Static and Free Vibration Analysis of Planar Curved Composite Beams on Elastic Foundation

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Abstract: - In this study, static and free vibration analysis of planar curved composite Timoshenko beam on elastic foundation is investigated via mixed finite element formulation. Torsional rigidity of composite cross-sections is determined over warping. In the analysis, two-nodded curved element is used with 24 degrees of freedom. At each node, the unknowns are three translations, three rotations, two shear forces, one axial force, two bending moment and one torque. First, the numerical calculation of the torsional rigidity is verified with the literature and the commercial finite element program results SAP2000 and ANSYS. Next, static and free vibration analysis of planar curved Timoshenko beam with (Winkler foundation) and without foundation is verified with the results available in the literature. In the case of Winkler foundation, the rocking effect is considered. Finally, as an original example, planar curved composite beams resting on Pasternak foundation are analyzed (static and free vibration) using the present mixed finite element formulation.

Key-Words: - Timoshenko beam theory, curved composite beam, elastic foundation, finite element method

1 Introduction

Beam structures are widely used in several fields such as, defense, transportation, aerospace structures *etc*. Curved beams are also preferred in many engineering fields due to architectural or structural reasons, besides straight beams. The literature on vibration of planar curved beams, rings, and arches is reviewed in [1]. A literature review on the vibration of straight and curved composite beams between 1989 and 2012 is given by [2].

[3] presented elastic and viscoelastic foundation models. [4] studied the natural out-of-plane vibrations curved beams on elastic foundation. [5] formulated a new horizontally curved three-noded isoparametric beam element with or without elastic foundation and applied the curved beam and ring problems. [6] and [7] presented the free vibration problem of curved beams resting on Winkler and Pasternak foundation, respectively. [8] studied the static and free vibration of circular rings on tensionless Winkler foundation. [9] derived the governing differential equations for free vibrations of shear deformable curved beams on Winkler foundation and solved numerically. [10] investigated the natural frequencies of non-symmetric thin-walled curved beams on Winkler and Pasternak type foundations. Static and free vibration analysis of straight and curved Timoshenko beams on elastic foundation is studied in [11]. [12] presented a static solution for space curved beams on Winkler foundation using transfer matrix method. By using the Hamiltonian structural analysis method, the static analysis of curved Timoshenko beams with or without generalized two-parameter elastic foundation are carried out by [13]. [14] investigated the flexural behaviour of a curved orthotropic beam on elastic foundation. The free vibration analysis of functionally graded circular curved beams resting on elastic foundation is presented by using the differential quadrature method in [15].

In order to handle Saint-Venant torsion problem for irregular cross-sections and non-homogenous materials it is necessary to use numerical approximate solutions. The Saint-Venant torsion problem can numerically be calculated by either by Prandtl stress function or by means of the warping function approach. As an incomplete list of finite element solutions, there exists displacement type elements by [16-21], stress function solution by [22-24], hybrid model by [25] and mixed type elements by [26-27], homogenized formulation by Nouri and Gay 1994 [28]. In the case of laminates and thin-walled sections and multiply-connected sections, the warping function approach by finite elements is much simpler [29]. In the literature for the Saint-

Venant torsion problem, there exist some other numerical approaches, namely, the boundary element method [30-37], the finite difference method [38-39].

In this study, the mixed finite element method (MFEM) is utilized in the static and free vibration analysis of planar curved composite beams resting on elastic foundation (Winkler and Pasternak). Timoshenko beam theory considers the shear influence and the rotary inertia (in dynamic analysis). The constitutive equations of layered orthotropic beams are derived by reducing the constitutive relations of orthotropic materials for three-dimensional body [40]. First, the torsional rigidity of composite cross-sections is calculated, and the results are verified with the literature and the commercial program results SAP2000 and ANSYS. Next, static and free vibration of composite semicircular curved beams is analyzed and the results are compared with the commercial program results ANSYS. The influence of elastic foundation on the natural frequencies of isotropic curved beam having the central angle of the arc on the plane of the elastic foundation is investigated and results are compared with the literature [11]. Also, the contribution of the rocking influence is considered. Finally, as an original problem, the static and free vibration analysis of the curved composite beam on Winkler and Pasternak foundation is handled as a contribution to the literature.

2 Formulation

2.1 Torsional rigidity

Torsional rigidity of the composite cross-section is determined over warping by using finite element formulation in [29].

2.2 The constitutive relations for composites

The constitutive equation yields

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon} \tag{1}$$

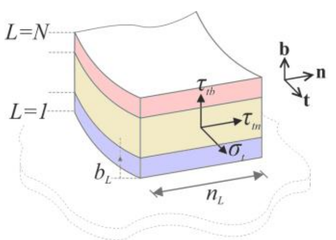

Fig.1 The stresses in the Frenet Coordinate System (N: Total number of layers)

$\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ is the strain tensor and $\mathbf{E}$ is the function of elastic constants. In order to derive the constitutive equations of a composite beam, firstly the assumptions made on stress, in accordance with beam geometry [41], secondly some reductions made on the constitutive relation of orthotropic materials for the three dimensional body by incorporating the Poisson's ratio [40].

In Frenet coordinate system (see Fig.1), paying attention to $\sigma_n = \sigma_b = \tau_{nb} = 0$, the constitutive relations yield

$$\begin{Bmatrix} \sigma_t \\ \tau_{bt} \\ \tau_{tn} \end{Bmatrix} = [\boldsymbol{\beta}] \begin{Bmatrix} \varepsilon_t \\ \gamma_{bt} \\ \gamma_{tn} \end{Bmatrix} \tag{2}$$

In (2), $[\boldsymbol{\beta}]_{3\times3}$ matrix is the function of orthotropic material constants. Timoshenko beam theory requires shear correction factors and it is assumed to be $5/6$ for a general rectangular cross-section. By means of the kinematic equations

$$\begin{aligned} u_t^* &= u_t + b\,\Omega_n - n\,\Omega_b \\ u_n^* &= u_n - b\,\Omega_t \\ u_b^* &= u_b + n\,\Omega_t \end{aligned} \tag{3}$$

u_t^*, u_n^*, u_b^* are displacements at the beam continuum and u_t, u_n, u_b are displacements on the beam axis and Ω_t, Ω_n and Ω_b present the rotations of the beam cross-section around the *t*, *n* and *b* Frenet coordinates, respectively. The strains which are derived from (3)

$$\begin{Bmatrix} \varepsilon_t \\ \gamma_{bt} \\ \gamma_{tn} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_t}{\partial t} \\ \dfrac{\partial u_t}{\partial b} + \dfrac{\partial u_b}{\partial t} \\ \dfrac{\partial u_t}{\partial n} + \dfrac{\partial u_n}{\partial t} \end{Bmatrix} + b \begin{Bmatrix} \dfrac{\partial \Omega_n}{\partial t} \\ 0 \\ -\dfrac{\partial \Omega_t}{\partial t} \end{Bmatrix} + n \begin{Bmatrix} -\dfrac{\partial \Omega_b}{\partial t} \\ \dfrac{\partial \Omega_t}{\partial t} \\ 0 \end{Bmatrix} \tag{4}$$

By obtaining strains for beam geometry due to displacements [42], the forces and moments for a layer can be derived by analytical integration of the stresses in each layer through the thickness of the cross-section, respectively.

$$T_t = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} \sigma_t \mathrm{d}b \right) \mathrm{d}n \right) \tag{5}$$

$$T_b = \sum_{L=1}^{N} \left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} \tau_{bt} \mathrm{d}b \right) \mathrm{d}n \right) \tag{6}$$

$$T_n = \sum_{L=1}^{N}\left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} \tau_{tn} \mathrm{d}b \right) \mathrm{d}n \right) \tag{7}$$

$$\begin{aligned} M_t = &\sum_{L=1}^{N}\left(-\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} b\,\tau_{tn} \mathrm{d}b \right) \mathrm{d}n \right) \\ &+ \sum_{L=1}^{N}\left(\int_{b_{L-1}}^{b_L} \left(\int_{-0.5n_L}^{0.5n_L} n\,\tau_{tb} \mathrm{d}n \right) \mathrm{d}b \right) \end{aligned} \tag{8}$$

$$M_n = \sum_{L=1}^{N}\left(\int_{-0.5n_L}^{0.5n_L} \left(\int_{b_{L-1}}^{b_L} b\sigma_t \mathrm{d}b \right) \mathrm{d}n \right) \tag{9}$$

$$M_b = -\sum_{L=1}^{N}\left(\int_{b_{L-1}}^{b_L} \left(\int_{-0.5n_L}^{0.5n_L} n\sigma_t \mathrm{d}n \right) \mathrm{d}b \right) \tag{10}$$

N is the number of the layer, n_L is the width of the layer, b_L and b_{L-1} are the directed distances to the bottom and the top of the L^{th} layer where b is positive upward. The constitutive equation in a matrix form:

$$\begin{Bmatrix} T_t \\ T_n \\ T_b \\ M_t \\ M_n \\ M_b \end{Bmatrix} = \sum_{L=1}^{N} \begin{bmatrix} \mathbf{E}_m^L & \mathbf{E}_{mf}^L \\ \mathbf{E}_{fm}^L & \mathbf{E}_f^L \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_t}{\partial t} \\ \dfrac{\partial u_t}{\partial n} + \dfrac{\partial u_n}{\partial t} \\ \dfrac{\partial u_t}{\partial b} + \dfrac{\partial u_b}{\partial t} \\ \dfrac{\partial \Omega_t}{\partial t} \\ \dfrac{\partial \Omega_n}{\partial t} \\ \dfrac{\partial \Omega_b}{\partial t} \end{Bmatrix} \tag{11}$$

or, since $[\mathbf{C}] = [\mathbf{E}]^{-1}$, in accordance with (2) and (4), (11) yields to the form

$$\begin{Bmatrix} \varepsilon_t \\ \gamma_{tn} \\ \gamma_{bt} \\ \kappa_t \\ \kappa_n \\ \kappa_b \end{Bmatrix} = \begin{bmatrix} \mathbf{C}_m & \mathbf{C}_{mf} \\ \mathbf{C}_{fm} & \mathbf{C}_f \end{bmatrix} \begin{Bmatrix} T_t \\ T_n \\ T_b \\ M_t \\ M_n \\ M_b \end{Bmatrix} \tag{12}$$

$\kappa_t, \kappa_n, \kappa_b$ are curvatures.

2.3 The field equations and functional

In Frenet coordinate system, the field equations and functional for the isotropic homogenous spatial Timoshenko beam exist in [43,44,21]. The field equations and the functional are extended to laminated composite beams in [45-46]. Winkler and Pasternak foundation terms inserted to the field equations of spatial beam in [47]. The foundation rocking terms are inserted to the field equations of spatial beam as follows

$$\left.\begin{aligned} -\mathbf{T}_{,s} - \mathbf{q} + (\mathbf{k}_W)^T \mathbf{u} - (\mathbf{k}_P)^T \mathbf{u}_{,ss} + \rho A \ddot{\mathbf{u}} = \mathbf{0} \\ -\mathbf{M}_{,s} - \mathbf{t} \times \mathbf{T} - \mathbf{m} + (\mathbf{k}_R)^T \mathbf{\Omega} + \rho \mathbf{I} \ddot{\mathbf{\Omega}} = \mathbf{0} \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} \mathbf{u}_{,s} + \mathbf{t} \times \mathbf{\Omega} - \mathbf{C}_m \mathbf{T} - \mathbf{C}_{mf} \mathbf{M} = \mathbf{0} \\ \mathbf{\Omega}_{,s} - \mathbf{C}_{fm} \mathbf{T} - \mathbf{C}_f \mathbf{M} = \mathbf{0} \end{aligned}\right\} \tag{14}$$

s is the arc axis of the spatial beam, $\mathbf{u}(u_t, u_n, u_b)$ is the displacement vector, $\mathbf{\Omega}(\Omega_t, \Omega_n, \Omega_b)$ is the cross section rotation vector. $\mathbf{k}_W(k_{Wt}, k_{Wn}, k_{Wb})$ and $\mathbf{k}_P(k_{Pt}, k_{Pn}, k_{Pb})$ are foundation vectors of Winkler and Pasternak, respectively. $\mathbf{k}_R(k_{Rt}, k_{Rn}, k_{Rb})$ is foundation rocking stiffness vector. $\ddot{\mathbf{u}}$ and $\ddot{\mathbf{\Omega}}$ are the accelerations of the displacement and rotations, $\mathbf{T}(T_t, T_n, T_b)$ defines the force vector, $\mathbf{M}(M_t, M_n, M_b)$ is the moment vector, ρ is the material density. A is the area of the cross section, $\mathbf{I}$ stores the moments of inertia, $\mathbf{C}_m$, $\mathbf{C}_f$, $\mathbf{C}_{mf}$ and $\mathbf{C}_{fm}$ are compliance matrices where $\mathbf{C}_{mf}$, $\mathbf{C}_{fm}$ are coupling matrices [48]. $\mathbf{q}$ and $\mathbf{m}$ are the distributed external force and moment vectors, respectively. Once the motion is considered as harmonic for the free vibration of the beam, the conditions $\mathbf{q} = \mathbf{m} = \mathbf{0}$ are satisfied. Incorporating Gateaux differential in terms of (13)-(14) with potential operator concept [49] yields to the following functional.

$$\begin{aligned} \mathbf{I}(\mathbf{y}) = &-[\mathbf{u}, \mathbf{T}_{,s}] - [\mathbf{M}_{,s}, \mathbf{\Omega}] + [\mathbf{t} \times \mathbf{\Omega}, \mathbf{T}] - \tfrac{1}{2}\{[\mathbf{C}_m \mathbf{T}, \mathbf{T}] \\ &+ [\mathbf{C}_{mf} \mathbf{M}, \mathbf{T}] + [\mathbf{C}_{fm} \mathbf{T}, \mathbf{M}] + [\mathbf{C}_f \mathbf{M}, \mathbf{M}]\} \\ &- \tfrac{1}{2}\rho A \omega^2 [\mathbf{u}, \mathbf{u}] - \tfrac{1}{2}\rho \omega^2 [\mathbf{I}\mathbf{\Omega}, \mathbf{\Omega}] + \tfrac{1}{2}[(\mathbf{k}_W)^T \mathbf{u}, \mathbf{u}] \\ &+ \tfrac{1}{2}[(\mathbf{k}_P)^T \mathbf{u}_{,s}, \mathbf{u}_{,s}] + \tfrac{1}{2}[(\mathbf{k}_R)^T \mathbf{\Omega}, \mathbf{\Omega}] \\ &+ [(\mathbf{T} - \hat{\mathbf{T}}), \mathbf{u}]_\sigma + [(\mathbf{M} - \hat{\mathbf{M}}), \mathbf{\Omega}]_\sigma - [\mathbf{k}_P \hat{\mathbf{u}}_{,s}, \mathbf{u}]_\sigma \\ &+ [\hat{\mathbf{u}}, \mathbf{T}]_\varepsilon + [\hat{\mathbf{\Omega}}, \mathbf{M}]_\varepsilon - [\mathbf{k}_P \mathbf{u}_{,s}, (\mathbf{u} - \hat{\mathbf{u}})]_\varepsilon \end{aligned} \tag{15}$$

For a static analysis, the above functional needs to be modified by excluding the terms $\tfrac{1}{2}\rho A \omega^2 [\mathbf{u}, \mathbf{u}]$, $\tfrac{1}{2}\rho \omega^2 [\mathbf{I}\mathbf{\Omega}, \mathbf{\Omega}]$ and inserting $[\mathbf{q}, \mathbf{u}]$, $[\mathbf{m}, \mathbf{\Omega}]$. In (15), the square brackets indicate the inner product, the terms with hats are known values on the boundary and the subscripts ε and σ represent the geometric and dynamic boundary conditions, respectively.

2.4 Mixed finite element formulation

The linear shape functions are used in the finite element formulation. The curvatures are satisfied exactly at the nodal points and linearly interpolated through the element [44]. Calculation of the natural free vibration frequencies of a structural system yields to the following standard eigenvalue problem,

$$\left([\mathbf{K}]-\omega^2[\mathbf{M}]\right)\{\mathbf{u}\}=\{\mathbf{0}\} \tag{16}$$

where, $[\mathbf{K}]$ and $[\mathbf{M}]$ are the system and mass matrix of the entire domain, respectively. $\mathbf{u}$ is the eigenvector (mode shape) and ω depicts the natural angular frequency of the system.

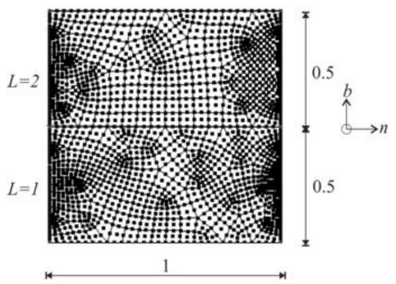

Fig.1 The square composite cross-section with 448 nine-node quadrilateral mesh elements.

3 Numerical Examples

3.1 Calculation of torsional rigidity

A new computer program based on the finite element (FE) formulation is developed using FORTRAN language in order to calculate the torsional rigidity of composite sections by the approach given in [29]. [29] considers the warping of the cross-sections. The torsional rigidities of an isotropic and composite square cross-section with two layers are obtained. The results are compared by the literature, SAP2000 and ANSYS in order to verify our developed FE program.

The non-dimensional width of the cross-section and the bottom layer's shear modulus are unity. The torsional rigidities of the section are obtained for top layer's shear modulus 1, 2, 3, respectively. The developed FE program is used with 448 nine-node quadratic quadrilateral mesh elements on the cross-section (Fig.1). The results are compared by [39,50], SAP2000 and ANSYS and given in Table 1. For ANSYS solution a $35\,\text{m}$ long composite solid beam under 0.01Nm torque is used. Element size of the mesh is $0.2\,\text{m}$.

Table 1. The torsional rigidities of a square composite cross-section

	GI_t			Diff. %		
G_2/G_1	1	2	3	1	2	3
This Study	0.1406	0.1970	0.2395			
[50]	0.1406	0.1970	0.2394	0.00	0.00	0.04
[39]	0.1388	0.1941	0.2358	1.28	1.47	1.54
SAP2000	0.1406	0.1970	0.2395	0.00	0.00	0.00
ANSYS	0.1402	0.1963	0.2385	0.28	0.36	0.42

3.2 Free vibration analysis of a planar curved beam on Winkler foundation

The fixed-fixed boundary condition is used. The material and geometric properties of the beam are: the modulus of elasticity is $E=47.24\,\text{GPa}$, Poisson's ratio is $\upsilon=0.2$, the density of material is $\rho=5000\,\text{kg/m}^3$, the radius of curved beam is $R=7.63\,\text{m}$, the dimensions of rectangular cross-section are $b=h=0.762\,\text{m}$. The component of Winkler foundation constant in the direction b is $k_{Wb}=23.623\,\text{MPa}$, the foundation rocking stiffness constant in the direction t is $k_{Rt}=1143\,\text{kNm/m}$. The first five natural frequencies for a curved beam on Winkler foundation having various opening angles θ ($45°, 90°, 135°, 180°, 225°, 270°$) are calculated and the results are tabulated together with the literature results [11] in Table 2. ANSYS results also exist in [11]. MFEM results determined using 80 mixed FEs are a good agreement with [11].

As the opening angles of the planar curved beam on elastic foundation increase, a reduction in the natural frequencies of the curved beam beams is observed. If the fundamental natural frequencies in each opening angles θ are compared with respect to the results of $\theta=45°$, the percent reduction for the cases $\theta=90°, 135°, 180°, 225°$ and $270°$ are in between $66\% \sim 76\%$.

Table 2. The first natural frequencies for the curved beam with the fixed-fixed boundary conditions

Opening angle (θ)		ω_1	ω_2	ω_3	ω_4	ω_5
		(in Hz)				
45°	[11] ANSYS	61.36	151.07	171.48	267.24	333.91
	[11]	61.38	151.11	180.68	267.31	351.81
	MFEM	61.30	150.97	172.92	267.14	334.65
90°	[11] ANSYS	20.79	44.04	81.39	94.26	128.67
	[11]	20.79	44.05	81.42	99.33	128.71
	MFEM	20.76	43.94	81.27	96.87	128.55
135°	[11] ANSYS	15.66	23.05	38.97	61.50	71.32
	[11]	15.66	22.97	38.98	61.52	75.16
	MFEM	15.66	22.99	38.62	61.37	74.74
180°	[11] ANSYS	14.72	17.22	24.49	36.53	52.43
	[11]	14.72	17.22	24.50	36.55	52.45
	MFEM	14.73	17.20	24.41	36.42	52.30
225°	[11] ANSYS	14.49	15.40	18.77	25.43	35.11
	[11]	14.49	15.41	18.78	25.44	35.12
	MFEM	14.49	15.40	18.72	25.34	34.99
270°	[11] ANSYS	14.41	14.78	16.37	20.07	26.08
	[11]	14.41	14.78	16.37	20.07	26.09
	MFEM	14.41	14.78	16.35	20.01	25.99

3.3 Static and free vibration analysis of a planar composite curved beam

The static and free vibration analysis of composite curved beams with and without elastic foundation is carried out. The fixed-fixed boundary condition is employed. The composite circular beam having rectangular composite cross-section which is made of steel on the bottom and concrete on the top as shown in Fig.2 is considered. The material properties and geometrical properties are as follows: the modulus of elasticity for steel is $E_s = 210\,\text{GPa}$, Poisson's ratio is $\upsilon_s = 0.3$ and the material density is $\rho_s = 7850\,\text{kg/m}^3$. The modulus of elasticity for concrete $E_c = 30\,\text{GPa}$, Poisson's ratio is $\upsilon_c = 0.2$ and the material density is $\rho_c = 2400\,\text{kg/m}^3$. The radius of composite curved beam is $R = 1.2\,\text{m}$, the opening angle is 180°. The dimensions of rectangular cross-section in Fig.2 are $b = 0.15\,\text{m}$, $h_1 = 0.02\,\text{m}$, $h_2 = 0.10\,\text{m}$. The planar curved beam is subjected to a uniformly distributed vertical load $q = 560\,\text{N/m}$. 80 mixed FEs are employed in the following numerical examples. The calculation of torsional rigidity of composite cross-section (see Fig.2) is carried out by using the FE program which is mentioned and verified in section 3.1.

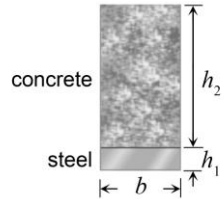

Fig.2 Composite cross-section

3.3.1 The curved beam without foundation

The maximum u_b displacement and fixed end reactions (T_b: shear force, M_t and M_n: moments) and the first five natural frequencies of composite curved beam are used for the numerical comparisons in static and free vibration analysis. The all results are compared with the commercial program results ANSYS and presented in Tables 3-4. The u_b displacements along the span of curved beam and the mode shapes of first five natural frequencies are given in Figs. 3-4, respectively.

Table 3. The static analysis results of curved beam

	u_b (mm)	T_b (N)	M_t (Nm)	M_n (Nm)
MFEM	0.3533	1055.58	239.76	806.51
ANSYS	0.3529	1055.60	240.79	807.45
Dif.%	0.11	0.00	-0.43	-0.12

Table 4. The first five natural frequencies (in Hz) of curved beam

	ω_1	ω_2	ω_3	ω_4	ω_5
MFEM	29.749	83.942	87.768	174.508	185.187
ANSYS	29.715	83.946	87.500	173.260	185.540
Dif.%	0.11	0.00	0.31	0.72	-0.19

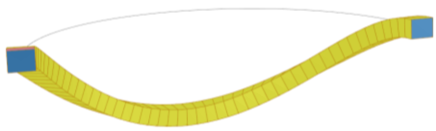

Fig.3 The u_b displacements along the span of curved beam

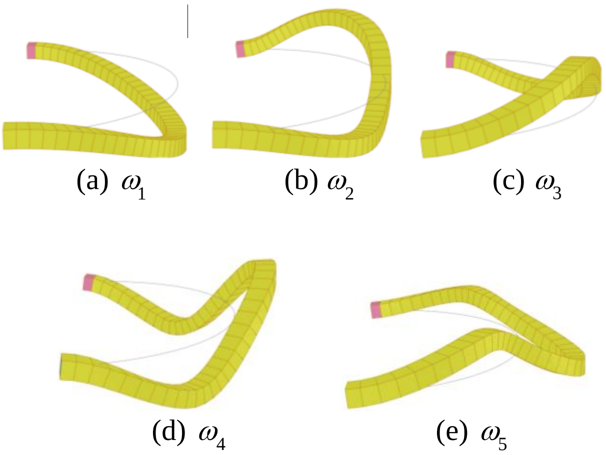

Fig.4 The mode shapes first five natural frequencies of curved beam without foundation

3.3.2 The curved beam with foundation

The components of Winkler and Pasternak foundation constants in the direction b are $k_{Wb} = 100\,\text{kN/m}^2$ and $k_{Pb} = 200\,\text{kN}$, respectively. The maximum u_b displacement and fixed end reactions (T_b: shear force, M_t and M_n: moments) and the first five natural frequencies of composite curved beam are tabulated in Tables 5-6.

Table 5. The static analysis of curved beam on elastic foundation

k_{W_b} (kN/m²)	k_{P_b} (kN)	u_b (mm)	T_b (N)	M_t (Nm)	M_n (Nm)
100	0	0.329	1023.49	229.17	774.03
	200	0.304	1055.08	212.83	730.38

Table 6. The first five natural frequencies (in Hz) of curved beam on elastic foundation

k_{W_b} (kN/m²)	k_{P_b} (kN)	ω_1	ω_2	ω_3	ω_4	ω_5
100	0	30.80	84.70	88.43	175.85	186.01
	200	32.02	85.41	89.30	177.01	186.50

If the maximum u_b displacements for the curved beam on resting Winkler and Pasternak foundation are compared with respect to the results of the curved beam without foundation, the percent reduction for Winkler and Pasternak foundations are 6.9% and 14.0%, respectively (see Tables 3 and 5). When a similar comparison is made for free vibration analysis of the curved beam without foundation, the percent increases for Winkler and Pasternak foundations are 3.5% and 7.6%, respectively (see Tables 4 and 6).

4 Conclusion

Static and free vibration analysis of a planar curved beam on resting Pasternak foundation having the composite cross-section is performed via the mixed finite element method. The finite element solutions are compared with the literature and the commercial program ANSYS. The following remarks can be given:

- The finite element formulation which is verified with the literature is used to calculate the torsional rigidity of the composite cross-section.
- The influence of the opening angle of a curved beam on resting Winkler foundation on the natural frequencies is investigated and verified with the literature.
- The static and free vibration analysis of composite curved beams without elastic foundation is carried out and verified with the commercial program results ANSYS.
- The static and free vibration analysis of composite curved beams on resting Pasternak foundation is investigated and the results are presented as an original example.

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