Investigation of Finite Element Methods in Electromagnetics

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Abstract: - With the advancement of technology, it has progressed to the same extent in engineering studies in parallel with it. It was seen that the analytical and experimental methods used in the past were insufficient, and new searches were made. These new pursuits are united under the main headings of speed and accuracy. Various methods have been tried with this merger. These methods are called numerical methods. Numerical methods are superior to other methods in terms of accuracy and processing time. These methods, which were applied manually in the early times, have changed with the development of electronic machines. Quick results were obtained from the designs made with various computer programs (Ansys HFSS, CST, Comsol etc.), making it easier to use in practice. In particular, the difficulty in processing electromagnetic problems has decreased, and the researchers have reached a solution easily under the specified conditions.

1 Introduction

Scientists and engineers use methods listed in three main groups to solve the problems they encounter. These are analytical methods, experimental methods and numerical methods. In the studies carried out, one of these methods is selected and the road continues. However, each method group has advantages and disadvantages. Although analytical methods are advantageous in terms of accuracy, they are disadvantageous in terms of time and complexity. Similarly, experimental methods are time-consuming and also costly and sometimes dangerous. Numerical methods provide efficient and fast results and facilitate the solution of problems [1,2]

Although analytical methods are preferred to get clear results in simple and time-consuming transactions, numerical methods are generally used to solve the problems encountered. Even if this is the case, analytical methods should be used to a certain extent for the use of numerical methods and a solution should be made. The solution of the problems with analytical methods is examined under four main headings.

These are;
• Separation of variables
• Extending sequences
• Coherent mapping
• Integral solution (Laplace and Fourier)
• Complex methods.

The most commonly used numerical methods can be examined under five main headings among themselves.

These are;
• Finite Difference Method
• Moment Method
• Monte Carlo Method
• Line Method
• Finite Element Method.

Numerical methods are usually approximation results. However, the accuracy of these results is sufficient to solve the problems. With these methods, the solution of electromagnetic problems through computers was found in the mid-1960s. With these programs, people can solve problems even if they do not know high-level mathematics or physics [3,4,5].

2 Electromagnetic Theory

Electromagnetic is the discipline in which electric and magnetic phenomena caused by stationary or moving electric charges are studied [6]. Electric fields express the force that one electric charge creates on another electric charge. The magnetic field, on the other hand, is the field that occurs when electric charges are displaced, that is, when electric current circulates [7]. Electromagnetic fields are calculated by equations called Maxwell's equations. Two theorems are generally used to solve these Maxwell equations. These are the divergence (2.1) and Stokes (2.2) theorems.

$$\int_{v} \nabla.\mathrm{A}dv = \oint_{S} \mathrm{A}.\mathrm{ds} \qquad (2.1)$$

$$\int_{S} (\nabla x\mathrm{A}).\mathrm{ds} = \oint_{C} \mathrm{A}.\mathrm{dl} \qquad (2.2)$$

The divergence theorem (Equation 2.1) means that if we take the volume integral of the divergence of a vector field, it is equal to the total outward flux of that vector on the surface bounding the region [8]. The meaning of Stokes' theorem is that the surface integral of the rotation of a vector field on an open surface is equal to the closed line integral of the vector along the path bounding the surface [8].

2.1 Electrostatic Fields

Electrostatic fields have two basic laws, Gauss's (2.3) and Coulomb's (2.4) laws. The electrostatic force between two point charges is directly proportional to the scalar product of the charges and inversely proportional to the square of the distance between them. If the signs of the charges of the two objects are the same (like positive-positive), they repel each other, if they are different, they attract each other [9].
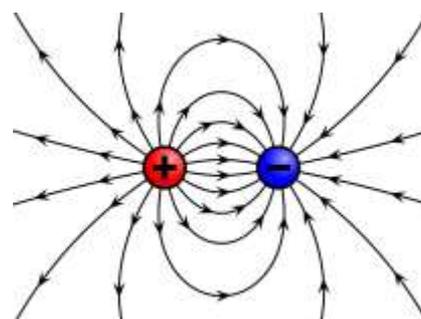

Fig. 1. Electrostatic field

$$\oint D.ds = \int p_v dv \qquad (2.3)$$

$$\oint E.dl = 0 \qquad (2.4)$$

D : Electric Flux Density (C/m^2)
p_v: Volume Charge Density (C/m^3)
E : Electric Field Intensity (V/m)

If equation (2.1) is applied to equation (2.3) and equation (2.2) to equation (2.4)

$$\nabla.D = p_v \qquad (2.5)$$

$$\nabla x E = 0 \qquad (2.6)$$

obtained.

It is expressed as electric flux density (D) and electric field density (E) (2.7). The expression between electric potential (V) and electric field intensity (E) is expressed as in (2.8).

$$D = \varepsilon E \qquad (2.7)$$

$$E = -\nabla V \qquad (2.8)$$

$$V = -\int E.dl \qquad (2.9)$$

If equations (2.5), (2.7) and (2.8) are combined, the Poisson equation (2.10) is obtained.

$$\nabla^2 V = -\frac{p_v}{\varepsilon} \qquad (2.10)$$

If the volume charge density (p_v) is then Laplace's equation (2.11) is obtained.

$$\nabla^2 V = 0 \qquad (2.11)$$

2.2 Magnetostatic Fields

The basic laws of magnetostatic fields are Ampere's law (2.12) and Biot-Savart's law (2.13). The Biot-Savart law is also called Magnetostatic Gaussian

$$\oint_L H.dl = \int_S J.dS \qquad (2.12)$$

$$\oint B.ds = 0 \qquad (2.13)$$

H : Magnetic Field Intensity (A/m)
J : Electric Current Density (A/m^2)
B : Magnetic Flux Density (Tesla or Weber/m^2)

If equation (2.2) is applied to equation (2.12) and equation (2.1) to equation (2.13)

$$\nabla.H = J_c \qquad (2.14)$$

$$\nabla.B = 0 \qquad (2.15)$$

obtained.

If B and H are associated with permeability (μ) (H/m), they are expressed as (2.16). J and E are expressed as (2.17) and their relationship with conductivity (σ). (2.17) is also known as Ohm's law.

$$B = \mu H \qquad (2.16)$$

$$J = \sigma E \qquad (2.17)$$

A in terms of magnetic vector potential

$$B = \nabla x A \qquad (2.18)$$

is displayed as.

It can derive an equation (2.19) from the equation sets obtained.

$$\nabla x(\nabla x F) = \nabla(\nabla.F) - \nabla^2 F \qquad (2.19)$$

f the conditions of equations (2.14), (2.18) and (∇.A=0) are combined, the Poisson equation (2.20) is obtained for magnetic fields. In this equation (2.20), if J=0, Laplace's equation emerges.

$$\nabla^2 A = -\mu J \qquad (2.20)$$

2.3 Boundary Conditions

The material environment in which the electromagnetic field is present is usually characterized by the parameters "μ" and "σ". The medium can be defined as: E and H; If μ is independent of σ and ε, the medium is linear. Otherwise, the environment is not linear. [10]

μ, σ and ε; If it is not a function of space variables, it is a homogeneous environment. Otherwise, the environment is not homogeneous. μ, σ and ε; direction-independent (scalar) is isotropic medium. Otherwise, it is an anisotropic medium.

Boundary conditions separate two different environments. The parameters μ, σ and ε are used for this separation.

2.4 Time-Changing Fields

The simultaneous presence of electric and magnetic fields. Maxwell's equations then undergo some modifications. The newly formed equations are given in (2.22), (2.23), (2.24), (2.25).

$$\nabla.D = p_v \qquad (2.22)$$

$$\nabla.B = 0 \qquad (2.23)$$

$$\nabla x E = -\frac{\partial B}{\partial t} - J_m \qquad (2.24)$$

$$\nabla x H = J_e + \frac{\partial D}{\partial t} \qquad (2.25)$$

The conversion of these equations into integral form is given in equations (2.26), (2.27), (2.28), (2.29).

$$\oint_S D \, . \, dS = \int_v p_v dv \qquad (2.26)$$

$$\oint_S B. \, ds = 0 \qquad (2.27)$$

$$\oint_L E. \, dl = -\int_S \left(\frac{\partial B}{\partial t} + J_m\right) . \, dS \qquad (2.28)$$

$$\oint_L H. \, dl = \int_S J_e + \frac{\partial D}{\partial t}). \, dS \qquad (2.29)$$

In addition to these four Maxwell's equations, there are equations that vary depending on the environment. These equations are given in (2.30), (2.31), (2.32), (2.33).

$$D = \varepsilon E \qquad (2.30)$$

$$B = \mu H \qquad (2.31)$$

$$J_e = \sigma E \qquad (2.32)$$

$$J_m = \sigma * M \qquad (2.33)$$

2.5 Wave Equation

When solving boundary value problems, it is difficult to apply Maxwell's equations combined with first-order differential equations. To overcome this difficulty, wave equations are created to separate first-order differential equations [10]. Wave equations, which are quadratic differential equations, are more useful in solving these problems. $p_v=0$ in linear, isotropic, homogeneous and weldless medium; J=0 is assumed. (2.24)'s curl is taken from both sides (2.34). If the J condition is applied in (2.25), (2.35) is obtained. The equation formed as a result of combining these two equations is given in (2.36).

$$\nabla x \nabla x E = \mu \frac{\partial}{\partial t}(\nabla x H) \qquad (2.34)$$

$$\nabla x H = \varepsilon \frac{\partial E}{\partial t} \qquad (2.35)$$

$$\nabla x \nabla x E = -\mu \frac{\partial^2 E}{\partial t^2} \qquad (2.36)$$

When the required mathematical equations are applied, the equation becomes (2.37) and finally the wave equation (2.38).

$$\nabla(\nabla. E) - \nabla^2 E = -\mu\varepsilon \frac{\partial^2 E}{\partial t^2} \qquad (2.37)$$

$$\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \qquad (2.38)$$

The equation given in (2.38) is simply the wave equation. If we apply similar steps to (2.25), we obtain the wave equation for H (2.39). If we are operating in empty space, we can further simplify the equation by applying (2.40).

$$\nabla^2 H - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0 \qquad (2.39)$$

$$u = c = \frac{1}{\sqrt{\mu\varepsilon}} \qquad (2.40)$$

3 Application of the Finite Element Methods to Equations

3.1 Finite Element Method in Laplace Problems

To find the potential distribution in the two-dimensional region, the region is divided into a finite number of elements. These regions can be triangular or quadrilateral. However, for practical applications, the same number of edges throughout the region is preferred. Look at the region to find the potential distribution and Ve. Equation (3.1) is used for the solution of this approach in triangular areas. N denotes the number of angular regions in the split region. The most common approach to find Ve in elements is the polynomial approach. The approximation polynomial used in quadrilateral regions is given in (3.2) [10].
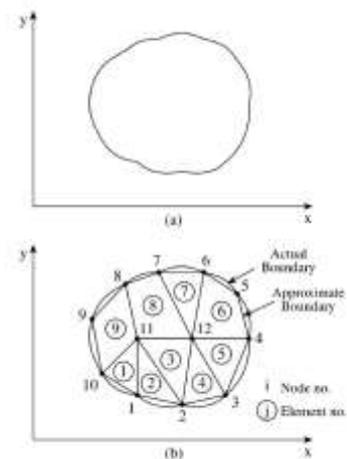

Fig.2. (a) Solution region and (b) finite element separation

The constants "a, b, c and d" are determined. And it's usually not zero within the elements. Triangular

shapes are preferred instead of quadrangular shapes, as they leave the spaces at the ends of the region too much.

$$V_e(x, y) = a + bx + cy \quad (3.1)$$

$$V_e(x, y) = a + bx + cy + dxy \quad (3.2)$$

Considering a normal triangular region, the potentials at nodes 1,2 and 3 are considered Ve1, V_{e2}, V_{e3} respectively. The matrix created using these potentials is given in (3.4).

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (3.4)$$

The solution is made by leaving the coefficients a, b and c alone. The conversion of the resulting matrix to the equation is given in (3.5).

$$V_e = \sum_{i=1}^{3} \alpha_i(x, y) V_{ei} \quad (3.5)$$

One of the necessary coefficients, α coefficients, is calculated and the process continues (3.6), (3.7), (3.8).

$$\alpha_1 = \frac{1}{2A} [(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \quad (3.6)$$

$$\alpha_2 = \frac{1}{2A} [(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \quad (3.7)$$

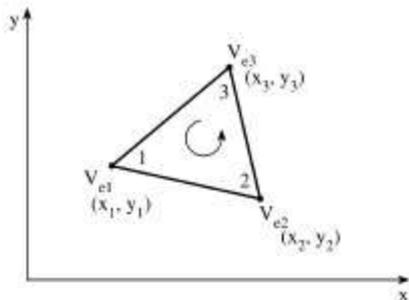$$\alpha_3 = \frac{1}{2A} [(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \quad (3.8)$$

Fig. 3. Numbering the nodes of a triangular element

$$A = \frac{1}{2} [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] \quad (3.9)$$

The equation corresponding to Laplace's equation is given below (3.10).

$$W_e = \frac{1}{2} \int \varepsilon |E_e|^2 dS = \frac{1}{2} \int \varepsilon |\nabla V_e|^2 dS \quad (3.9)$$

$$W_e = \frac{1}{2} \varepsilon [V_e]^t [C^{(e)}][V_e] \quad (3.10)$$

$$[V_e] = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \quad (3.11)$$

$$[C^{(e)}] = \begin{bmatrix} C_{11}^{(e)} & C_{12}^{(e)} & C_{13}^{(e)} \\ C_{21}^{(e)} & C_{22}^{(e)} & C_{23}^{(e)} \\ C_{31}^{(e)} & C_{32}^{(e)} & C_{33}^{(e)} \end{bmatrix} \quad (3.12)$$

C numbers are calculated as follows (3.13). After calculating all C numbers, some numbers are equal as in (3.14).

$$C_{12}^{(e)} = \frac{1}{4A} [(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)] \quad (3.13)$$

$$C_{21}^{(e)} = C_{12}^{(e)}; \ C_{31}^{(e)} = C_{13}^{(e)}; \ C_{32}^{(e)} = C_{23}^{(e)} \quad (3.14)$$

C numbers vary according to the number of nodes in the environment. But the operation logic is exactly the same. If our operation has multiple connections, which we call a global matrix (for example, 5 nodes), the C numbers go up to the 5th row and column (3.15).
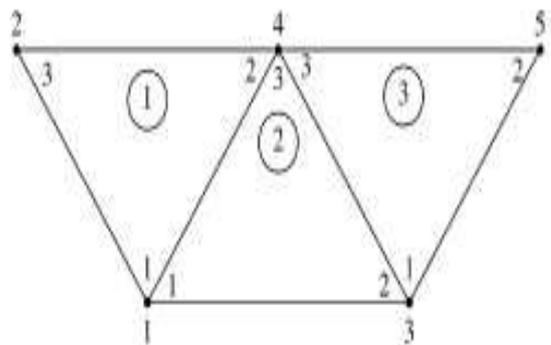

Fig. 4. Locally and globally numbered solution zone with three zones

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix} \tag{3.15}$$

After this stage, the connections between global nodes and local nodes are examined. For example, the C_{11} node means: the local nodes coming to the general 1st node are collected. In the C_{12} node, the elements from the general 1st node to the 2nd node are collected. Continuing like this, the coefficients matrix is created according to the geometry and the node numbers we have given. Our sample matrix formed after these operations is given in (3.16).

$$\begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{13}^{(1)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} \\ C_{31}^{(1)} & C_{33}^{(1)} & 0 & C_{32}^{(1)} \\ C_{21}^{(2)} & 0 & C_{22}^{(2)} + C_{11}^{(3)} & C_{23}^{(2)} + C_{13}^{(3)} \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{23}^{(1)} & C_{32}^{(2)} + C_{31}^{(3)} & C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)} \\ 0 & 0 & C_{21}^{(3)} & C_{23}^{(3)} \end{bmatrix}$$

(3.16)

W is calculated using equation (3.10). The principle of minimization of energy is applied (3.17). Equations are written and solved for the desired nodes by entering the given Dirichlet boundary conditions. As a result, voltages are reached.

$$\frac{\partial W}{\partial V_e} = C_{e1}V_1 + C_{e2}V_2 + C_{e3}V_3 + C_{e4}V_4 \tag{3.17}$$
$$+ C_{e5}V_5 = 0$$

3.2 Finite Element Method in Poisson Equation

The use of finite elements in the Poisson equation is almost the same as in Laplace. Equation (2.10) is used. The p_v values are known when using this equation. Equation (3.17) is written.

$$F(V_e) = \frac{1}{2} \int_S [\varepsilon|\nabla V_e|^2 - 2\rho_{ve}V_e]dS \tag{3.18}$$

$$F(V_e) = \frac{1}{2}\varepsilon[V_e]^t[C^{(e)}][V_e] \tag{3.19}$$
$$- [V_e]^t[T^{(e)}][\rho_e]$$

The T coefficient is calculated as in (3.20).

$$T_{ij}^{(e)} = \begin{cases} \dfrac{A}{12}, i \neq j \\ \dfrac{A}{6}, i = j \end{cases} \tag{3.20}$$

With the help of equation (3.19), the value of F is written and the principle of minimization of energy is applied. Equations are written and solved for the desired nodes by entering the given Dirichlet boundary conditions. As a result, voltages are reached.

$$\frac{\partial F}{\partial V_e} = \varepsilon[C_{e1}V_1 + C_{e2}V_2 + C_{e3}V_3 + C_{e4}V_4 \tag{3.21}$$
$$+ C_{e5}V_5] - [T_{e1}\rho_1$$
$$+ T_{e2}\rho_2 + T_{e3}\rho_3 + T_{e4}\rho_4$$
$$+ T_{e5}\rho_5] = 0$$

3.3 Finite Element Method in Wave Equation

The typical wave equation is the inhomogeneous scalar Helmholtz equation (3.22). Here the amount of area Φ is determined. The source function is g and k is the wave number of the medium (3.23).

Three separate exceptions occur.

• Laplace's equation if k=0=g
• Poisson's equation if k=0
• If k is unknown and g=0, the process continues with the scalar Helmholtz equation.

$$k = \omega\sqrt{\mu\varepsilon} \tag{3.22}$$

The equation for the boundary conditions and the minimization principle is:

$$I(\Phi) = \frac{1}{2} \iint [|\nabla\Phi|^2 - k^2\Phi^2 + 2\Phi g]dS \tag{3.23}$$

The Φ value and the source function expression g on the triangular elements are as in (3.24) and (3.25).

$$\Phi_e(x,y) = \sum_{i=1}^{3} \alpha_i\, \Phi_{ei} \tag{3.24}$$

$$g_e(x,y) = \sum_{i=1}^{3} \alpha_i\, g_{ei} \tag{3.25}$$

$$I(\Phi_e) = \frac{1}{2}[\Phi_e]^t\big[C^{(e)}\big][\Phi_e] \tag{3.26}$$

$$-\frac{k^2}{2}[\Phi_e]^t\big[T^{(e)}\big][\Phi_e]$$

$$+[\Phi_e]^t\big[T^{(e)}\big][g_e]$$

In (3.26) $[\Phi e] = [\Phi_{e1}, \Phi_{e2}, \Phi_{e3}]t$; $[G_e] =$ means $[g_{e1}, g_{e2}, g_{e3}]$.

In equation (3.26), a special case occurs when g=0. Equation takes the form (3.27).

$$I \tag{3.27}$$

$$= \frac{1}{2}[\Phi_f \quad \Phi_p]\begin{bmatrix} C_{ff} & C_{fp} \\ C_{pf} & C_{pp} \end{bmatrix}\begin{bmatrix} \Phi_f \\ \Phi_p \end{bmatrix}$$

$$-\frac{k^2}{2}[\Phi_f \quad \Phi_p]\begin{bmatrix} T_{ff} & T_{fp} \\ T_{pf} & T_{pp} \end{bmatrix}\begin{bmatrix} \Phi_f \\ \Phi_p \end{bmatrix}$$

If (3.28) is set to zero for the energy minimization principle, the result is reached.

$$\frac{\partial I}{\partial \Phi_f} = [C_{ff} \quad C_{fp}]\begin{bmatrix} \Phi_f \\ \Phi_p \end{bmatrix} \tag{3.27}$$

$$-k^2[T_{ff} \quad T_{fp}]\begin{bmatrix} \Phi_f \\ \Phi_p \end{bmatrix} = 0$$

4 Conclusions

When the results of the applied finite element method processing processes and analytical method results are examined and compared; It is seen that the results are close to each other. In addition, the accuracy rate increases when the element (number of divided regions) used in the finite element method is increased. However, it is seen that the computer program slows down due to the increase in the number of processes.

In order to obtain optimum, fast and easy results, operations must be performed with the number of elements at the normal level and the verification phase of the draft drawing must be passed. At this stage, our approved draft should then be modeled with a higher number of elements than the number of elements normally used, and the best results should be achieved. It is seen that following such a way, especially in the implementation of processes that require high cost and fast processes, will easily reach the goals of the engineers.

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