Derivation and Analysis of the Navier-Stokes Equations in Coordinate System Rotating Around the Vertical Axis

IVAN V. KAZACHKOV
Department of Information Technology and Data Analysis
Nizhyn Mykola Gogol State University
16600 Grafska str. 2, Nizhyn
UKRAINE
kazachkov.iv@ndu.edu.ua, ivan.kazachkov@energy.kth.se
Yevgen V. CHESNOKOV
Institute of Cybernetics of the Ukrainian Acad. Sci.,
02091 Trostyanetska str. 12, apt.155 Kyiv
UKRAINE
evchesnokov@ukr.net

Abstract: - The Navier-Stokes equations are derived in the coordinate system rotating around the vertical axis. As a starting point the corresponding differential equations are used in the Cartesian coordinates for inviscid liquid with account of possible compressibility and effect of the mass forces (gravity and other ones, e.g. electromagnetic). The transformation of the coordinates is performed and then the Navier-Stokes equations are derived. The obtained equations are analyzed for a number of physical situations.

Key-Words: - Rotation of Coordinate System; Navier-Stokes Equations; Analysis; Centrifugal Forces

1 Statement of the Problem

1.1 The Rotating Coordinate System

The first Cartesian coordinate system x_1, x_2, x_3 with the vertical axis x_3 is immovable, the other coordinate system y_1, y_2, y_3 is rotating around the vertical axis, with the rotation speed ω as shown in Fig. 1:

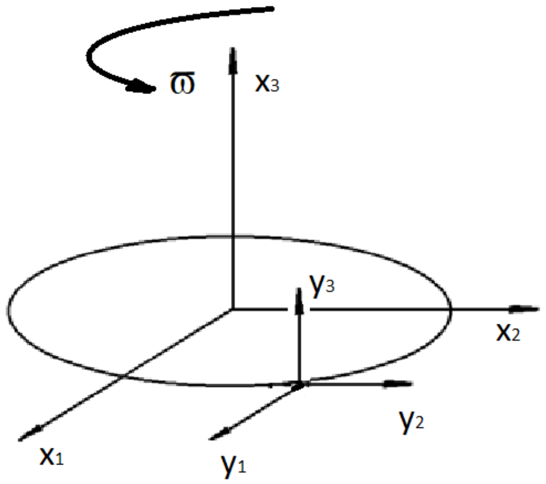


Fig. 1 Cartesian immovable and rotating around the vertical axis coordinate systems

The rotating coordinate system can be done with the vertical axis coinciding with x_3 or shifted from the central axis ($x_1 = x_2 = 0$) on some distance R_0.

The peculiarities and dynamics of the flows in the inertial rotating coordinate system are analyzed in the present paper. Different vortex flows and particle motions, vortices and rotating flows have always wondered and sometimes scared people. Examples of the swirling flows at all scales in nature are: the spiral galaxies, the atmospheric hurricanes, sea and river vortices, and even stirring tea in a teacup (see for example Figs 2-4) [1-3]:

Fig. 2 Spiral Galaxy in Ursa Major M101

Fig. 3 Hurricane Rita at its peak in the Gulf of Mexico in September 2005

Fig. 4 Giant vortex in southern US lake

Interesting that all in Figs 2-4 examples are with counter clockwise rotations.

Intensive rotational movement and mixing are fascinating phenomena and may be very effective in a number of engineering and technological applications [4-11]. Many theoretical aspects have been studied for the diverse rotational flows [12-19]. Nevertheless, the problem still remains insufficiently studied for many theoretical, as well as practical applications.

1.2 Basic Navier-Stokes Equations

As the outgoing differential equations the following Navier-Stokes equation array for the inviscid liquid is taken:

$$\frac{\partial \rho}{\partial t} + div(\rho \boldsymbol{V}) = 0, \qquad (1)$$

$$\frac{\partial}{\partial t}\rho \boldsymbol{V} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\rho\boldsymbol{V} = -\boldsymbol{\nabla}P.$$

The momentum equation can be written in more general form, accounting the available mass forces:

$$\frac{\partial}{\partial t}\rho \boldsymbol{V} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\rho\boldsymbol{V} = -\boldsymbol{\nabla}P + \rho\boldsymbol{g} + \rho\boldsymbol{f} + \rho\boldsymbol{\nu}\boldsymbol{\Delta}\boldsymbol{V}. \qquad (2)$$

Here ρ is density of liquid, P- its pressure, t- time, ν- kinematic viscosity coefficient, $\boldsymbol{V} = \{V_1, V_2, V_3\}$, $\boldsymbol{f} = \{f_1, f_2, f_3\}$, $\boldsymbol{g} = \{g_1, g_2, g_3\}$- vectors of the velocity, volumetric mass forces and gravity, respectively. And $\boldsymbol{\nabla}$, $\boldsymbol{\Delta}$ are the vector gradient and Laplace operator.

Many natural and technical systems deal with the fluid flows in the situation when all coordinate system rotates: river flow on earth rotating itself and around the sun, the flow of liquids in rotating parts of a car, etc. In some cases rotation is very intensive and affects the fluid flow in the rotation system a lot. Therefore, this paper aims to derive the Navier-Stokes equation array and analyze it in the rotating coordinate system. The focus is on the peculiarities appearing in a flow due to rotation of the system.

1.3 The Rotating Coordinates

The rotating coordinates y_1, y_2, y_3 are expressed through the outgoing immovable coordinates x_1, x_2, x_3 as follows

$$y_1 = cos(\omega t)x_1 - sin(\omega t)x_2,$$
$$y_2 = sin(\omega t)x_1 + cos(\omega t)x_2, \qquad (3)$$
$$y_3 = x_3.$$

This rotation in the plane (x_1, x_2) with the constant speed of rotation ω.

2 Transformation of the Coordinates and Derivatives

2.1 Partial Spatial Derivatives of Coordinates

The derivatives by the spatial coordinates are transformed from the immovable Cartesian coordinate system to the rotational one as follows

$$\frac{\partial Q}{\partial x_1} = \cos(\omega t)\frac{\partial Q}{\partial y_1} + \sin(\omega t)\frac{\partial Q}{\partial y_2},$$
$$\frac{\partial Q}{\partial x_2} = -\sin(\omega t)\frac{\partial Q}{\partial y_1} + cos(\omega t)\frac{\partial Q}{\partial y_2}, \qquad (4)$$
$$\frac{\partial Q}{\partial x_3} = \frac{\partial Q}{\partial y_3}.$$

According to the equations (3) and (4), transformation of the coordinates and spatial derivatives depend on time and rotation speed. Only the vertical coordinate remains the same.

2.2 Transformation of Velocity Components

The velocity components in the outgoing immovable Cartesian coordinates can be presented as follows

$$V_1 = \frac{dx_1}{dt} = \frac{d}{dt}(\ cos(\omega t) y_1 + \sin(\omega t) y_2),$$

$$V_2 = \frac{dx_2}{dt} = \frac{d}{dt}(-sin(\omega t) y_1 + cos(\omega t) y_2),$$

$$V_3 = \frac{dx_3}{dt} = \frac{dy_3}{dt}.$$

Or

$$V_1 = -\omega(\sin(\omega t) y_1 - cos(\omega t) y_2) + cos(\omega t)\dot{y}_1 + \sin(\omega t)\dot{y}_2,$$

$$V_2 = -\omega(cos(\omega t) y_1 + sin(\omega t) y_2) - sin(\omega t)\dot{y}_1 + cos(\omega t)\dot{y}_2,$$

$$V_3 = \dot{y}_3, \qquad (5)$$

where the dots over coordinates y_1, y_2, y_3 assign the derivatives by time.

For the velocity components in rotational coordinate system the following assignments are taken:

$$U_1 = \dot{y}_1,\ U_2 = \dot{y}_2,\ U_3 = \dot{y}_3.$$

Thus, from (5) yields

$$V_1 = \omega(-\sin(\omega t) y_1 + cos(\omega t) y_2) + cos(\omega t) U_1 + \sin(\omega t) U_2,$$

$$V_2 = -\omega(cos(\omega t) y_1 + sin(\omega t) y_2) - sin(\omega t) U_1 + cos(\omega t) U_2,$$

$$V_3 = U_3. \qquad (6)$$

2.3 Transformation of Derivatives of the Velocity Components

Derivatives from the velocity components are transformed in the floowing way:

$$\frac{\partial}{\partial t} V_1 = -\omega^2 (cos(\omega t) y_1 + sin(\omega t) y_2)$$
$$+2\omega(-\sin(\omega t)\dot{y}_1 + cos(\omega t)\dot{y}_2) + cos(\omega t)\ddot{y}_1 + \sin(\omega t)\ddot{y}_2,$$

$$\frac{\partial}{\partial t} V_2 = -\omega^2(-\sin(\omega t) y_1 + cos(\omega t) y_2)$$
$$- 2\omega(cos(\omega t)\dot{y}_1 + sin(\omega \cdot t)\dot{y}_2)$$
$$-sin(\omega \cdot t)\ddot{y}_1 + cos(\omega \cdot t)\ddot{y}_2,$$

$$\frac{\partial}{\partial t} V_3 = \ddot{y}_3. \qquad (7)$$

The second order derivatives by time are assigned by two dots over the corresponding value.

The expressions (7) with account of the above are the following:

$$\frac{\partial}{\partial t} V_1 = -\omega^2 (cos(\omega t) y_1 + sin(\omega t) y_2)$$
$$+2\omega(-\sin(\omega t) U_1 + cos(\omega t) U_2) + cos(\omega t)\dot{U}_1 + \sin(\omega t)\dot{U}_2,$$

$$\frac{\partial}{\partial t} V_2 = -\omega^2(-\sin(\omega t) y_1 + cos(\omega t) y_2)$$
$$- 2\omega(cos(\omega t) U_1 + sin(\omega t) U_2)$$
$$-sin(\omega t)\dot{U}_1 + cos(\omega t)\dot{U}_2,$$

$$\frac{\partial}{\partial t} V_3 = \dot{U}_3. \qquad (8)$$

And then

$$\frac{\partial}{\partial t}(\rho V_1) = -\omega^2 \rho (cos(\omega t) y_1 + sin(\omega t) y_2)$$
$$+ 2\omega\rho(-\sin(\omega t) U_1 + cos(\omega t) U_2)$$
$$+\rho\big(cos(\omega t)\dot{U}_1 + \sin(\omega t)\dot{U}_2\big) +$$
$$+(\omega(-\sin(\omega t) y_1 + cos(\omega t) y_2) + cos(\omega t) U_1 + \sin(\omega t) U_2)\dot{\rho},$$

$$\frac{\partial}{\partial t}(\rho V_2) = -\omega^2 \rho(-\sin(\omega t) y_1 + cos(\omega t) y_2)$$
$$- 2\omega\rho(cos(\omega t) U_1 + sin(\omega t) U_2)$$
$$+\rho\big(-sin(\omega t)\dot{U}_1 + cos(\omega t)\dot{U}_2\big) +$$
$$+(-\omega(+cos(\omega t) y_1 + sin(\omega t) y_2) - sin(\omega t) U_1 + cos(\omega t) U_2)\dot{\rho},$$

$$\frac{\partial}{\partial t}(\rho V_3) = \rho \dot{U}_3 + U_3 \dot{\rho}, \qquad (9)$$

where are:

$\frac{\partial}{\partial t}(\rho V_1) = \rho \frac{\partial}{\partial t} V_1 + V_1 \frac{\partial}{\partial t}\rho$, $\quad \frac{\partial}{\partial t}(\rho V_2) = \rho \frac{\partial}{\partial t} V_2 + V_2 \frac{\partial}{\partial t}\rho$, $\frac{\partial}{\partial t}(\rho V_3) = \rho \frac{\partial}{\partial t} V_3 + V_3 \frac{\partial}{\partial t}\rho$.

3 Derivation of the Navier-Stokes Equations in Rotating Coordinate System

3.1 The Spatial and Temporal Derivatives in the Rotational Coordinate System

According to the equations (4) the spatial derivatives can be written as follows

$$\frac{\partial(\rho V_1)}{\partial x_1} = (\omega(-sin(\omega t) y_1 + cos(\omega t) y_2)$$
$$+ cos(\omega t) U_1 + sin(\omega t) U_2) \cdot$$

$$\cdot\left(cos(\omega t)\frac{\partial}{\partial y_1}+sin(\omega t)\frac{\partial}{\partial y_2}\right)\rho +$$
$$+\rho\left(cos(\omega t)\frac{\partial}{\partial y_1}+sin(\omega t)\frac{\partial}{\partial y_2}\right)\cdot(cos(\omega t)U_1 + sin(\omega t)U_2),$$

$$\frac{\partial(\rho V_1)}{\partial x_2}=(\omega(-sin(\omega t)y_1+cos(\omega t)y_2) + cos(\omega t)U_1+sin(\omega t)U_2)\cdot$$
$$\cdot\left(-sin(\omega t)\frac{\partial}{\partial y_1}+cos(\omega t)\frac{\partial}{\partial y_2}\right)\rho +$$
$$+\rho\left[\left(-sin(\omega t)\frac{\partial}{\partial y_1}+cos(\omega t)\frac{\partial}{\partial y_2}\right)\cdot(cos(\omega t)U_1 + sin(\omega t)U_2)+\omega\right],$$

$$\frac{\partial(\rho V_1)}{\partial x_3}=\frac{\partial(\rho V_1)}{\partial y_3}; \qquad (10)$$

$$\frac{\partial(\rho V_2)}{\partial x_1}=(-\omega(cos(\omega t)y_1+sin(\omega t)y_2) - sin(\omega t)U_1+cos(\omega t)U_2)\cdot$$
$$\cdot\left(cos(\omega t)\frac{\partial}{\partial y_1}+sin(\omega t)\frac{\partial}{\partial y_2}\right)\rho +$$
$$+\rho\left[\left(cos(\omega t)\frac{\partial}{\partial y_1}+sin(\omega t)\frac{\partial}{\partial y_2}\right)\cdot(-sin(\omega t)U_1 + cos(\omega t)U_2)-\omega\right],$$

$$\frac{\partial(\rho V_2)}{\partial x_2}=(-\omega(cos(\omega t)y_1+sin(\omega t)y_2) - sin(\omega t)U_1+cos(\omega t)U_2)\cdot$$
$$\cdot\left(-sin(\omega t)\frac{\partial}{\partial y_1}+cos(\omega t)\frac{\partial}{\partial y_2}\right)\rho$$
$$+\rho\left(-sin(\omega t)\frac{\partial}{\partial y_1}+cos(\omega t)\frac{\partial}{\partial y_2}\right)\cdot(-sin(\omega t)U_1+cos(\omega t)U_2),$$

$$\frac{\partial(\rho V_2)}{\partial x_3}=\frac{\partial(\rho V_2)}{\partial y_3};$$

$$\frac{\partial(\rho V_3)}{\partial x_1}=V_3\left(cos(\omega t)\frac{\partial\rho}{\partial y_1}+sin(\omega t)\frac{\partial\rho}{\partial y_2}\right)+\rho\left(cos(\omega t)\frac{\partial V_3}{\partial y_1}+sin(\omega t)\frac{\partial V_3}{\partial y_2}\right),$$

$$\frac{\partial(\rho V_3)}{\partial x_2}=V_3\left(cos(\omega t)\frac{\partial\rho}{\partial y_2}-sin(\omega t)\frac{\partial\rho}{\partial y_1}\right)+\rho\left(cos(\omega t)\frac{\partial V_3}{\partial y_2}-sin(\omega t)\frac{\partial V_3}{\partial y_1}\right),$$

$$\frac{\partial(\rho V_3)}{\partial x_3}=\frac{\partial(\rho V_3)}{\partial y_3};$$

$$\frac{\partial P}{\partial x_1}=cos(\omega t)\frac{\partial P}{\partial y_1}+sin(\omega t)\frac{\partial P}{\partial y_2},$$

$$\frac{\partial P}{\partial x_2}=-sin(\omega t)\frac{\partial P}{\partial y_1}+cos(\omega t)\frac{\partial P}{\partial y_2},$$

$$\frac{\partial P}{\partial x_3}=\frac{\partial P}{\partial y_3}.$$

As it is observed from the equations (10), remarkably for incompressible liquid, the flow gradient by the first coordinate x_1, $\partial(\rho V_1)/\partial x_1$, transforms to zero in the rotational coordinate system, while by the second coordinate x_2 it is just equal to $\omega\rho$. For the second velocity component, gradient $\partial(\rho V_2)/\partial x_2$ is, respectively -$\omega\rho$ and 0 by the coordinates x_1, x_2.

The vertical velocity component remains non-zero, it does not depend on rotation of the coordinate system. And the vertical gradients for all velocity components are not transformed in the new coordinate system.

3.2 Transformation of the Navier-Stokes Equations

Now let us write the equation array (1), (2) in the following coordinate form:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho V_1)}{\partial x_1}+\frac{\partial(\rho V_2)}{\partial x_2}+\frac{\partial(\rho V_3)}{\partial x_3}=0,$$

$$\frac{\partial}{\partial t}(\rho V_1)+V_1\frac{\partial(\rho V_1)}{\partial x_1}+V_2\frac{\partial(\rho V_1)}{\partial x_2}+V_3\frac{\partial(\rho V_1)}{\partial x_3}=-\frac{\partial P}{\partial x_1}+\rho g_1+\rho f_1,$$

$$\frac{\partial}{\partial t}(\rho V_2)+V_1\frac{\partial(\rho V_2)}{\partial x_1}+V_2\frac{\partial(\rho V_2)}{\partial x_2}+V_3\frac{\partial(\rho V_2)}{\partial x_3}=-\frac{\partial P}{\partial x_2}+\rho g_2+\rho f_2,$$

$$\frac{\partial}{\partial t}(\rho V_3)+V_1\frac{\partial(\rho V_3)}{\partial x_1}+V_2\frac{\partial(\rho V_3)}{\partial x_2}+V_3\frac{\partial(\rho V_3)}{\partial x_3}=-\frac{\partial P}{\partial x_3}+\rho g_3+\rho f_3,$$

and substitute the obtained expressions (6), (8-10):

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho W_1)}{\partial y_1}+\frac{\partial(\rho W_2)}{\partial y_2}+\frac{\partial(\rho W_3)}{\partial y_3}=0,$$

$$\frac{\partial}{\partial t}(\rho W_1)+W_1\frac{\partial(\rho W_1)}{\partial y_1}+W_2\frac{\partial(\rho W_1)}{\partial y_2}+W_3\frac{\partial(\rho W_1)}{\partial y_3}=$$
$$=-\frac{\partial P}{\partial y_1}+\rho(cos(\omega t)g_1-sin(\omega t)g_2)+\rho(cos(\omega t)f_1-sin(\omega t)f_2), \qquad (11)$$

$$\frac{\partial}{\partial t}(\rho W_2)+W_1\frac{\partial(\rho W_2)}{\partial y_1}+W_2\frac{\partial(\rho W_2)}{\partial y_2}+W_3\frac{\partial(\rho W_2)}{\partial y_3}=$$

$$-\frac{\partial P}{\partial y_2}+\ \rho(sin(\omega t)g_1+cos(\omega t)g_2)+$$
$$\rho(sin(\omega t)f_1+cos(\omega t)f_2),$$
$$\frac{\partial}{\partial t}(\rho W_3)+W_1\frac{\partial(\rho W_3)}{\partial y_1}+W_2\frac{\partial(\rho W_3)}{\partial y_2}+W_3\frac{\partial(\rho W_3)}{\partial y_3}=$$
$$-\frac{\partial P}{\partial y_3}+\ \rho g_3+\ \rho f_3,$$

where are: $W_1=U_1-\omega y_2$, $W_2=U_2+\omega y_1$, $W_3=U_3$.

Thus, in the new variables, the system of the Navier-Stokes equations has similar to the above (1), (2) form represented in the immovable coordinate system. Therefore, in the appropriate variables the equations remain the same but the right parts contain the volumetric forces written in the new coordinate system (rotating one).

The effect of rotation is hidden in the additive replace of the variable similar to [20, p. 60], when $\boldsymbol{W}=\boldsymbol{U}+[\boldsymbol{\omega}\mathrm{x}\boldsymbol{r}]$.

3.3 Analysis of the Navier-Stokes Equations

3.3.1 Equations for the incompressible liquid

Despite the cumbersome view of the general case equations (11), they become much simpler for a number of specific conditions. For example, for incompressible liquid it yields:

$$\frac{\partial W_1}{\partial y_1}+\frac{\partial W_2}{\partial y_2}+\frac{\partial W_3}{\partial y_3}=0,$$
$$\frac{\partial W_1}{\partial t}+W_1\frac{\partial W_1}{\partial y_1}+W_2\frac{\partial W_1}{\partial y_2}+W_3\frac{\partial W_1}{\partial y_3}=$$
$$=-\frac{1}{\rho}\frac{\partial P}{\partial y_1}+cos(\omega t)g_1-sin(\omega t)g_2+$$
$$cos(\omega t)f_1-sin(\omega t)f_2, \qquad (12)$$
$$\frac{\partial W_2}{\partial t}+W_1\frac{\partial W_2}{\partial y_1}+W_2\frac{\partial W_2}{\partial y_2}+W_3\frac{\partial W_2}{\partial y_3}=$$
$$=-\frac{1}{\rho}\frac{\partial P}{\partial y_2}+sin(\omega t)g_1+cos(\omega t)g_2+$$
$$sin(\omega t)f_1+cos(\omega t)f_2,$$
$$\frac{\partial W_3}{\partial t}+W_1\frac{\partial W_3}{\partial y_1}+W_2\frac{\partial W_3}{\partial y_2}+W_3\frac{\partial W_3}{\partial y_3}=-\frac{1}{\rho}\frac{\partial P}{\partial y_3}+$$
$$g_3+f_3,$$

or in the normal rotating coordinates:

$$\frac{\partial U_1}{\partial y_1}+\frac{\partial U_2}{\partial y_2}+\frac{\partial U_3}{\partial y_3}=0, \qquad (13)$$
$$\frac{\partial U_1}{\partial t}+U_1\frac{\partial U_1}{\partial y_1}+U_2\frac{\partial U_1}{\partial y_2}+U_3\frac{\partial U_1}{\partial y_3}=$$
$$=\omega(y_2\frac{\partial U_1}{\partial y_1}-y_1\frac{\partial U_1}{\partial y_2})+\omega(2U_2+\omega y_1)-\frac{1}{\rho}\frac{\partial P}{\partial y_1}+$$
$$cos(\omega t)g_1-sin(\omega t)g_2+cos(\omega t)f_1-$$
$$sin(\omega t)f_2,$$
$$\frac{\partial U_2}{\partial t}+U_1\frac{\partial U_2}{\partial y_1}+U_2\frac{\partial U_2}{\partial y_2}+U_3\frac{\partial U_2}{\partial y_3}=$$
$$=\omega(y_2\frac{\partial U_2}{\partial y_1}-y_1\frac{\partial U_2}{\partial y_2})+\omega(\omega y_2-2U_1)-\frac{1}{\rho}\frac{\partial P}{\partial y_2}+$$
$$sin(\omega t)g_1+cos(\omega t)g_2+sin(\omega t)f_1+$$
$$cos(\omega t)f_2,$$
$$\frac{\partial U_3}{\partial t}+U_1\frac{\partial U_3}{\partial y_1}+U_2\frac{\partial U_3}{\partial y_2}+U_3\frac{\partial U_3}{\partial y_2}=\omega(y_2\frac{\partial U_3}{\partial y_1}-$$
$$y_1\frac{\partial U_3}{\partial y_2})-\frac{1}{\rho}\frac{\partial P}{\partial y_3}+g_3+f_3,$$

The first peculiarity of the equation array (13) is absence of the stationary solution if any of the gravitational components g_1, g_2 or any other mass force f_1, f_2 is acting in the plane y_1y_2. Only vertical components of these forces do not cause non-stationary flow regimes. The situation is very unusual: any (even constant!) external force in this plane makes the fluid flow regime principally non-stationary.

Now let us consider the system (13) without external forces and take the sum of the three momentum equations, which results in

$$\frac{\partial U_1}{\partial y_1}+\frac{\partial U_2}{\partial y_2}+\frac{\partial U_3}{\partial y_3}=0, \qquad (14)$$
$$\frac{\partial U_s}{\partial t}+U_1\frac{\partial U_s}{\partial y_1}+U_2\frac{\partial U_s}{\partial y_2}+U_3\frac{\partial U_s}{\partial y_3}=$$
$$=\omega\left(y_2\frac{\partial U_s}{\partial y_1}-y_1\frac{\partial U_s}{\partial y_2}\right)+\omega(2U_2-2U_1+\omega(y_1+$$
$$y_2))-\frac{1}{\rho}divP,$$
$$\frac{\partial U_2}{\partial t}+U_1\frac{\partial U_2}{\partial y_1}+U_2\frac{\partial U_2}{\partial y_2}+U_3\frac{\partial U_2}{\partial y_3}=$$
$$=\omega(y_2\frac{\partial U_2}{\partial y_1}-y_1\frac{\partial U_2}{\partial y_2})+\omega(\omega y_2-2U_1)-\frac{1}{\rho}\frac{\partial P}{\partial y_2},$$
$$\frac{\partial U_3}{\partial t}+U_1\frac{\partial U_3}{\partial y_1}+U_2\frac{\partial U_3}{\partial y_2}+U_3\frac{\partial U_3}{\partial y_2}=\omega(y_2\frac{\partial U_3}{\partial y_1}-$$
$$y_1\frac{\partial U_3}{\partial y_2})-\frac{1}{\rho}\frac{\partial P}{\partial y_3},$$

where $U_s=U_1+U_2+U_3$.

In case of $U_s=const$ from the second equation (14) follows

$$divP=\omega\rho(2(U_2-U_1)+\omega(y_1+y_2)), \quad (15)$$

so that pressure in a flow is totally determined by the centrifugal and Coriolis forces independent of the value U_s. Only the position of the point in a flow (y_1+y_2) and the velocity difference U_2-U_1 (not their absolute values!) determine the pressure.

3.3.2 Equations for the compressible liquid

In more general case, the equations (11) for the flow of compressible liquid yield:

$$\frac{\partial\rho}{\partial t}+\rho\left(\frac{\partial U_1}{\partial y_1}+\frac{\partial U_2}{\partial y_2}+\frac{\partial U_3}{\partial y_3}\right)+U_1\frac{\partial\rho}{\partial y_1}+U_2\frac{\partial\rho}{\partial y_2}+U_3\frac{\partial\rho}{\partial y_3}=\omega(y_2\frac{\partial\rho}{\partial y_1}-y_1\frac{\partial\rho}{\partial y_2}),$$

$$\rho(\frac{\partial U_1}{\partial t}+U_1\frac{\partial U_1}{\partial y_1}+U_2\frac{\partial U_1}{\partial y_2}+U_3\frac{\partial U_1}{\partial y_3})=(\omega y_2-U_1)\frac{\partial\rho}{\partial t}-(\omega y_2-U_1)^2\frac{\partial\rho}{\partial y_1}+(\omega y_2-U_1)(U_2+\omega y_1)\frac{\partial\rho}{\partial y_2}+(\omega y_2-U_1)U_3\frac{\partial\rho}{\partial y_3}+\omega\rho(y_2\frac{\partial U_1}{\partial y_1}-\frac{\partial U_1}{\partial y_2})+\omega\rho(2U_2+\omega y_1)-\frac{\partial P}{\partial y_1}+\rho(cos(\omega t)g_1-sin(\omega t)g_2+cos(\omega t)f_1-sin(\omega t)f_2), \quad (16)$$

$$\rho(\frac{\partial U_2}{\partial t}+U_1\frac{\partial U_2}{\partial y_1}+U_2\frac{\partial U_2}{\partial y_2}+U_3\frac{\partial U_2}{\partial y_3})=$$
$$=-(U_2+\omega y_1)\frac{\partial\rho}{\partial t}+(\omega y_2-U_1)(U_2+\omega y_1)\frac{\partial\rho}{\partial y_1}-(U_2+\omega y_1)^2\frac{\partial\rho}{\partial y_2}-(U_2+\omega y_1)U_3\frac{\partial\rho}{\partial y_3}+\omega\rho(y_2\frac{\partial U_2}{\partial y_1}-y_1\frac{\partial U_2}{\partial y_2})+\omega\rho(\omega y_2-2U_1)-\frac{\partial P}{\partial y_2}+\rho(sin(\omega t)g_1+cos(\omega t)g_2+sin(\omega t)f_1+cos(\omega t)f_2),$$

$$\rho\left(\frac{\partial U_3}{\partial t}+U_1\frac{\partial U_3}{\partial y_1}+U_2\frac{\partial U_3}{\partial y_2}+U_3\frac{\partial U_3}{\partial y_2}\right)=-U_3\frac{\partial\rho}{\partial t}+(\omega y_2-U_1)U_3\frac{\partial\rho}{\partial y_1}-(U_2+\omega y_1)U_3\frac{\partial\rho}{\partial y_2}-U_3{}^2\frac{\partial\rho}{\partial y_3}+\omega\rho(y_2\frac{\partial U_3}{\partial y_1}-y_1\frac{\partial U_3}{\partial y_2})-\frac{\partial P}{\partial y_3}+\rho(g_3+f_3).$$

As shows analysis of the obtained equation array (16), in contrast to (13), for the compressible liquid, the stationary flow regime is impossible even in absence of any external mass forces because of the term $\frac{\partial\rho}{\partial t}$ present in all 3 momentum equations, except very specific case when this term is zero and there is only the spatial density distribution.

The flow regime (15) is unique for the incompressible flow and is not available for compressible one as clearly seen from the (16).

The first equation of the system (16) contains the "density vortex" to the right, which shows that the higher is rotation speed, the more intensive is density variation by time and space. This, in turn, may cause further growing of the "density vortex", and so on. By the high rotation speed despite the small coordinate values the term $\omega(y_2\frac{\partial\rho}{\partial y_1}-y_1\frac{\partial\rho}{\partial y_2})$ can be big due to substantial density gradients, which may grow with time creating the singularity point in the centre of rotation.

If we present the first equation in the form

$$\omega=\frac{\frac{\partial\rho}{\partial t}+\rho\left(\frac{\partial U_1}{\partial y_1}+\frac{\partial U_2}{\partial y_2}+\frac{\partial U_3}{\partial y_3}\right)+U_1\frac{\partial\rho}{\partial y_1}+U_2\frac{\partial\rho}{\partial y_2}+U_3\frac{\partial\rho}{\partial y_3}}{(y_2\frac{\partial\rho}{\partial y_1}-y_1\frac{\partial\rho}{\partial y_2})}, \quad (17)$$

we can consider it as the equation for the vortex formation in the compressible fluid flow due to the density spatial gradients. What is very interesting from this correlation that even in case of zero velocities and zero velocity gradients, the vortex formation can start just due to density gradients: $\frac{\partial\rho}{\partial t}$, $\frac{\partial\rho}{\partial y_1}$, $\frac{\partial\rho}{\partial y_2}$. The signs of the terms $\frac{\partial\rho}{\partial t}$ and $y_2\frac{\partial\rho}{\partial y_1}-y_1\frac{\partial\rho}{\partial y_2}$ determine the direction of the rotation.

3.4 Vortex flow creation and evolution

3.4.1 Conditions for vortex creation

It is interesting to underline that zero value of the term $(y_2\frac{\partial\rho}{\partial y_1}-y_1\frac{\partial\rho}{\partial y_2})$ in the equation (17) excludes influence of the rotation in the first equation (16), while any small values of this term in (17) may create the reason for an abrupt grow of the rotation (vortex) in a compressible fluid, which may start a vortex flow in the immovable liquid or create a vortex in the fluid flow.

For example, let us consider the vortex birth in a volume of liquid or gas being initially in the rest. From (17) follows

$$\omega=\frac{\frac{\partial\rho}{\partial t}}{(y_2\frac{\partial\rho}{\partial y_1}-y_1\frac{\partial\rho}{\partial y_2})}, \quad (18)$$

which shows that the vortex cannot be created in case of symmetrical density variation around some point. Naturally such conditions may happen due to abrupt local heating causing the remarkable variation of the liquid or gas density in time and space e.g. from a solar radiation concentrated in local region.

Then assuming that gravitation force is acting in the vertical direction and substituting (18) into (16) yields:

$$\frac{\partial\rho}{\partial t}+\rho\left(\frac{\partial U_1}{\partial y_1}+\frac{\partial U_2}{\partial y_2}+\frac{\partial U_3}{\partial y_3}\right)+U_1\frac{\partial\rho}{\partial y_1}+U_2\frac{\partial\rho}{\partial y_2}+U_3\frac{\partial\rho}{\partial y_3}=\omega(y_2\frac{\partial\rho}{\partial y_1}-y_1\frac{\partial\rho}{\partial y_2}),$$

$$\rho(\frac{\partial U_1}{\partial t}+U_1\frac{\partial U_1}{\partial y_1}+U_2\frac{\partial U_1}{\partial y_2}+U_3\frac{\partial U_1}{\partial y_3})=(\omega y_2-U_1)\frac{\partial\rho}{\partial t}-(\omega y_2-U_1)^2\frac{\partial\rho}{\partial y_1}+(\omega y_2-U_1)(U_2+\omega y_1)\frac{\partial\rho}{\partial y_2}+(\omega y_2-U_1)U_3\frac{\partial\rho}{\partial y_3}+\omega\rho(y_2\frac{\partial U_1}{\partial y_1}-y_1\frac{\partial U_1}{\partial y_2})+\omega\rho(2U_2+\omega y_1)-\frac{\partial P}{\partial y_1}, \quad (19)$$

$$\rho(\frac{\partial U_2}{\partial t}+U_1\frac{\partial U_2}{\partial y_1}+U_2\frac{\partial U_2}{\partial y_2}+U_3\frac{\partial U_2}{\partial y_3})=$$

$$= -(U_2 + \omega y_1)\frac{\partial \rho}{\partial t} + (\omega y_2 - U_1)(U_2 + \omega y_1)\frac{\partial \rho}{\partial y_1} - (U_2 + \omega y_1)^2\frac{\partial \rho}{\partial y_2} - (U_2 + \omega y_1)U_3\frac{\partial \rho}{\partial y_3} + \omega\rho(y_2\frac{\partial U_2}{\partial y_1} - y_1\frac{\partial U_2}{\partial y_2}) + \omega\rho(\omega y_2 - 2U_1) - \frac{\partial P}{\partial y_2},$$

$$\rho\left(\frac{\partial U_3}{\partial t} + U_1\frac{\partial U_3}{\partial y_1} + U_2\frac{\partial U_3}{\partial y_2} + U_3\frac{\partial U_3}{\partial y_2}\right) = -U_3\frac{\partial \rho}{\partial t} + (\omega y_2 - U_1)U_3\frac{\partial \rho}{\partial y_1} - (U_2 + \omega y_1)U_3\frac{\partial \rho}{\partial y_2} - {U_3}^2\frac{\partial \rho}{\partial y_3} + \omega\rho(y_2\frac{\partial U_3}{\partial y_1} - y_1\frac{\partial U_3}{\partial y_2}) - \frac{\partial P}{\partial y_3} + \rho g_3.$$

where the initial vortex birth is supposed to the stated initial density gradients according to the equation (18). Then the equation array (19) is solved for the flow field and density distribution. For the gas flow it is added the equation of state to close the system (19), for example: $P = \rho RT$, R=286 J/(kg*K)- the universal gas constant for the air.

The condition (18) can be considered as the origin for the vortex flow formation by different density variation at the initial time. And then solve the equation array (19) under simplification that at the initial moment the vortex formation is starting as the two-dimensional process. The equations (19) yield:

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial U_1}{\partial y_1} + \frac{\partial U_2}{\partial y_2}\right) + U_1\frac{\partial \rho}{\partial y_1} + U_2\frac{\partial \rho}{\partial y_2} = \omega(y_2\frac{\partial \rho}{\partial y_1} - y_1\frac{\partial \rho}{\partial y_2}),$$

$$\rho(\frac{\partial U_1}{\partial t} + U_1\frac{\partial U_1}{\partial y_1} + U_2\frac{\partial U_1}{\partial y_2}) = (\omega y_2 - U_1)\frac{\partial \rho}{\partial t} - (\omega y_2 - U_1)^2\frac{\partial \rho}{\partial y_1} + (\omega y_2 - U_1)(U_2 + \omega y_1)\frac{\partial \rho}{\partial y_2} + \omega\rho(y_2\frac{\partial U_1}{\partial y_1} - y_1\frac{\partial U_1}{\partial y_2}) + \omega\rho(2U_2 + \omega y_1) - \frac{\partial P}{\partial y_1}, \qquad (20)$$

$$\rho(\frac{\partial U_2}{\partial t} + U_1\frac{\partial U_2}{\partial y_1} + U_2\frac{\partial U_2}{\partial y_2}) =$$

$$= -(U_2 + \omega y_1)\frac{\partial \rho}{\partial t} + (\omega y_2 - U_1)(U_2 + \omega y_1)\frac{\partial \rho}{\partial y_1} - (U_2 + \omega y_1)^2\frac{\partial \rho}{\partial y_2} + \omega\rho(y_2\frac{\partial U_2}{\partial y_1} - y_1\frac{\partial U_2}{\partial y_2}) + \omega\rho(\omega y_2 - 2U_1) - \frac{\partial P}{\partial y_2}.$$

The boundary problem (18), (20) was solved numerically in the region circle of the given radius a with the following initial and boundary conditions:

$$t = 0,\ \frac{\partial \rho}{\partial t} = d\rho 0, \frac{\partial \rho}{\partial y_1} = d\rho 1, \frac{\partial \rho}{\partial y_2} = d\rho 2,$$

$$U_i = 0,\ \omega = \omega_0 = \frac{d\rho 0}{a(d\rho 1 - d\rho 2)}; \qquad (21)$$

$$y_2 = y_1 = a,\ U_1 = U_2 = 0. \qquad (22)$$

3.4.2 Numerical simulation of the vortex creation

The numerical solution of the boundary problem (20)-(22) is presented below in Figs 5-10:

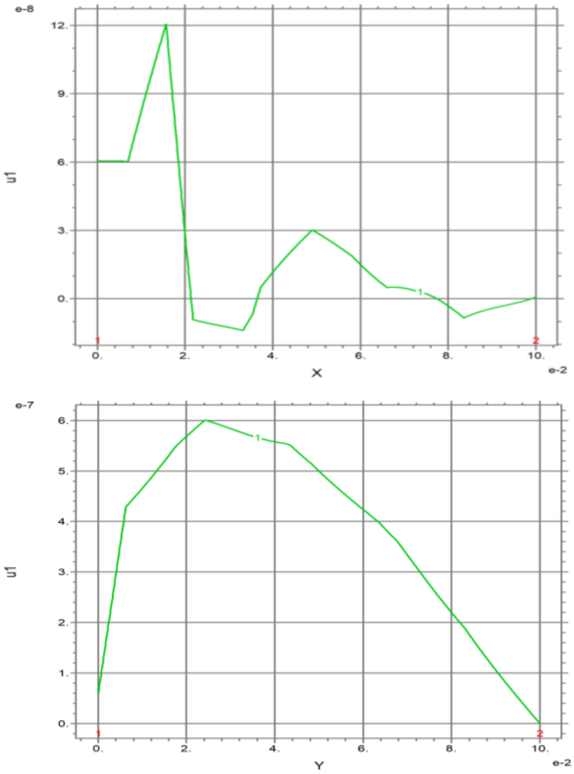


Fig. 5 Velocity field u_1 by y_1 y_2 (X,Y) at t=1

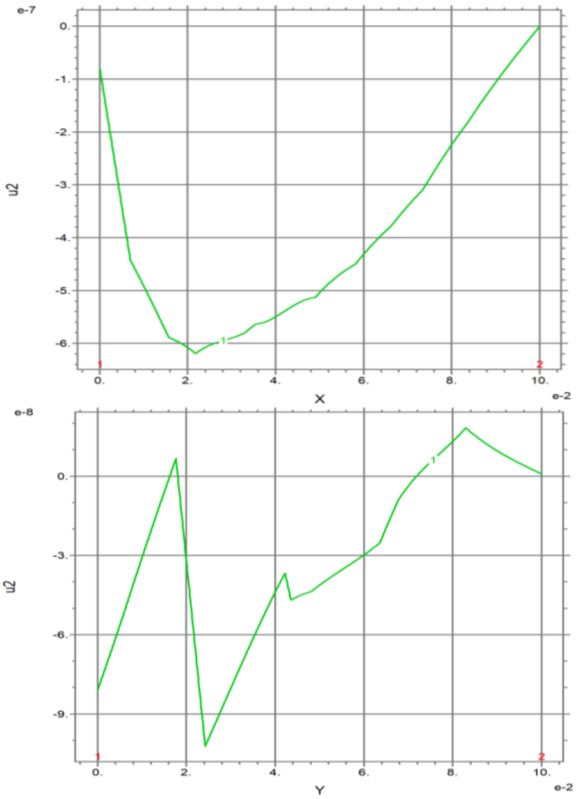


Fig. 6 Velocity field u_2 by y_1 y_2 (X,Y) at t=1

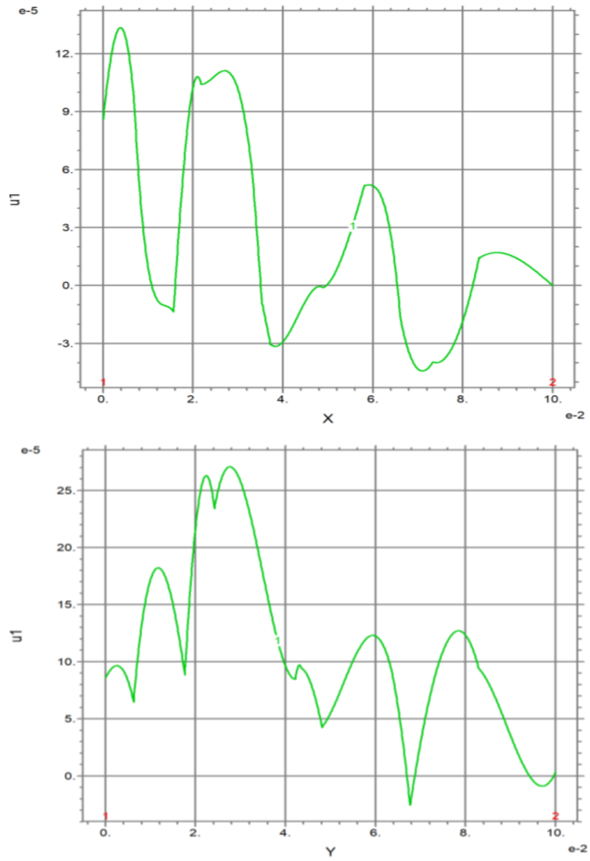


Fig. 7 Velocity field u_1 by y_1 y_2 (X,Y) at t=10

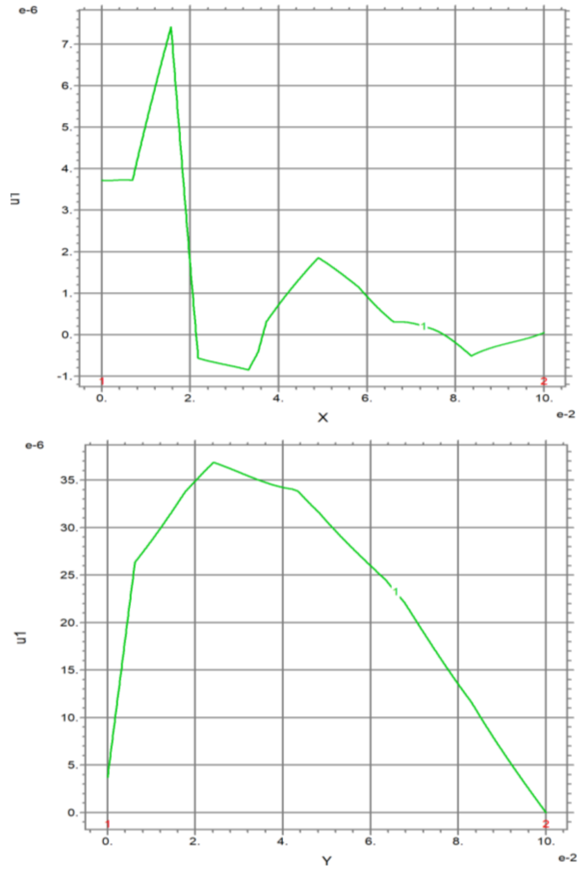


Fig. 9 Velocity field u_1 by y_1 y_2 (X,Y) at t=60

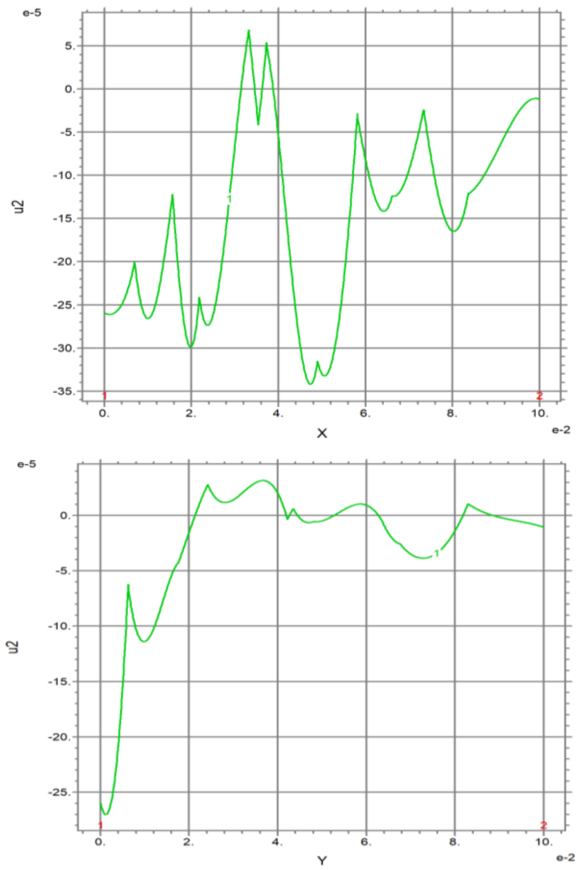


Fig. 8 Velocity field u_2 by y_1 y_2 (X,Y) at t=10

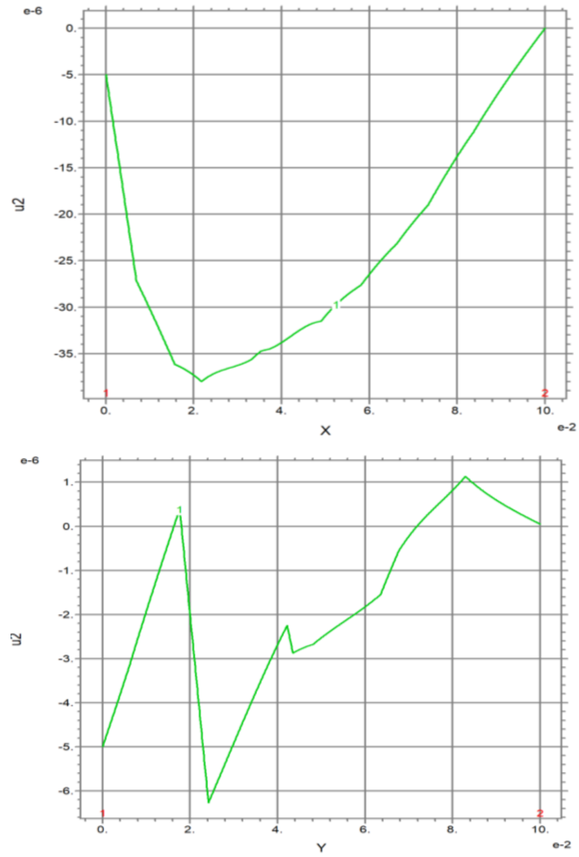


Fig. 10 Velocity field u_2 by y_1 y_2 (X,Y) at t=60

It was revealed high sensitivity to the fidelity of the initial data. A few results are presented in Figs 5, 6 for the initial moment *t*=1.

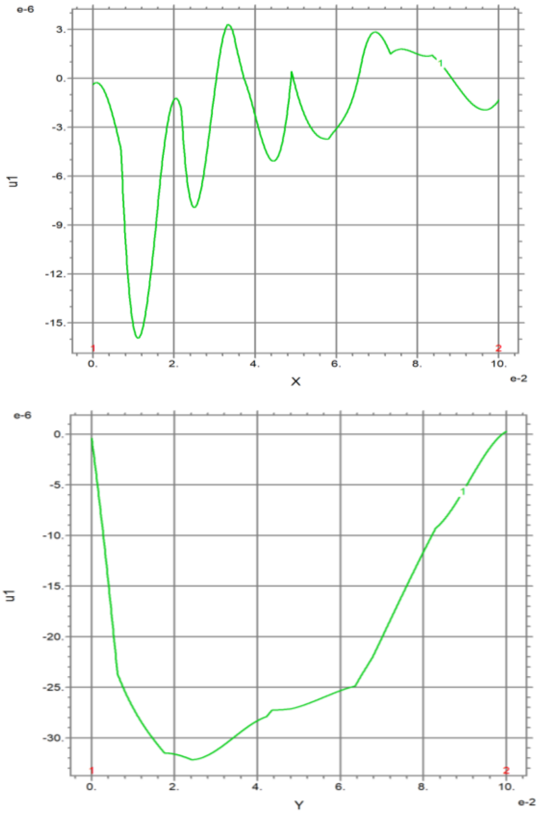


Fig. 11 Velocity field u_1 by y_1 y_2 (X,Y) at *t*=600

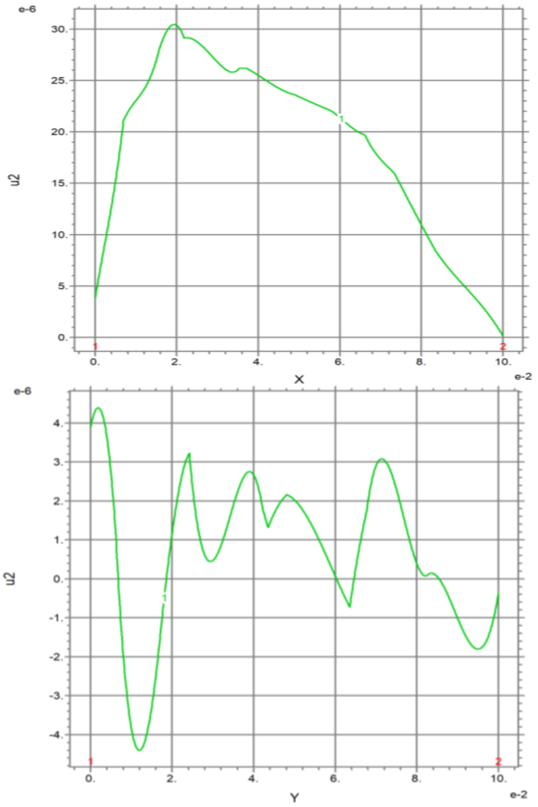


Fig. 12 Velocity field u_2 by y_1 y_2 (X,Y) at *t*=600

The density and pressure fields do not change during this small time interval from zero to *t*=1, as well as up to *t*=60, as clearly shown from the data presented in Figs 7, 8 and Figs 9, 10 for the intervals *t*=10 and *t*=60, respectively. The velocities are growing since starting point but at *t*=60 they are decreasing. Then flow is gradually decreasing as shown in Figs 11, 12 for the *t*=600 and Figs 13, 14 for the *t*=3600.

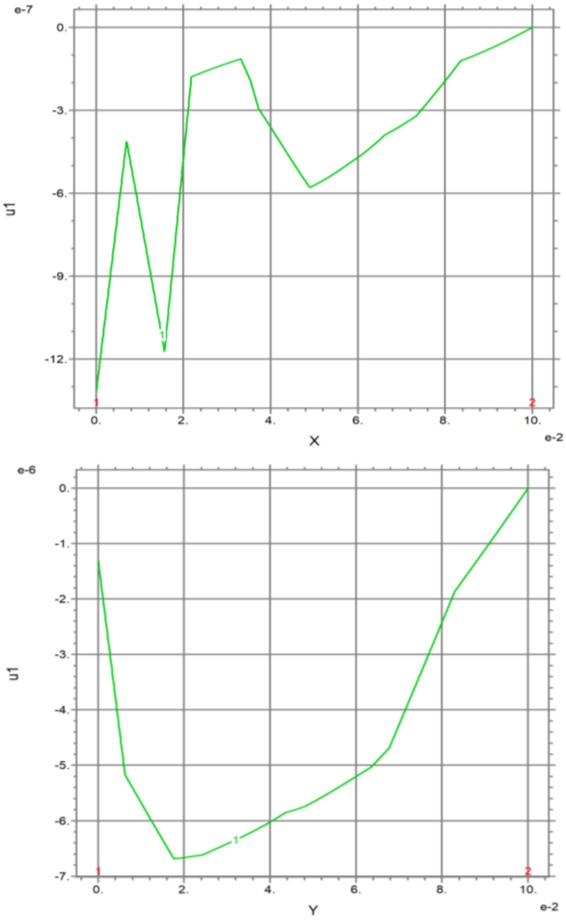
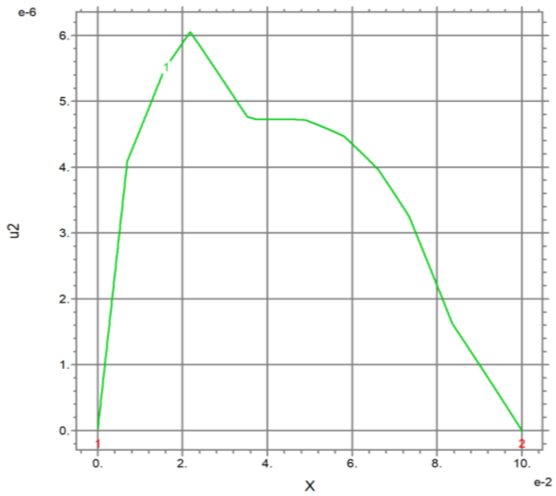


Fig. 13 Velocity field u_1 by y_1 y_2 (X,Y) at *t*=3600



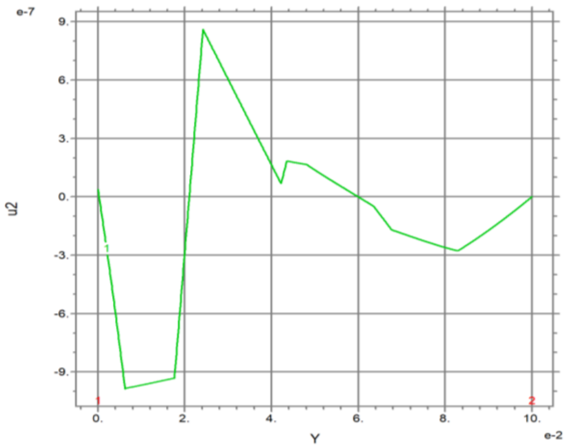


Fig. 14 Velocity field u_2 by y_1 y_2 (X,Y) at t=3600

Remarkably during all process the density and pressure are not prone to any changes.

4 Conclusion

Mathematical model obtained allowed studying the rotational flows. The inertial rotational coordinate system was applied and the Navier-Stokes equations were analyzed. The computer simulation for the air rotational flow created by the initial density gradients has been done using the gas equation of state by the temperature 300K. It revealed starting the slow flow in a circle region of 10 sm radius under the initial density gradients by time 0.01K/s, and 0.25K/m, 0.05K/m by coordinates, which is growing up to the flow velocities about 0.25-0.35 mm/s and then is gradually fading after t=600 s. This is the first attempts to reveal the features of the rotational flows using the inertial coordinates systems, which will be continued in the future study.

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