Second Order Differential Equation of Motion in Railways: the Variance of the Dynamic Component of Actions due to the Sprung Masses of the Vehicles

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Abstract: - In this paper the second order differential equation of motion is presented for the case of a railway vehicle rolling on a railway track with defects/faults and its solution is presented for the Sprung (Suspended) Masses of the vehicle that act indirectly on the track through the springs and dampers of the vehicle.

Key-Words: - Static/Dynamic Stiffness Coefficient, Sprung Masses, Unprung Masses, Fourier Transform, Spectral Density, Variance, Standard Deviation, Dynamic Component of Actions.

1 Introduction

The motion of a railway vehicle on the rail running table/surface or the motion of a road vehicle on the road, the response of the structures to earthquakes, etc, is a forced oscillation with a forcing excitation (force), and damping expressed by a random, non-periodic function. The motion is described by equations and, in railway engineering, it is illustrated through the simplified form of a spring-mass-damper system as depicted in Fig. 1, with a railway vehicle running on a track with longitudinal defects (Fig. 1 left) and the forces exerted on the vehicle's "car-body" (Fig. 1 right).

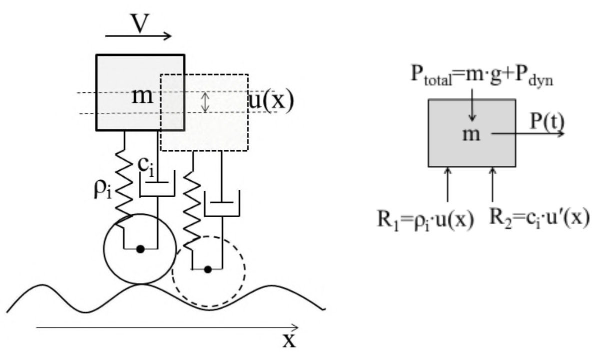

Fig. 1 A spring-mass-damper-system in railway engineering: (left) a railway vehicle on a railway track with longitudinal defects and (right) the forces exerted on the "car-body".

In this simplified model, with the wheel rolling over a surface with defects but undeflected itself, the acting forces are:

a. the weight of the vehicle $m{\cdot}g$;
b. the dynamic component of the Load P_{dyn};
c. the motive force P(t) which is equal to the difference between the tractive force of the locomotive minus the friction and it is positive in the case of increase of speed (accelerated motion) or negative in the case of braking (decrease of speed) since it is equal to zero if the motive force is equal to friction;
d. the reaction R_1 provided by the system "vehicle-track" equal to a spring constant or coefficient of elasticity ρ_i (given in kN/mm) multiplied by the subsidence u(x) of the center of gravity of the vehicle;
e. the reaction R_2 provided by the system "vehicle-track" equal to a damping constant c_i multiplied by the first derivative of the subsidence u(x) of the center of gravity of the vehicle.

In practice, the circulation of a railway vehicle on a railway track differs significantly from this simplified model, since the support/railway track is not undeflected and the railway vehicle has the Sprung and the Unsprung Masses as described below. The behavior of the Unsprung Masses (the masses located under the primary suspension of the vehicle) is approached with the track simulated –with the observer situated on the wheel– as an elastic mean with damping as illustrated in the simplified model in Fig. 2. An analysis of the behavior of the Non Suspended Masses was presented in [1], [2]. For the simulation of the Sprung (Suspended) Masses of the vehicles (that is of the masses located over the primary suspension of the vehicle), the forces resulting from the forcing excitation imposed by the rail running table –in the most of the cases– have small effect on the rolling of the wheel, but for defects of very long wavelength, the oscillations of the Suspended Masses become predominant since the oscillations of the Non Suspended Masses decrease. The Suspended/Sprung Masses are examined in the present paper.

2 Problem Formulation

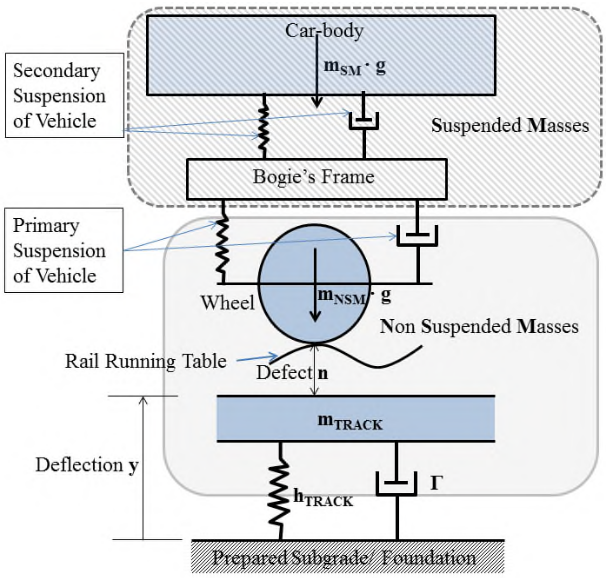

Fig. 2 Model of a Vehicle running the Rail Running Table: the Non Suspended Masses (NSM) and the Suspended Masses (SM), the primary and secondary suspensions are depicted.

2.1 Simulation of the System "Railway Vehicle-Railway Track"

In Fig. 2 a simplified model of the system "Railway Track – Railway Vehicle" is depicted; in practice it is an ensemble of springs and dashpots. The railway track is represented by the resultants of springs and dashpots, as described in [1] and [2]. The railway vehicle has two levels of suspensions the primary suspension between the axles and the bogie's frame and the secondary suspension between the car-body and the bogie's frame. Under the primary suspension there are the Unsprung (Non-Suspended) Masses (axle, wheels and a fraction of the semi-suspended motive electromotor in the locomotives) which act without any damping directly on the track panel. On the contrary the Sprung (Suspended) Masses that are cited above the primary suspension of the vehicle, act through a combination of springs and dampers on the track. A part of the track mass is also added to the unsprung masses, which participates in their motion ([3], [4]).

The rail running table has the shape of a wave that is not completely "rectilinear", that is, it does not form a perfectly straight line but contains faults/ defects, varying from a few fractions of a millimeter to a few millimeters, and imposes forced oscillation on the railway vehicles that move on it; the faults/defects are represented by the ordinate n in Fig. 2. Moreover during the rolling of the wheel, a deflection y of the rail running table appears (see [5]), since the support (track) is not undeflected.

The general form of the equation of a vehicle moving on an infinitely long beam, on elastic ground without damping, is described in the next paragraph 2.2..

2.2 Second Order Differential Equation of Motion for the Railway Track

In this paragraph, the simulation of the track as an elastic media with damping as shown in Fig. 2, depicting the rolling wheel on the rail running table (see relevantly also [6]), is attempted. Forced oscillation is caused by the irregularities of the rail running table (like an input random signal) –which are represented by n–, in a gravitational field with acceleration g. There are two suspensions on the vehicle for passenger comfort purposes: primary and secondary suspension, as described in the previous paragraph 2.1.. Moreover, a section of the mass of the railway track participates in the motion of the Non-Suspended (Unsprung) Masses of the vehicle. These Masses are situated under the primary suspension of the vehicle.

If the random excitation (track irregularities) is given, it is difficult to derive the response, unless the system is linear and invariable. In this case the input signal can be defined by its spectral density and from this we can calculate the spectral density of the response. The theoretical results confirm and explain the experimental verifications ([7], p.39, 71).

The equation for the interaction between the vehicle's axle and the track-panel becomes ([1], [5], [6]):

$$\left(m_{NSM}+m_{TRACK}\right)\cdot\frac{d^2y}{dt^2}+\Gamma\cdot\frac{dy}{dt}+h_{TRACK}\cdot y=$$
$$=-m_{NSM}\cdot\frac{d^2n}{dt^2}+\left(m_{NSM}+m_{SM}\right)\cdot g \qquad (1)$$

where: m_{NSM} the Non-Suspended/Unsprung Masses (NSM) of the vehicle in tonnes-mass, m_{TRACK} the mass of the track that participates in the motion of the NSM (for its calculation see Ref. [3] and [4]), m_{SM} the Suspended/Sprung Masses (SM) of the vehicle that are cited above the primary suspension of the vehicle, Γ damping constant of the track, h_{TRACK} the total dynamic stiffness coefficient of the track (its calculation Eqn. 3 below), n the fault ordinate of the rail running table, g the acceleration of gravity and y the total deflection of the track. Furthermore:

$$\frac{1}{\rho_{total}}=\frac{1}{\rho_{rail}}+\frac{1}{\rho_{pad}}+\frac{1}{\rho_{sleeper}}+\frac{1}{\rho_{ballast}}+\frac{1}{\rho_{subgrade}} \qquad (2)$$

where: ρ_i the static stiffness coefficients of the constitutive layers of the track, the quasi spring constants of the layers and ρ_{total} the resultant total static coefficient of the track and htrack the total dynamic stiffness coefficient of the track given by:

$$h_{TRACK}=\rho_{dyn-total}=\frac{1}{2\cdot\sqrt{2}}\cdot\sqrt[4]{E\cdot J\cdot\frac{\rho_{total}}{\ell}} \qquad (3)$$

with E, J the modulus of elasticity and the moment of inertia of the rail (steel) and ℓ the distance among the sleepers.

The phenomena of the wheel-rail contact and of the wheel hunting, particularly the equivalent conicity of the wheel and the forces of pseudo-glide, are non-linear. In any case the use of the linear system's approach is valid for speeds lower than the $V_{critical}$≈500 km/h. The integration for the non-linear model (wheel-rail contact, wheel-hunting and pseudoglide forces) is performed through the Runge Kutta method ([7], p.94-95, 80, [8], p.98, see also [9], p.171, 351).

In Fig. 2 the rail running table depicts a longitudinal fault/ defect of the rail surface. In the above equation, the oscillation of the axle is damped after its passage over the defect. Viscous damping, due to the ballast, enters the above equation under the condition that it is proportional to the variation of the deflection dy/dt. To simplify the investigation, if we ignore the track mass (for its calculation Ref. [3] and [4]) in relation to the much larger Vehicle's Non Suspended Mass and bearing in mind that y+n is the total subsidence of the wheel during its motion (since the y and n are added algebraically), we can approach the problem of the random excitation, from cosine defect (V<< $V_{critical}$=500 km/h).

2.3 Fourier Transform and the Second Order Differential Equation of Motion

We begin from the hypothesis of a cosine form defect on the rail running table of the form:

$$\eta=a\cdot\cos\omega t=a\cdot\cos\left(2\pi\cdot\frac{V\cdot t}{\lambda}\right) \qquad (4)$$

where: η the ordinate of the defect along the track (abscissa x), V the speed of the vehicle, t the time and λ the wavelength of the defect, so:

$$T=\frac{2\pi}{\omega}\Rightarrow\omega t=\frac{2\pi}{T}t\Rightarrow\omega t=\frac{2\pi Vt}{\lambda} \qquad (5a)$$

since the wheel overpasses the wavelength λ of the defect, in:

$$T=\frac{\lambda}{V}\Rightarrow\lambda=T\cdot V \qquad (5b)$$

If we set:

$$y=z+\frac{m_{SM}+m_{NSM}}{h_{TRACK}}\cdot g\Rightarrow\frac{dy}{dt}=\frac{dz}{dt}$$

its second derivative will be:

$$\frac{d^2y}{dt^2}=\frac{d^2z}{dt^2}$$

where the quantity $\frac{m_{NSM}+m_{SM}}{h_{TRACK}}\cdot g$ represents the subsidence due to the static loads only, and z random (see [20]) due to the dynamic loads. Eqn (1) becomes:

$$m_{NSM}\frac{d^2z}{dt^2}+\Gamma\cdot\frac{dz}{dt}+h_{TRACK}\cdot z=-m_{NSM}\cdot\frac{d^2n}{dt^2}\Rightarrow \qquad (6a)$$

$$\Rightarrow m_{NSM}\left(\frac{d^2z}{dt^2}+\frac{d^2n}{dt^2}\right)+\Gamma\cdot\frac{dz}{dt}+h_{TRACK}\cdot z=0 \qquad (6b)$$

Since, in this case, we are examining the dynamic loads only (derived from the actions of the Suspended and Non Suspended Masses) , in order to approach their effect, we could narrow the study of equation (6b), by changing the variable:

$$u=n+z\Rightarrow\frac{d^2u}{dt^2}=\frac{d^2n}{dt^2}+\frac{d^2z}{dt^2}$$

Equation (6) becomes:

$$m_{NSM}\frac{d^2u}{dt^2}+\Gamma\cdot\frac{dz}{dt}+h_{TRACK}\cdot z=0\Rightarrow \quad (7a)$$

$$\Rightarrow m_{NSM}\frac{d^2u}{dt^2}+\Gamma\cdot\frac{d(u-n)}{dt}+h_{TRACK}\cdot(u-n)=0 \quad (7b)$$

where, u is the trajectory of the wheel over the vertical fault (of ordinate n) in the longitudinal profile of the rail.

If we apply the Fourier transform to the equation (6a) (see relevantly Ref. [10] for solving second order differential equations with the Fourier transform):

$$(i\omega)^2\cdot Z(\omega)+\frac{\Gamma\cdot(i\omega)}{m_{NSM}}\cdot Z(\omega)+\frac{h_{TRACK}}{m_{NSM}}\cdot Z(\omega)=$$
$$=-(i\omega)^2\cdot N(\omega)\Rightarrow \quad (8a)$$

$$H(\omega)=\frac{Z(\omega)}{N(\omega)}, \quad (8b)$$

$$\left|H(\omega)\right|^2=\frac{m_{NSM}^2\cdot\omega^4}{\left(m_{NSM}\cdot\omega^2-h_{TRACK}\right)^2+\Gamma^2\cdot\omega^2} \quad (8c)$$

H(ω) is a complex transfer function, *called frequency response function* [10], that makes it possible to pass from the fault n to the subsidence Z. If we apply the Fourier transform to equation (7a):

$$(i\omega)^2\cdot U(\omega)+\Gamma\cdot(i\omega)\cdot Z(\omega)+h_{TRACK}\cdot(i\omega)^0\cdot Z(\omega)=0\Rightarrow$$

$$G(\omega)=\frac{U(\omega)}{Z(\omega)},\quad \left|G(\omega)\right|^2=\frac{h_{TRACK}^2+\Gamma^2\cdot\omega^2}{m_{NSM}^2\cdot\omega^4} \quad (9)$$

G(ω) is a complex transfer function, *the frequency response function,* that makes it possible to pass from Z to Z+n.

If we name U the Fourier transform of u, N the Fourier transform of n, $p=2\pi i\nu=i\omega$ the variable of frequency and ΔQ the Fourier transform of ΔQ and apply the Fourier transform at equation (7b):

$$Eq.(7)\Rightarrow m_{NSM}\frac{d^2u}{dt^2}+\Gamma\cdot\frac{du}{dt}+h_{TRACK}\cdot u=\Gamma\cdot\frac{dn}{dt}+h_{TRACK}\cdot n\Rightarrow$$

$$\left(m_{NSM}\cdot p^2+\Gamma\cdot p+h_{TRACK}\right)\cdot U=\left(\Gamma\cdot p+h_{TRACK}\right)\cdot N\Rightarrow$$

$$U(\omega)=\underbrace{\frac{\Gamma\cdot p+h_{TRACK}}{m_{NSM}\cdot p^2+\Gamma\cdot p+h_{TRACK}}}_{B(\omega)}\cdot N(\omega) \quad (10a)$$

where:

$$\left|B(\omega)\right|^2=\frac{\Gamma^2\cdot\omega^2+h_{TRACK}^2}{\left(m_{NSM}\cdot\omega^2-h\right)^2+\Gamma^2\cdot\omega^2} \quad (10b)$$

B(ω) is a complex transfer function, *the frequency response function,* that makes it possible to pass from the fault n to the u=n+Z. Practically it is verified also by the equation:

$$\left|B(\omega)\right|^2=\left|H(\omega)\right|^2\cdot\left|G(\omega)\right|^2=$$
$$=\frac{h_{TRACK}^2+\Gamma^2\cdot\omega^2}{\left(m_{NSM}\cdot\omega^2-h_{TRACK}\right)^2+\Gamma^2\cdot\omega^2} \quad (10c)$$

passing from n to Z through H(ω) and afterwards from Z to n+Z through G(ω). This is a formula that characterizes the transfer function between the wheel trajectory and the fault in the longitudinal level and enables, thereafter, the calculation of the transfer function between the dynamic load and the track defect (fault).

The transfer function B(ω) allows us to calculate the effect of a spectrum of sinusoidal faults, like the undulatory wear. If we replace $\omega/\omega_n=\rho$, where ω_n= the circular eigenfrequency (or natural cyclic frequency) of the oscillation, and:

$$\omega_n^2=\frac{h_{TRACK}}{m_{NSM}},\quad \omega=\frac{2\pi V}{\lambda},\quad 2\zeta\omega_n=\frac{\Gamma}{m_{NSM}},\quad \beta=\frac{\omega}{\omega_n}$$

where ζ is the damping coefficient. Eqn (10b) is transformed:

$$\left|B(\omega)\right|^2=\left|B_n(\beta)\right|^2=\frac{1+4\zeta^2\cdot\beta^2}{\left(1-\beta^2\right)^2+4\zeta^2\cdot\beta^2} \quad (10d)$$

The transfer function C(ω) of the second derivative of (Z+n) in relation to time: $\frac{d^2(Z+n)}{dt^2}$, –that is the acceleration γ–, will be equal to *ω·B(ω)*:

$$p^2\cdot U(\omega)=p^2\left|B(\omega)\right|\cdot p^2\cdot N(\omega)\Rightarrow$$
$$\Rightarrow U(\omega)=\underbrace{p^2\left|B(\omega)\right|}_{C(\omega)}\cdot N(\omega) \quad (11a)$$

that is:

$$\left|C(\omega)\right|=\omega^2\cdot\left|B(\omega)\right|=\omega_n^2\cdot\frac{\omega^2}{\omega_n^2}\cdot\left|B(\omega)\right|\Rightarrow$$
$$\Rightarrow\left|C(\omega)\right|=\omega_n^2\cdot\beta^2\cdot\left|B(\omega)\right| \quad (11b)$$

The increase of the vertical load on the track due to the Non Suspended Masses, according to the principle *force = mass x acceleration*, is given by:

$$\Delta Q = m_{NSM} \cdot \frac{d^2 u}{dt^2} = m_{NSM} \cdot \frac{d^2 (n+Z)}{dt^2} \quad (12)$$

If we apply the Fourier transform to Eqn. (12):

$$\hat{\Delta}Q = m_{NSM} \cdot p^2 \cdot U(\omega) = m_{NSM} \cdot p^2 \cdot \hat{f}_{Z+n}(\omega) \Rightarrow \quad (13a)$$

$$\left|\hat{\Delta}Q\right| = m_{NSM} \cdot \left|C(\omega)\right| \cdot \left|N(\omega)\right| = m_{NSM} \cdot |p|^2 \cdot \left|B(\omega)\right| \cdot \left|N(\omega)\right|$$

$$\left|\hat{\Delta}Q\right| = m_{NSM} \cdot \beta^2 \cdot \omega_n^2 \cdot \left|B(\omega)\right| \cdot \left|N(\omega)\right| \quad (13b)$$

2.4 Input and Output Power Spectral Density and Variance

The excitation (rail irregularities) in reality is random and neither periodic nor analytically defined, like the eq. (4). It can be defined by its autocorrelation function in space and its spectral density ([7], p.58; [11], p.700; [12]). If f(x) is a signal with determined total energy and F(ν) its Fourier transform, from Parseval's modulus theorem [13], the total energy is [10]:

$$\int_{-\infty}^{+\infty} \left| f(x) \right|^2 \cdot dx = \int_{-\infty}^{+\infty} \left| F(\nu) \right|^2 \cdot d\nu \quad (14a)$$

where, $F(\nu)=A(\nu)\cdot e^{i\varphi(\nu)}$ and the power spectral density:

$$S(\omega) = \left| F(\nu) \right|^2 = A^2(\nu) \quad (15)$$

Reference [10] solves equation (14a) as:

$$\int_{-\infty}^{+\infty} f(t)^2 \cdot dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)^2 \cdot d\omega \quad (14b)$$

The square of the modulus F(ω) is called the energy spectrum of the signal because $F^2(\omega)\cdot\Delta(\omega)$ represents the amount of energy in any ΔΩ segment of the frequency spectrum, and the integral of $F^2(\omega)$ over (-∞, +∞)gives the total energy of the signal. An input signal -like the running rail table- creates through the vehicle an output signal: the wheel trajectory. The output spectral density and the input spectral density of the excitation are related through equation [14], [5]:

$$S_{OUTPUT}(\bar{\omega}) = \left| H(i\bar{\omega}) \right|^2 \cdot S_{INPUT}(\bar{\omega}) \quad (15a)$$

In order to relate the temporal spectrum with the spectrum in space we use the following equation:

$$\omega \cdot t = \frac{2\pi V t}{\lambda} \Rightarrow \omega = \frac{2\pi}{\lambda} \cdot V \Rightarrow \omega = \Omega \cdot V \quad (15b)$$

where λ is the wavelength of the defect. This means that circular frequency in space Ω is the wave number k of the equation of oscillation, and [13]:

$$\int_0^{\infty} S(\Omega) \cdot d\Omega = \int_0^{\infty} s(\omega) \cdot d\omega \Rightarrow$$

$$\Rightarrow \mathbb{F}\left[f(ax) \right] = \frac{1}{|a|} \cdot \hat{f}\left(\frac{\nu}{a} \right) \Rightarrow$$

$$\Rightarrow S(\omega) = S\left(\frac{\omega}{V} \right) = \frac{1}{V} \cdot S(\Omega) \quad (16a)$$

where $\mathbb{F}$ is the symbol for the application of the Fourier transform of f and $\hat{f}$ the function after the transform. This is a property of the Fourier transform.

Eqn (16a) applied in the case that the power spectrum of the vertical defects along the track (for the NSM) in the space domain is S(Ω) then the power spectrum of the excitation of the wheel in the time domain will result after a replacement of Ω by the ω/V.

$$s_\upsilon(\omega) = \frac{1}{V} S\left(\frac{\omega}{V} \right) \quad (16b)$$

The Variance or mean square value $\sigma^2(x)$ of the function is given by [15], [16]:

$$\sigma^2(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S(\omega) \cdot d\omega = \bar{x}^2 \quad (17)$$

where σ(x) is the standard deviation of the function.

The Power Spectral density and the variance of a function are depicted in Fig. 3.

From equation (17) we derive:

$$\sigma^2(n) = \frac{1}{\pi} \int_0^{+\infty} S_n(\omega) \cdot d\omega$$

$$\sigma^2(z) = \frac{1}{\pi} \int_0^{+\infty} S_z(\omega) \cdot d\omega \quad (18)$$

$$\sigma^2(\Delta Q) = \frac{1}{\pi} \int_0^{+\infty} S_{\Delta Q}(\omega) \cdot d\omega$$

where n is the random variable of the defect (input), z the subsidence of the wheel (output) and ΔQ the dynamic component of the Load that is added to the Static Load of the wheel due to the Non Suspended Masses (output also).

From these equations and the analytic form of the spectrum of the defects/faults, we can calculate the mean square value of the dynamic component of the Load due to the Non Suspended Masses that is added to the relevant dynamic component of the

Suspended Masses and the total dynamic component of the load is added to the Static Load of the wheel.

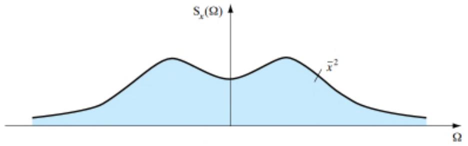

Fig. 3 Power Spectral density S(Ω) –the black curve– Variance (mean square value $\overline{x}^2$) –the shaded area– of a function [5].

From the power spectral density and the variance functions and their definitions [5]:

$$S_{\Delta Q}(\omega) = S_n(\omega) \cdot \left|B(\omega)\right|^2 \tag{19a}$$

$$\Delta Q = m_{NSM} \cdot \Delta\gamma \Rightarrow \sigma^2(\Delta Q) = m_{NSM} \cdot \sigma^2(\gamma) \tag{19b}$$

$$\Rightarrow \sigma^2(\gamma) = \frac{\sigma^2(\Delta Q)}{m_{NSM}} \tag{19c}$$

and using the eqs. (19) and (11-12b):

$$\sigma^2(\gamma) = \frac{1}{m_{NSM}} \cdot \sigma^2(\Delta Q) \Rightarrow$$

$$\sigma^2(\gamma) = \frac{1}{m_{NSM}^2 \cdot \pi} \cdot \int_0^{+\infty} \left|B(\omega)\right|^2 \cdot S_n(\omega) \cdot d\omega \tag{20}$$

$$\sigma^2(\gamma) = \frac{m_{NSM}^2}{m_{NSM}^2 \cdot \pi} \cdot \int_0^{+\infty} \beta^4 \cdot \omega_n^4 \cdot \left|B(\omega)\right|^2 \cdot S_n(\omega) \cdot d\omega =$$

$$\sigma^2(\gamma) =$$

$$= \frac{1}{\pi} \cdot \int_0^{+\infty} \beta^4 \cdot \omega_n^4 \cdot \frac{1 + 4\zeta^2 \cdot \beta^2}{\left(1-\beta^2\right)^2 + 4\zeta^2 \cdot \beta^2} \cdot S_n(\omega) \cdot d\omega \tag{21}$$

From the above equations and the analytical form of the spectrum of the longitudinal defects/ faults of the track we could effectively calculate the variance (mean square value) of the dynamic component of the Loads on the track panel due to the Non Suspended Masses. All the results of measurements on track in the French railways network show that the spectrum of defects in the longitudinal level has the form [6], [17]:

$$S_n(\Omega) = \frac{A}{\left(B+\Omega\right)^3} \tag{22}$$

This implies that the mean square value or variance of the defects is given by:

$$\sigma^2(z) = \frac{1}{\pi} \cdot \int_0^{+\infty} \frac{A}{\left(B+\Omega\right)^3} \cdot d\Omega \Rightarrow$$

$$\Rightarrow \sigma^2(z) = \frac{A}{\pi} \cdot \int_0^{+\infty} \frac{1}{(x)^3} \cdot dx = -\frac{A}{2\pi}\left[\frac{1}{\left(B+\Omega\right)^2}\right]_0^{+\infty} \Rightarrow$$

$$\sigma^2(z) = -\frac{A}{2\pi} \cdot \left[\frac{1}{B^2 + 2B\Omega + \Omega^2}\right]_0^{+\infty} \Rightarrow$$

$$\Rightarrow \sigma^2(z) = -\frac{A}{2\pi}\left[0 - \frac{1}{B^2}\right] \Rightarrow$$

$$\sigma^2(z) = \frac{1}{2\pi} \cdot \frac{A}{B^2} \tag{23}$$

If we examine only the much more severe case, for the case of the Non Suspended Masses, of the defects of short wavelength, consequently large Ω –like the undulatory wear– then we can omit the term B, and using Eqn. (15b):

$$S_n(\Omega) = \frac{A}{\Omega^3} = \frac{A}{\frac{1}{V^3} \cdot \omega^3} = \frac{A \cdot V^3}{\omega^3} \tag{24}$$

The term B characterizes the defects with large wavelengths, for which the maintenance of track is effective, and when we examine this kind of defects term B should be taken into account. Suspended Masses should be examined for long wavelength defects.

For the line "*Les Aubrais – Vierzon*", the parameters values are: B=0,36, A=2,1·10^{-6} and S(Ω) is calculated in m^3 and σ(z)=1,57 mm. The eigenfrequency of the Non Suspended Masses of the vehicles is approximately 30 – 40 Hz and even for speeds of 300 km/h there are wavelengths less than 3 m [17].

From equations (16) and (24):

$$S_n(\omega) = \frac{1}{V} \cdot S(\Omega) = \frac{1}{V} \cdot \frac{A \cdot V^3}{\omega^3} = \frac{A \cdot V^2}{\omega_n^3 \cdot \beta^3} \tag{25}$$

3 The Case of Non Suspended Masses

This case has been analyzed in [1], [2], [5], [18]. The interested reader should read the relevant texts.

4 The Variance of the Suspended Masses

If we assume that the defects of the two rails –constituting the cross-section of a track– are quite the same at the same time and presenting the same phase, or if we examine the trajectory of one wheel, then the conclusion that will be derived can be used for more complicated cases of rolling of vehicles, motion of car-bodies etc. Furthermore we consider the simplified model of Fig. 1 with one-floor mass-spring-damper system rolling on a rail's surface.

In order to calculate the power spectrum density of the excitation $s_E(\omega)$ from the excitation spectrum of the wheel $s_\upsilon(\omega)$, we apply the Eqn. 15a with the Eqns. 16, Eqn. 22 (the parameter B is not omitted) and Eqns. 10 (in order to pass from the defect n to n+Z):

$$s_E(\omega) = \left|B(\omega)\right|^2 \cdot s_\upsilon(\omega) \Rightarrow$$

$$\Rightarrow s_E(\omega) = \frac{1 + 4\zeta^2 \cdot \dfrac{\omega^2}{\omega_n^2}}{\left[1 - \dfrac{\omega^2}{\omega_n^2}\right]^2 + 4\zeta^2 \cdot \dfrac{\omega^2}{\omega_n^2}} \cdot \frac{AV^2}{(BV+\omega)^3} \quad (26)$$

where: $s_E(\omega)$ is the power spectrum density of the excitation, ω_n is always the eigenfrequency of the Non Suspended Masses, ζ the damping coefficient of the track, $s_\upsilon(\omega)$ the spectrum of the excitation of the wheel due to the track defects/faults and $\left|B(\omega)\right|$ the modulus of the transfer function of the motion of the wheel.

From the Eqns. 10 and the Eqns. 11, 12 and 13, with the analysis cited above, we keep that, C(ω) ις the transfer function of the second derivative of (Z+n) in relation to time: $\dfrac{d^2(Z+n)}{dt^2}$, that is the acceleration γ and it is equal to $\omega \cdot B(\omega)$.

In the case of the Suspended Masses $\left|C(\omega)\right|$ is the modulus of the transfer function of the accelerations of the car-body. Consequently for the spectrum of the accelerations of the car-body we will use Eqn. 15a substituting the parameters ζ and ω_n of the track with the relevant parameters ζ' (damping coefficient) and ω'_n (eigenfrequency) of the car-body.

For the railway vehicles the eigenfrequencies ω'_n of the car-body are in the area of 1 Hz, since with the development of high-speeds it could arrive 10 Hz. For the damping coefficient of the car-body of the railway vehicles two characteristic values of ζ' could be used with reliability: 0,15 and 0,20 (see relevantly [17] and [7]).

$$s_\gamma(\omega) = \left|C(\omega)\right|^2 \cdot s_E(\omega) \Rightarrow$$

$$s_\gamma(\omega) = \omega_n^4 \cdot \frac{AV^2}{(BV+\omega)^3} \cdot \frac{1+4\zeta^2 \cdot \beta^2}{\left(1-\beta^2\right)^2 + 4\zeta^2 \cdot \beta^2} \cdot$$

$$\cdot \frac{1+4\zeta'^2 \cdot \beta'^2}{\left(1-\beta'^2\right)^2 + 4\zeta'^2 \cdot \beta'^2} \quad (27)$$

The variance of the accelerations of the car-body of the railway vehicles is given by (Eqns. 18):

$$\sigma(\gamma)^2 = \frac{1}{\pi} \cdot \int_0^\infty s_\gamma(\omega) \cdot d\omega \quad (28)$$

which converges for ω infinite. Consequently the variance of the part of the dynamic component of the load due to the Suspended/Sprung Masses of the vehicle is given by (see Eqn. 19b):

$$\Delta Q_{SM} = m_{SM} \cdot \Delta\gamma \Rightarrow \sigma^2(\Delta Q_{SM}) = m_{SM} \cdot \sigma^2(\gamma) \quad (29)$$

Finally an approximation could be used for the calculation of the variance of this part of the dynamic component of the load (see [5], [6]):

$$\sigma(\Delta Q_{SM}) = \frac{V-40}{1000} \cdot N_L \cdot Q_{wheel} \quad (30)$$

where: Q_{wheel} is the static wheel load, V is the operational speed, and the coefficient N_L is the mean standard deviation of the longitudinal level condition of the track, on a 300 m length approximately, for both rails is the mean standard deviation of the longitudinal level condition of the track, on a 300 m length fluctuating between 0,7–1,5 mm or more (see [6]; [19], p. 335–336); for the Greek network N_L is estimated to fluctuate –mainly– between 1 and 1,5 [5].

In more details N_L, the average of the brutal signal on a basis of approximately 300 m for the vertical and horizontal defects of the two rails, is the convolution:

$$N_L(x_0) = \frac{1}{300} \int_{-\infty}^{+\infty} \eta l(x) e^{\left(\frac{x-x_0}{300}\right)} dx \quad (31)$$

where ηl(x) is the value of the primary signal; in practice a weighted average index which "crashes" less the isolated defects than one classic average and simulates roughly the "memory" of the vehicle (see relevantly [19]).

5 Conclusions

In this paper the second order differential equation of motion of the Sprung (Suspended) Masses of a railway vehicle rolling on the track with defects/faults is investigated and its solution is approached. The investigation is performed through the Fourier transform and the solution is verified from findings from a research program performed by the Greek railways in collaboration with the French state railways (SNCF) and further research performed by the author.

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