Two-Dimensional Flow of a Pressure-Dependent Viscosity Fluid in Porous Media with a Specified Streamfunction

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Abstract: - In this paper, the Two-dimensional flow of a fluid with pressure-dependent viscosity through a porous structure is considered. Exact solutions are obtained for a Riabouchinsky type flow. Streamfunction is specified in such a way that the resulting flow is irrotational.

Key-Words: - Applied and Theoretical Mechanics, Irrotational flow, Pressure-dependent viscosity.

1 Introduction

It has long been recognized that fluid viscosity increases exponentially with pressure. This realization dates back to the nineteenth Century and the works of Stokes [1] and Barus [2]. Barus [2] formulated an exponentially increasing relationship between viscosity and pressure of the form

$$\mu = \mu_0 e^{\alpha(p-p_0)} \quad (1)$$

where μ is the viscosity, p is the pressure, μ_0 is the viscosity at reference pressure p_0, and $\alpha > 0$ is the pressure-dependence coefficient, [3].

For small values of α or small pressure differences, relationship (1) between viscosity and pressure can be expressed as, [3]

$$\mu = \mu_0[1 + \alpha(p - p_0)] \quad (2)$$

While the above relationships are suitable for fluid with small molecules, long chain molecules (the likes of polymers and some oil mixtures) require other forms of viscosity-pressure relations, [3,4]. It has also been realized that the effect of pressure on density of a fluid is small as compared to the effect of pressure on viscosity, [3]. Accordingly, it suffices in the current work to study incompressible fluid flow with pressure-dependent viscosity as governed by the equation of continuity and the Navier-Stokes equations, written here for steady flow, in the absence of body forces, in the following forms, respectively

$$\nabla \cdot \vec{u} = 0 \quad (3)$$

$$\rho\vec{u} \cdot \nabla\vec{u} = -\nabla p + \nabla \cdot \vec{T} \quad (4)$$

where

$$\vec{T} = -p\vec{I} + 2\mu(p)\vec{K} \quad (5)$$

is the Cauchy stress tensor in which

$$\vec{K} = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T) \quad (6)$$

is the symmetric part of the velocity gradient, p is the pressure, $\mu(p)$ is the viscosity as a function of pressure, $\vec{u}$ is the velocity field, and ρ is the constant fluid density.

Over the past quarter of a century, there has been an increasing interest in flow of pressure-dependent fluids through porous media. While models of flow through porous media have been in existence since the mid-nineteenth Century, fluid viscosity had been mainly treated as constant. Emerging applications of flow through porous media have given rise to the need for flow models with pressure-dependent viscosity. These applications arise in a number of natural and industrial settings that involve chemical and process technologies, such as enhanced oil recovery, carbon sequestration, crude oil pumping, lubrication mechanisms with porous linings, groundwater pumping, filtration problems and microfluidics, to name only a few, [3, 5, 6, 7, 8, 9, 10]. These and other applications underscore the fundamental importance of accurate modelling of flow through porous media of fluids with pressure-dependent viscosities and providing solutions to initial and boundary value problems.

The modelling aspect asserts that the form of momentum equations used must include a pressure-dependent drag function to account for pore-level effects of the porous matrix (as in the generalized Darcy's equation and Forchheimer's equations) and a shear viscosity that is a function of pressure (as in the case of generalized Brinkman's equation, where viscous shear effects are important). A number of models describing the flow of fluids with pressure-dependent viscosities through porous media are available in the literature, and have been derived using homogenization techniques, thermodynamic balance, mixture theory, and intrinsic volume averaging (cf. [9-13]). Of interest to the current work is the generalized Brinkman equation, derived in [9], and takes the following form

$$\rho\vec{u} \cdot \nabla\vec{u} = -\nabla p + \nabla \cdot \vec{T} - \alpha(p)\vec{u} \quad (7)$$

where $\alpha(p)$ is the drag coefficient that is also pressure-dependent. Various forms and combinations of $\mu(p)$ and $\alpha(p)$ have been suggested and usd in the literature [11, 14], popular among which are the following forms:

$$\mu(p) = Ae^{ap} \quad , \quad \alpha(p) = Be^{bp} \quad (8)$$

$$\mu(p) = Ae^{ap} \quad \alpha(p) = B(p/p_0)^m \quad (9)$$

$$\mu(p) = A(p/p_0)^n \quad \alpha(p) = Be^{bp} \quad (10)$$

$$\mu(p) = A(p/p_0)^n \quad \alpha(p) = B(p/p_0)^m \quad (11)$$

Solutions to flow problems involving Navier-Stokes equations with pressure-dependent viscosity depend on whether one deals with internal flow, external flow, or unconfined flow with infinite domains. Kalogirou, Poyiadji, and Georgiou, [15], compiled analytical solutions for internal, Poiseuille-type, steady flows. Although solutions to external flows are rare, [3], some elegant solutions have been provided in the literature, notable among which is the creeping flow past a sphere for fluids with pressure-dependent viscosity was also studied by Housiadas, Georgiou, and Tanner [3]. Study of compressible flow has also been accomplished, where Housiadas and Georgiou, [16], provided new solutions for weakly compressible Newtonian Poiseuille flows with pressure-dependent viscosity.

Solutions to flow problems with pressure-dependent viscosity in unconfined domains have been obtained under various assumptions, and using a plethora of techniques, have been reviewed, reported and implemented in the work of Naeem [17], and the vast literature reported in his work. These include methods of solution where the streamfunction is specified or the vorticity distribution is given.

Solutions to the generalized Brinkman equation are challenging since inherent within this model are nonlinearities of the Navier-Stokes equations, with the added difficulty of handling viscosity variations and drag function as functions of pressure. Danish et.al., [18], provided first exact solutions for mixed boundary value problems concerning the motions of fluids with exponential dependence of viscosity on pressure. Hron et.al., [4], and the references therein, provided solutions for unidirectional and special flows. Pažanin et.al. [19] provided asymptotic analysis of the generalized Brinkman's equation for two-dimensional flow. Alharbi et.al, [20], used a technique based on the concept of Riabouchinsky flow in which the streamfunction of a two-dimensional flow is assumed to be a function of one space variable, and combinations of functions of single variables. They, [20], provided solutions of a special form of the generalized Brinkman's equation, one in which the drag function is a function of pressure scaled by variable permeability of the porous medium. The permeability function introduces an additional variable in the governing equations, thus rendering the system of governing equations under-determined. In order to circumvent this situation, Alharbi et.al. [20] introduced a condition, derived from the specified streamfunction, that was satisfied by the permeability function.

The Riabouchinsky approach has received considerable success in the understanding of flow phenomena and the introduction of methodologies based on this approach. Naeem [17] successfully implemented this approach in his study of Navier-Stokes flow of a fluid with pressure-dependent viscosity. In this work, we follow Naeem's work [17] and assume the form streamfunction in order to obtain solutions to equation (7). We deliberately chose the form of streamfunction so that the resulting flow is of zero vorticity. This implies that the flow is both viscous and irrotational. Irrotational flow of viscous fluids has recently received attention in the literature (cf. [21] and [ ] and the references therein). In their elegant analysis, Sirakov et.al. [21] described the main features of steady irrotational flow in a fluid with constant viscosity. Their goal was is to illustrate the different roles played by the identically zero viscous forces and the non-zero viscous stresses, in addition to studying the behavior of stagnation pressure, that is constant along a streamline, and stagnation enthalpy, which changes through the flow. In their work, [21] they also discussed the role viscous stresses on the boundary play in creating these behaviors. In the current work, the examples used result in irrotational flow even for non-constant viscosity.

2 Governing Equations

The flow of a fluid with pressure-dependent through a porous sediment, in the absence of heat transfer effects and body forces, is governed by the continuity equation (1) and momentum equations (7), above. Assuming the flow is in two space dimensions, we let $\vec{u} = (u, v)$, and write equations (1) and (7) in the following components' form:

$$u_x + v_y = 0$$
(12)

$$\frac{1}{2}(q^2)_x - v(v_x - u_y) = -P_x + \frac{1}{\rho}[(2\mu u_x)_x + (\mu u_y + \mu v_x)_y] - \frac{\alpha(p)}{\rho}u$$
(13)

$$\frac{1}{2}(q^2)_y + u(v_x - u_y) = -P_y + \frac{1}{\rho}[(2\mu v_y)_y + (\mu u_y + \mu v_x)_x] - \frac{\alpha(p)}{\rho}v$$
(14)

where subscript notation denotes partial differentiation, and

$$P = \frac{p}{\rho}$$
(15)

$$q^2 = u^2 + v^2$$
(16)

with q^2 being the square of the speed.

Assuming that viscosity and drag depend on the pressure according to the following positive, and increasing relationships:

$\mu(p) = a\rho P$
(17)

$\alpha(p) = b\rho P$
(18)

where a and b are known positive constants, then (13) and (14) can be written, respectively, as

$\frac{1}{2}(q^2)_x - v(v_x - u_y) = (2au_x - 1)P_x - aP(v_x - u_y)_y + aP_y(u_y + v_x) - bPu$
(19)

$\frac{1}{2}(q^2)_y + u(v_x - u_y) = -(2au_x + 1)P_y + aP(v_x - u_y)_x + aP_x(u_y + v_x) - bPv$ (20)

Continuity equation (12) implies the existence of the streamfunction $\psi(x,y)$ such that:

$\psi_y = u$ (21)

$\psi_x = -v$ (22)

Vorticity, $\omega(x,y)$, is defined as:

$\omega = \nabla \times \vec{u} = v_x - u_y$

$= -\psi_{xx} - \psi_{yy} = -\nabla^2\psi$
(23)

Using equations (21) and (22) in equations (19) and (20), we obtain, respectively:

$\frac{1}{2}(\psi_x^2 + \psi_y^2)_x - \psi_x\nabla^2\psi = (2a\psi_{xy} - 1)P_x + aP\nabla^2\psi_y + aP_y(\psi_{yy} - \psi_{xx}) - bP\psi_y$
(24)

$\frac{1}{2}(\psi_x^2 + \psi_y^2)_y - \psi_y\nabla^2\psi = -(2a\psi_{xy} + 1)P_y - aP\nabla^2\psi_x + aP_x(\psi_{yy} - \psi_{xx}) + bP\psi_x$
(25)

The problem at hand is thus reduced to solving equations (24) and (25) for the streamfunction $\psi(x,y)$ and the pressure $P(x,y)$. Viscosity and drag functions, velocity components and vorticity, can be obtained using equations (17), (18), (21), (22) and (23).

3 Method of Solution

In order to solve (24) and (25), we assume a functional form of $\psi(x,y)$ and then solve for the pressure $P(x,y)$. In this work we consider the following two cases:

Case 1: The streamfunction is a function of one of the coordinate variables.

Case 2: The streamfunction is linear in one of the coordinate variables.

Case 1: Assuming that the streamfunction is a function of *y* only, namely

$\psi(x,y) = f(y)$
(26)

wherein "prime" notation denotes ordinary differentiation. Velocity components and vorticity, equations (21)-(23), take the following forms

$u = f'(y)$
(27)

$v = 0$
(28)

$\omega = -f''(y)$
(29)

Using (26) in (24) and (25) we obtain, respectively

$P_y = aP_x f''(y)$
(30)

$P_x = aPf'''(y) + aP_y f''(y) - bPf'(y)$
(31)

Using (30) in (31), we obtain

$$\frac{p_x}{P} = \frac{[af'''(y) - bf'(y)]}{[1 - a^2(f''(y))^2]} = \frac{[af'''(y)]}{[1 - a^2(f''(y))^2]} - \frac{[bf'(y)]}{[1 - a^2(f''(y))^2]} \quad (32)$$

Equation (32) can be written as

$$\frac{\partial lnp}{\partial x} = \frac{1}{2}\frac{d}{dy}ln\left[\frac{1+af''(y)}{1-af''(y)}\right] - \frac{bf'(y)}{[1-a^2(f''(y))^2]}$$
(33)

Integrating (33) with respect to x, we obtain

$$lnP = \left\{\frac{1}{2}\frac{d}{dy}ln\left[\frac{1+af''(y)}{1-af''(y)}\right] - \frac{bf'(y)}{[1-a^2(f''(y))^2]}\right\}x + lnF(y)$$
(34)

where $F(y)$ is an arbitrary function of y.

Equation (34) can be written as

$$P = F(y)exp\left(\frac{x}{2}G(y)\right)$$
(35)

where

$$G(y) = \frac{d}{dy}ln\left[\frac{1+af''(y)}{1-af''(y)}\right] - \frac{2bf'(y)}{[1-a^2(f''(y))^2]}$$
(36)

From (35), we obtain

$$P_x = \frac{F(y)G(y)}{2}exp\left(\frac{x}{2}G(y)\right)$$
(37)

$$P_y = [F'(y) + \frac{x}{2}F(y)G'(y)]exp\left(\frac{x}{2}G(y)\right)$$
(38)

Using (37) and (38) in (30), we obtain

$$\left[F'(y) + \frac{x}{2}F(y)G'(y)\right] = a\frac{F(y)G(y)}{2}f''(y)$$
(39)

By equating powers of x in (39), we obtain

$$F'(y) = a\frac{F(y)G(y)}{2}f''(y) \quad (40)$$

$$F(y)G'(y) = 0$$
(41)

Equations (40) and (41) yield:

$$F(y) = \exp\left(\frac{a}{2}\int f''(y)G(y)dy\right)$$
(42)

and

$$G(y) = C_1$$
(43)

where C_1 is a constant.

Using (43), and integrating the RHS of equation (42), then equation (42) is replaced by

$$F(y) = C_2\exp\left(\frac{a}{2}C_1 f'(y)\right) \quad (44)$$

where C_2 is a constant. In addition, using (43), equation (36) can be written as

$$\frac{d}{dy}ln\left[\frac{1+af''(y)}{1-af''(y)}\right] - \frac{2bf'(y)}{[1-a^2(f''(y))^2]} = C_1 \quad (45)$$

and the pressure equation (35) is replaced by:

$$P = F(y)exp\left(\frac{C_1 x}{2}\right) \quad (46)$$

Using (44) in (46) we obtain

$$P = C_2\exp\left(\frac{C_1}{2}(x + af'(y))\right)$$
(47)

The pressure distribution and flow quantities thus hinge on the function $f(y)$ that must be chosen such that (45) is satisfied. Equation (45) yields

$$f'''(y) + \frac{aC_2}{2}[f''(y)]^2 - \frac{bf'(y)}{a} = \frac{C_1}{2a}$$
(48)

Now, equation (48) is satisfied by the linear polynomial function

$$f(y) = Ay + B$$
(49)

where *A* and *B* are known constants.
Using (49) in (48), we obtain $C_1 = -2bA$ and equation (47) gives the following pressure distribution:

$$P = C_2\exp(-bA(x + aA)$$
(50)

Constant C_2 in the pressure distribution (50) can be determined with the imposition of a condition on the pressure. For instance, if $P(0,0) = P_0$ then (50) yields $C_2 = P_0 \exp(baA^2)$, and the pressure distribution, viscosity and drag functions become, respectively,

$$P = P_0 \exp(-bAx)$$
(51)

$$\mu(p) = a\rho P_0 \exp(-bAx)$$
(52)

$$\alpha(p) = b\rho P_0 \exp(-bAx)$$
(53)

The flow quantities of streamfunction, vorticity and velocity components are determined from equations (26)-(29) as:

$$\psi = Ay + B$$
(54)
$$u = A$$
(55)
$$v = 0$$
(56)
$$\omega = 0$$
(57)

Streamfunction is a linear function of y with a tangential velocity component $u = A > 0$, and a zero normal velocity component of velocity. Since vorticity is zero, the flow is irrotational. Pressure, viscosity and drag function are functions of x. With $a, b, A > 0$, pressure is bounded with

$$\lim_{x\to\infty} P_0 \exp(-bAx) = P_0$$
(58)

This implies that viscosity and drag functions are also bounded.

CASE 2: Assume that

$$\psi(x,y) = yh(x) = y(Ax + B)$$
(59)

Then

$$u = \psi_y = h(x) = Ax + B$$
(60)

$$v = -\psi_x = -yh'(x) = -Ay$$
(61)

$$\omega = -\psi_{xx} - \psi_{yy} = -yh''(x) = 0$$
(62)

Using (60)-(62) in (19) and (20), we obtain, respectively

$$P_x - \frac{(bAx+bB)}{(2aA-1)}P = \frac{A^2x+AB}{(2aA-1)}$$
(63)

$$(2aA+1)P_y = A(bP - A)y$$
(64)

Solution to (63) takes the form

$$P = \frac{A}{b} + F(y)\exp[-\frac{\left(\frac{bA}{2}x^2+bBx\right)}{(2aA-1)}]$$
(65)

with the condition that $aA \neq \frac{1}{2}$. The function $F(y)$ is an arbitrary function of y that is to be determined.

Differentiating (65) with respect to y, we obtain

$$P_y = F'(y)\exp[-\frac{\left(\frac{bA}{2}x^2+bBx\right)}{(2aA-1)}]$$
(66)

Equations (64) and (66) yield

$$\frac{F'}{F} = \frac{bA}{(2aA+1)}y$$
(67)

Solution to (67) is given by

$$F(y) = c_3 exp\left(\frac{bA}{2(2aA+1)}y^2\right)$$
(68)

where c_3 is an arbitrary constant.

Using (68) in (65), we obtain the pressure distribution

$$P = \frac{A}{b} + c_3 exp\left(\frac{\frac{bAy^2}{2}}{(2aA+1)} - \frac{\frac{bAx^2}{2}+bBx}{(2aA-1)}\right)$$
(69)

Letting $P(0,0) = P_0$, equation (69) gives $c_3 = P_0 - \frac{A}{b}$ and (69) becomes

$$P = \frac{A}{b} + (P_0 - \frac{A}{b})exp\left(\frac{\frac{bAy^2}{2}}{(2aA+1)} - \frac{\frac{bAx^2}{2}+bBx}{(2aA-1)}\right) \quad (70)$$

Using (70) in (17) and (18), we obtain the following viscosity and drag functions, respectively:

$$\mu(p) = a\rho\left[\frac{A}{b} + (P_0 - \frac{A}{b})exp\left(\frac{\frac{bAy^2}{2}}{(2aA+1)} - \frac{\frac{bAx^2}{2}+bBx}{(2aA-1)}\right)\right] \quad (71)$$

$$\alpha(p) = b\rho\left[\frac{A}{b} + (P_0 - \frac{A}{b})exp\left(\frac{\frac{bAy^2}{2}}{(2aA+1)} - \frac{\frac{bAx^2}{2}+bBx}{(2aA-1)}\right)\right] \quad (72)$$

Solution is thus determined by equations (59)-(62) and (70)-(72). Equation (59) gives the streamfunction of the flow as a function of x and y. The tangential velocity component is given by the linear function of x shown in equation (60) and the normal velocity component is given by the linear function of y, shown in equation (61). This flow is irrotational with a zero vorticity given by equation (61).

The pressure in (70) remains bounded when $A > 0$ and $aA \neq \frac{1}{2}$, with

$$\lim_{\substack{x\to\infty \\ y\to\infty}} P = \frac{A}{b} \quad (73)$$

4 Conclusion

In this work, we considered the two-dimensional flow of a fluid with pressure-dependent viscosity through a porous medium. We obtained a Riabouchinsky-type solution in which the streamfunction was specified. The forms of the streamfunction were chosen in such a way that the vorticity of the flow vanishes. The flow obtained is thus one in which a fluid with pressure-dependent viscosity is irrotational.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Both authors participated in pertinent literature review to identify missing state-of-the-art knowledge in this area of research. Both authors independently obtained the solutions to governing equations..
M.S. Abu Zaytoon double checked the equations for consistency with the literature.
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