u^2)/c D(z).$$

The retarded and advanced Green's functions are equal, respectively:

$$D_{ret}(z) = \frac{1}{4\pi} \frac{1}{|L|} \delta(|L| - c(t - \tau)), \quad (26)$$

$$D_{adv}(z) = \frac{1}{4\pi} \frac{1}{|L|} \delta(|L| + c(t - \tau)).$$

The Pauli function $D(z) = D_{ret}(z) - D_{adv}(z)$ is used to construct a solution to the Cauchy problem of the free d'Alembert equation (wave equation in the absence of sources). Note that

$$L^2 = (1 - u^2)(\delta^2 - (\mathbf{u} \cdot \boldsymbol{\delta})^2). \quad (27)$$

If we denote the angle between the vector $\mathbf{u}$ and $\boldsymbol{\delta}$ by the letter θ , then

$$L^2 = \delta^2(1 - u^2)(1 - u^2 \cos^2(\theta)).$$

This expression is positive for $u^2 < 1$ for any values of θ , and is also positive for $u^2 > 1$ for some θ such that $\cos^2(\theta) > \frac{1}{u^2}$. Thus, there always exists $L > 0$, which plays the role of the effective distance to the

source in classical electrodynamics. Below in Fig. 2 is a graph of the dependence of L on the angle θ in the range $0 < u^2 < 2$:

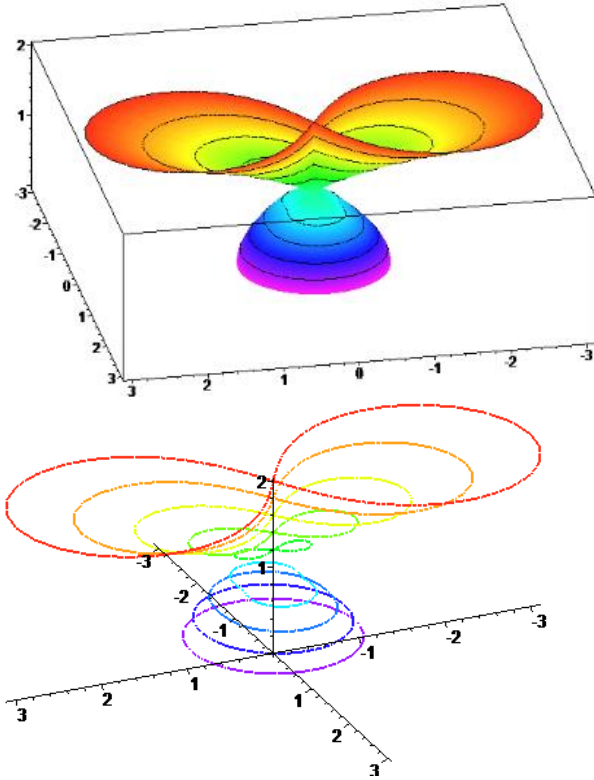


Fig. 2 Dependence of the effective distance to the source L on the angle θ for different $0 < u^2 < 2$

It is evident that for $u^2 > 1$, the effective distance L across the source velocity vector increases with the growth of the velocity u . Since L is included in the denominator of formulas (26), the amplitude of the vector potential will decrease in the direction across the source movement: the field is as if “pressed” to the axis of movement, which is observed in the Cherenkov Effect.

4 Light cone and its appearance

Using formulas (26) we can give the expressions for Green's functions a 4-dimensional form:

$$\begin{aligned} D_{ret}(z) &= \frac{\vartheta(t-\tau) \delta(|L| - c(t-\tau))}{4\pi |L|} = \\ &= \frac{\vartheta(t-\tau) 2\delta(|L| - c(t-\tau))}{4\pi (|L| + c(t-\tau))} = \frac{\vartheta(t-\tau)}{2\pi} \delta(L^2 - c^2(t-\tau)^2), \\ D_{adv}(z) &= \frac{\vartheta(\tau-t) \delta(|L| + c(t-\tau))}{4\pi |L|} = \\ &= \frac{\vartheta(\tau-t) 2\delta(|L| + c(t-\tau))}{4\pi (|L| - c(t-\tau))} = \frac{\vartheta(\tau-t)}{2\pi} \delta(L^2 - c^2(t-\tau)^2), \\ D(z) &= \frac{\text{sign}(t-\tau)}{2\pi} \delta(L^2 - c^2(t-\tau)^2). \end{aligned}$$

The light cone equation is $L^2 - c^2(t-\tau)^2 = 0$, where L^2 is calculated using formula (27).

Let us introduce the notation

$$R = |x - y|, \quad U = \frac{R}{c(t-\tau)},$$

$$U_z = U \cos(\theta), \quad U_\rho = U \sin(\theta),$$

then the equation of the light cone will have the form $U_\rho^2 + (1 - u^2)(U_z - u)^2 = \frac{1}{(1 - u^2)}$, which is the equation of an ellipsoid for $u^2 < 1$, and for $u^2 > 1$ – the equation of a two-sheet hyperboloid as shown in Fig. 3:

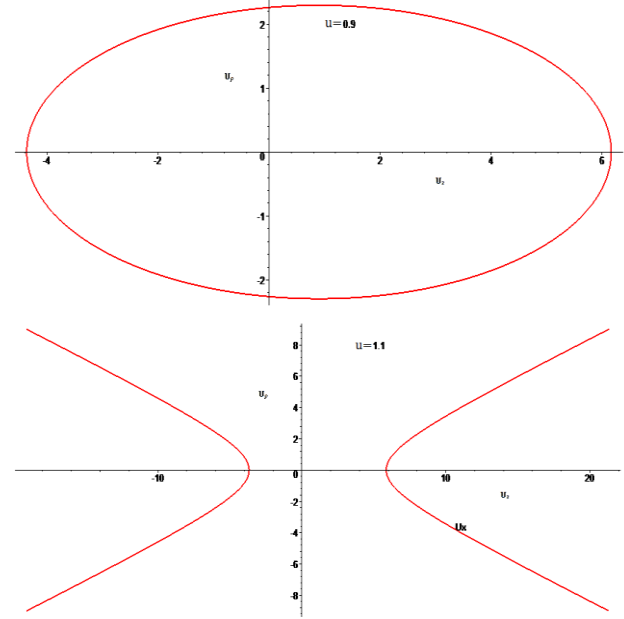


Fig. 3 The light cone for $u^2 < 1$ and for $u^2 > 1$

4.1 Calculation of the Green's Function for $v^2 = c^2$

In this case – when the source moves at the speed of light – according to formula (9)

$$\begin{aligned} s^2 - c^2 k^2 + [\mathbf{v} \times \mathbf{k}]^2 &= \\ &= s^2 - c^2 k^2 + v^2 k^2 - (\mathbf{v} \cdot \mathbf{k})^2 = s^2 - (\mathbf{v} \cdot \mathbf{k})^2, \\ \frac{1}{s^2 - c^2 k^2 + [\mathbf{v} \times \mathbf{k}]^2} &= \frac{1}{s^2 - (\mathbf{v} \cdot \mathbf{k})^2} = \\ &= \frac{1}{2\mathbf{v} \cdot \mathbf{k}} \left(\frac{1}{s - \mathbf{v} \cdot \mathbf{k}} - \frac{1}{s + \mathbf{v} \cdot \mathbf{k}} \right) = \\ &= \frac{1}{2\mathbf{v} \cdot \mathbf{k}} \left(\frac{1}{\omega} - \frac{1}{\omega - 2\mathbf{v} \cdot \mathbf{k}} \right) \\ &- \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{s^2 - (\mathbf{v} \cdot \mathbf{k})^2} d\omega = \frac{2\pi}{\mathbf{v} \cdot \mathbf{k}} e^{-i\mathbf{v} \cdot \mathbf{k} t} \sin(\mathbf{v} \cdot \mathbf{k} t). \end{aligned}$$

Thus, we need to calculate the integral

$$\int_{-\infty}^{+\infty} \frac{d^3 \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot ((x-y) - \mathbf{v}(t-\tau))} \frac{\sin(\mathbf{v} \cdot \mathbf{k}(t-\tau))}{\mathbf{v} \cdot \mathbf{k}}$$

over the entire space $\mathbf{k}$ or, using the notation $\mathbf{n} \equiv \frac{\mathbf{v}}{c}$:

$$G = -\frac{i}{2c} \left(\int_{-\infty}^{+\infty} \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot (\mathbf{x}-\mathbf{y})} - e^{i\mathbf{k} \cdot ((\mathbf{x}-\mathbf{y}) - 2\mathbf{n}c(t-\tau))}}{\mathbf{n} \cdot \mathbf{k}} \right),$$

where $n^2 = 1$, thus, $\mathbf{n}$ is a unit vector directed along the velocity vector $\mathbf{v}$.

An arbitrary vector $\mathbf{q}$ can be decomposed into components parallel to the unit vector $\mathbf{n}$ and perpendicular to it, using the formula

$$\mathbf{q} = \mathbf{n}(\mathbf{n} \cdot \mathbf{q}) + [\mathbf{n} \times [\mathbf{q} \times \mathbf{n}]],$$

or

$$\mathbf{q} = \mathbf{n}(\mathbf{n} \cdot \mathbf{q}) + \mathbf{q} - \mathbf{n}(\mathbf{n} \cdot \mathbf{q}).$$

The length of the longitudinal projection of the vector $q_{||} = (\mathbf{n} \cdot \mathbf{q})$, the transverse – $q_{\perp} = \sqrt{q^2 - (\mathbf{n} \cdot \mathbf{q})^2}$, or $q_{||} = q \cos(\theta)$, $q_{\perp} = q \sin(\theta)$, where θ – angle between vectors.

The scalar product of vectors – via a unit vector – can be represented as

$$(\mathbf{k} \cdot \mathbf{q}) = (\mathbf{k} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{q}) - [\mathbf{k} \times \mathbf{n}] \cdot [\mathbf{n} \times \mathbf{q}].$$

Thus, the problem is reduced to calculating the integral

$$I = \int_{-\infty}^{+\infty} \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot \mathbf{x}} - e^{i\mathbf{k} \cdot (\mathbf{x} - 2\mathbf{n}ct)}}{\mathbf{n} \cdot \mathbf{k}}, \quad (28)$$

which we will calculate in a spherical coordinate system, with the z-axis directed along $\mathbf{n}$:

$$k_1 = K \cos \varphi \sin \theta \quad k_2 = K \sin \varphi \sin \theta$$

$$k_3 = K \cos \theta;$$

$$n_1 = 0 \quad n_2 = 0$$

$$n_3 = 1;$$

$$x_1 = R \cos \beta \sin \alpha \quad x_2 = R \sin \beta \sin \alpha$$

$$x_3 = R \cos \alpha.$$

Let us denote $P = R \sin(\alpha)$, $Z = R \cos(\alpha)$.

Taking into account that

$$\mathbf{k} \cdot \mathbf{x} = K \sin(\theta) P \cos(\varphi - \beta) + K \cos(\theta) Z,$$

$$\int_0^{2\pi} e^{iK \sin(\theta) P \cos(\varphi - \beta)} d\varphi = 2\pi J_0(PK \sin(\theta)),$$

we have

$$I = \frac{2\pi}{(2\pi)^3} \int_0^{\infty} K dK \int_0^{\pi} \frac{\sin(\theta)}{\cos(\theta)} d\theta (e^{iK \cos(\theta) Z} - e^{iK \cos(\theta) (Z - 2ct)}) J_0(PK \sin(\theta)),$$

or, to show that there is no singularity when $\cos(\theta) = 0$:

$$I = \frac{4\pi i}{(2\pi)^3} \int_0^{\infty} K dK \int_0^{\pi} \frac{\sin(K \cos(\theta) ct)}{K \cos(\theta)} e^{iK \cos(\theta) (Z - ct)} J_0(PK \sin(\theta)) K \sin(\theta) d\theta.$$

If we switch to a cylindrical coordinate system:

$$K_z = K \cos(\theta), \quad K_{\rho} = K \sin(\theta),$$

then the double integral is factorized:

$$I = \frac{4\pi i}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dK_z}{K_z} \sin(K_z ct) e^{iK_z (Z - ct)} \int_0^{\infty} K_{\rho} dK_{\rho} J_0(PK_{\rho}),$$

$$I = \frac{4\pi i}{(2\pi)^3 P^2} \int_{-\infty}^{+\infty} \frac{dK_z}{K_z} \sin(K_z ct) e^{iK_z (Z - ct)} \int_0^{\infty} J_0(x) x dx.$$

Taking into account $\int_0^{\infty} J_0(x) x dx = 1$, we have

$$I = \frac{2\pi}{(2\pi)^3 P^2} \int_{-\infty}^{+\infty} \frac{dK_z}{K_z} (e^{iK_z Z} - e^{iK_z (Z - 2ct)}).$$

The sine remains from the exponent because it is the integral of an odd function,

$$\int_{-\infty}^{+\infty} \frac{\cos(K_z Z)}{K_z} dK_z = 0:$$

$$I = \frac{2\pi}{(2\pi)^3 P^2} \int_{-\infty}^{+\infty} \frac{dK_z}{K_z} \left(\frac{\sin(K_z Z)}{-\sin(K_z (Z - 2ct))} \right).$$

4.2 Three different solution areas

The integral from the above is tabular, the result is

$$I = \frac{1}{4\pi^2} (\text{sign}(Z) - \text{sign}(Z - 2ct)).$$

Fig. 4 shows the regions on the Z, ct plane where the function $\text{sign}(Z) - \text{sign}(Z - 2ct)$ takes on different values:

- the region where the function is +2 (absolute future), marked in red;
- the region where the function is -2 (absolute past), marked in pink;
- the regions where the function is zero (unreachable region), marked in green.

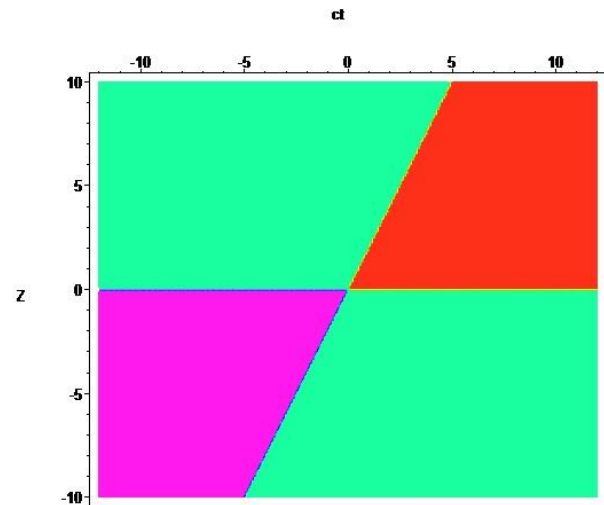


Fig. 4 Different values of function $\text{sign}(Z) - \text{sign}(Z - 2ct)$ on plane Z, ct

4.3 The Energy–Momentum tensor conservation law

For the sake of simplicity, we introduce new field variables according to

$$\boldsymbol{\beta} = \mathbf{B} + \mathbf{u} \times \mathbf{E} \quad (29)$$

$$\boldsymbol{\varepsilon} = \mathbf{E} - \mathbf{u} \times \mathbf{B}.$$

In this variables the system (3) looks like

$$sB_0 + \mathbf{k} \cdot \boldsymbol{\beta} = 0 \quad (30)$$

$$s\boldsymbol{\beta} + c(\mathbf{k} + \mathbf{u} \times [\mathbf{u} \times \mathbf{k}])B_0 + \\ + c\mathbf{k} \times (\boldsymbol{\varepsilon} - \mathbf{u}(\mathbf{u} \cdot \boldsymbol{\varepsilon})) = -i\mathbf{u} \times \mathbf{j}$$

$$sE_0 + \mathbf{k} \cdot \boldsymbol{\varepsilon} = 0$$

$$s\boldsymbol{\varepsilon} + c(\mathbf{k} + \mathbf{u} \times [\mathbf{u} \times \mathbf{k}])E_0 - \\ - c\mathbf{k} \times (\boldsymbol{\beta} - \mathbf{u}(\mathbf{u} \cdot \boldsymbol{\beta})) = i\mathbf{j}$$

Using the vector analysis formula

$$\nabla \cdot [\mathbf{a} \times \mathbf{b}] = \mathbf{b} \cdot [\nabla \times \mathbf{a}] - \mathbf{a} \cdot [\nabla \times \mathbf{b}],$$

we can obtain from system (29) the conservation law for the energy and momentum density tensor

$$\frac{1}{2} \frac{d}{dt} \left((1 - u^2)(B_0^2 + E_0^2 + \beta^2 + \varepsilon^2) + \right. \\ \left. + [\mathbf{u} \times \boldsymbol{\beta}]^2 + [\mathbf{u} \times \boldsymbol{\varepsilon}]^2 \right) + \\ + c\nabla \cdot \left((1 - u^2)(B_0\boldsymbol{\beta} + E_0\boldsymbol{\varepsilon}) + [\boldsymbol{\varepsilon} \times \boldsymbol{\beta}] + \mathbf{u} \right. \\ \left. \times [\mathbf{u} \times [\boldsymbol{\varepsilon} \times \boldsymbol{\beta}]] \right) = \\ = -(\boldsymbol{\varepsilon} - \mathbf{u} \times \boldsymbol{\beta} - \mathbf{u}(\mathbf{u} \cdot \boldsymbol{\varepsilon})) \cdot \mathbf{j}; \quad (31) \\ \frac{d}{dt} \left(-(1 - u^2)(B_0\boldsymbol{\beta} + E_0\boldsymbol{\varepsilon}) + [\boldsymbol{\varepsilon} \times \boldsymbol{\beta}] + \mathbf{u} \right. \\ \left. \times [\mathbf{u} \times [\boldsymbol{\varepsilon} \times \boldsymbol{\beta}]] \right)_i -$$

$$-(1 - u^2)c\nabla_j \left((\beta_i\beta_j + \varepsilon_i\varepsilon_j) \right. \\ \left. + \varepsilon_{ijl} \left((\varepsilon_l - u_l(\mathbf{u} \cdot \boldsymbol{\varepsilon}))B_0 \right. \right. \\ \left. \left. - (\beta_l - u_l(\mathbf{u} \cdot \boldsymbol{\beta}))E_0 \right) \right) + \\ + (1 - u^2)c(\nabla_i + u_i(\mathbf{u} \cdot \nabla)) \cdot \\ \left(\frac{-(1 - u^2)(B_0^2 + E_0^2 - \beta^2 - \varepsilon^2) + [\mathbf{u} \times \boldsymbol{\beta}]^2 + [\mathbf{u} \times \boldsymbol{\varepsilon}]^2}{2} \right) \\ = (1 - u^2)(E_0\mathbf{J} + [\boldsymbol{\beta} \times \mathbf{J}] - B_0\mathbf{u} \times \mathbf{J} + [\boldsymbol{\varepsilon} \times$$

$$[\mathbf{u} \times \mathbf{J}]) - \mathbf{u}((\boldsymbol{\beta} - \mathbf{u} \times \boldsymbol{\varepsilon}) \cdot [\mathbf{u} \times \mathbf{J}]). \quad (32)$$

Note, that the energy density

$$E = \frac{1}{2} \left((1 - u^2)(B_0^2 + E_0^2 + \beta^2 + \varepsilon^2) + [\mathbf{u} \times \boldsymbol{\beta}]^2 + [\mathbf{u} \times \boldsymbol{\varepsilon}]^2 \right) \quad (33)$$

can become negative by $u^2 > 1$. The result is quite unexpected and requires further study and discussion.

5 Conclusion

In the mathematical model for electromagnetic field considered in this paper, we have obtained solution for the wave front, which can change its direction, so that what was longitudinal will become transverse after turning. Therefore, we pass from a globally transverse EM field to fields equally polarized in four

directions: longitudinal, two transverse and "temporal".

The Mathematical modelling and computer simulations showed the peculiarities of the EM waves' spreading depending on the speed of movement of the source. The three specific areas were revealed in computer simulation, so-called: the area of absolute future, the area of absolute past, and the unreachable area.

The following is significantly new in our model:

- There is no scalar potential and static electric field as such.
- The light cone is no longer a cone, but a more complex surface of the 2nd order ellipsoid, which at $|\mathbf{v}| = c$ turns into a two-cavity hyperboloid (a similar situation was with the trajectories of celestial bodies in classical mechanics: closed elliptical orbits at the second cosmic velocity ~ 11.2 km/s turned into open hyperbolic ones).
- The wave energy can become negative at a velocity of $u^2 > 1$, which requires further understanding and a deeper explanation.

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