3D ISAR Imaging from Single View Using Bessel Function Expansion

AHMED S. ABUTALEB[1,2]
[1]Currently with Cairo University, School of Engineering,
Giza,
EGYPT

[2]MIT Lincoln Lab, Lexington,
Massachusetts, 02421,
USA

Abstract: We study the case of a moving target that is observed from a ground radar station. The target has relatively slow yaw angle and slow rotation angle. The coordinates of each reflector is studied and estimated from only one viewing ground station and for relatively short period of one second. The compensated range and the Doppler shift were interpreted as phase modulated (PM) signals. Bessel function expansion was used to find, for each reflector, an estimate for the unknowns: (1) yaw frequency, (2) rotation frequency, (3) x coordinate, (4) y coordinate, and (5) z coordinate. Only range data is used for the estimation. The unique Bessel functions parameters are guaranteed using the location of the peak of the spectrum of the signal. The cross range or Doppler data could be used to verify and adjust the estimated values. The approach was applied to simulated 3D inverse synthetic aperture radar (ISAR) imaging.

Keywords: ISAR, Bessel Functions Expansion, Phase Modulation, Radar Echo, Doppler Shift.

Received: May 26, 2025. Revised: August 9, 2025. Accepted: September 21, 2025. Published: April 15, 2026.

1 Introduction

Inverse synthetic aperture radar (ISAR) is an effective tool to image moving targets in all weather conditions. Conventional two-dimensional (2D) ISAR imaging methods can obtain a high range resolution by compressing the wide-band signal. Pulse compression schemes based on conventional Fourier techniques are efficient and robust; however, they often suffer from relatively poor resolution [1]. The relative motion of a target generates Doppler shift that is used, along with the range data, to generate 3-D ISAR. The relative motion of the target to the radar can generally be decomposed into two parts: translation and rotation [2]. Achieving high-resolution ISAR imaging requires precise compensation.

Techniques for 3D geometry reconstruction based on ISAR images can be divided into two groups, ISAR image 3D geometry reconstruction and single-sensor-based 3D geometry reconstruction [3, 4]. The focus of this paper is the single-sensor or single view.

Conventional single-sensor 3D geometry reconstruction necessitates prolonged observation and uses the relative kinematics between target and radar to acquire an ISAR image sequence. Subsequent processing extracts the third-dimensional information. The common technique for 3D reconstruction from image sequences is the factorization method [3]. For 3D imaging, one needs a sequence of preprocessing steps prior to reconstruction. These steps include the extraction of the echo of each reflector and trajectory association.

In this report we estimate the single reflector coordinates, including height, and both the yaw and rotation angles of the whole target. This is achieved within a relatively small time around one second. We use Bessel function expansion to overcome the problem of small observation angles and small observation time. For each reflector, the height (or z coordinate) is computed from the compensated range. The compensated range is considered as phase modulated PM signal. We take the cosine of this PM signal. The resultant signal is expanded in terms of Bessel functions. This expansion is acting as frequency amplifier i.e. the low frequency sinusoid is transformed into a high frequency sinusoid. The applied Bessel function expansion for this cosine PM signal is used to extract all the coordinates, the yaw angle, and the rotation angle. Thus, only the range compensated signal is needed. We could also use the phase of the Doppler shift signal to assist in the estimation of the reflector coordinates. Again applying the Bessel function approximation to the cosine of the phase, we are able to estimate the all the coordinates and the angles. Thus, all coordinates of the reflector are estimated. We repeat this process for all the reflectors. In the simulation, we add noise to show the robustness of the proposed method.

This report is divided as follows: Section 2 is problem formulation and the presentation of the motion equations of the target. In Section 3 we present the proposed approach based on Bessel function expansion of the cosine of the phase. Section 4 presents simulations, results, and conclusions. We have an appendix that describes Bessel functions and their properties.

2.Problem Formulation:

The target is composed of point scatterers. The ISAR received signal is a superposition of each return from the individual scatterers. In figure 1, we show a schematic of the incident signal and the target.

The target return from the kth pulse is written as [5]:

$$S_R(t,k)=\sum_i a_i S(t-kT-\tau_i) \quad (1)$$

Where T is the pulse width, a_i is the amplitude and τ_i is the time delay

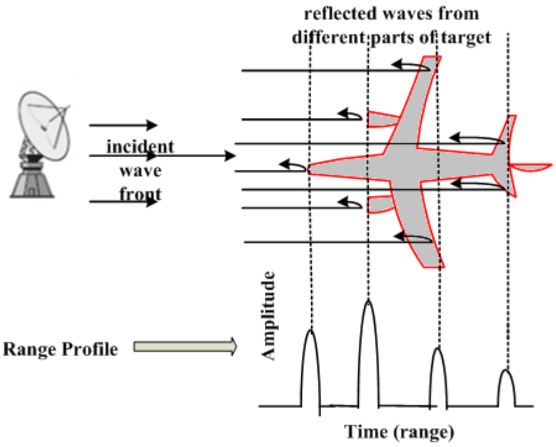


Fig.1, A schematic of incident radar signal and the radar echo [5].

corresponding to the range of the ith scatterer. The time delay could be expressed as:

$$\tau_i=\frac{2R_i(t)}{C} \quad (2)$$

Where $R_i(t)$ is the range to the ith scatterer, and C is the speed of light. Taking the Fourier transform of the reflected time domain signal, we get:

$$S_R(f,k)=S(f)\sum_i a_i e^{-j2\pi f\left(kT+\frac{2R_i(t}{C}\right)}$$

(3)

Where $S(f)$ is the Fourier transform of the transmitted signal.

When the object has both translational and rotational motion, the range of the ith scatterer can be expressed as [1, 2]

$$\begin{aligned}R_i(t)&=R_o+v_R t+a_R t^2/2\\&+x_i\sin\alpha(t)\sin\kappa(t)+y_i\cos\alpha(t)\sin\kappa(t)\\&+z_i\cos\kappa(t)\end{aligned}$$

(4)

Where R_o is the distance between the radar and the center of the target, v_R is the speed of the target, a_R is the acceleration of the target, $\alpha(t)=\Theta t$ is the angular speed or rotational (yaw) displacement, $\kappa(t)$ is the roll angle or roll displacement, (x_i, y_i, z_i) is the coordinates of the ith scattering point on the target.

In the case that we deal with only 2D echo, the reflected signal becomes:

$$S_R(f,k)=S(f)e^{-j2\pi f\left(kT+\frac{2R_0}{C}\right)}\sum_i a_i e^{-j\frac{4\pi}{C}f\left(v_R t+a_R t^2/2+x_i\sin\varepsilon(t)+y_i\cos\alpha(t)\right)},$$ 2D echo

(5)

For the ith scatterer, define the phase angle as:

$$\phi_i(t)=\frac{4\pi f R_i(t)}{C},\ C=f\lambda$$

$$=\frac{4\pi}{C}f\begin{pmatrix}R_0+v_R t+a_R t^2/2\\+x_i\cos\alpha(t)\sin\kappa(t)\\+y_i\sin\alpha(t)\sin\kappa(t)+z_i\cos\kappa(t)\end{pmatrix}$$

(6)

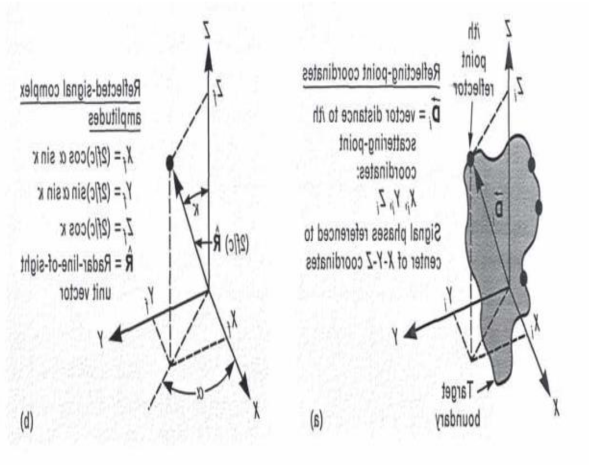


Fig. 2, Geometry of the reflections from one reflector. In the figure, f is the radar frequency and C is the speed of light. [1].

$\Phi t = \kappa(t)$, and $\Theta t = \alpha(t)$ in the figure

Compensating for the range, velocity, and acceleration, we obtain the reduced phase angle:

$$f_{Di}(t) = \frac{4\pi f}{C}\left[\begin{array}{l}(x_i \sin\Phi t)\sin\Theta t \\ +y_i(\sin\Phi t\cos\Theta t) \\ +z_i\cos\Phi t\end{array}\right], C = f\lambda \quad (7)$$

Applying the identity $\sin A\sin B = \frac{1}{2}(\cos(A-B)-\cos(A+B))$ we get:

$$f_{Di}(t) = \frac{4\pi}{\lambda}\left[\begin{array}{l}\frac{x_i}{2}(\cos(\Phi-\Theta)t-\cos(\Phi+\Theta)t) \\ +y_i(\sin\Phi t\cos\Theta t)+z_i\cos\Phi t\end{array}\right] \quad (8)$$

which could be approximated as:

$$f_{Di}(t) \approx \frac{4\pi}{\lambda}\left[\begin{array}{l}\frac{x_i}{2}\left(1-\sin((\Phi+\Theta)t+\frac{\pi}{2})\right) \\ +y_i(\sin\Phi t)+z_i\end{array}\right] \quad (9)$$

$$\approx \frac{4\pi}{\lambda}\left[\begin{array}{l}\frac{x_i}{2}\left(1+\sin(-(\Phi+\Theta)t+\frac{\pi}{2})\right) \\ +y_i(\sin\Phi t)+z_i\end{array}\right]$$

In summary, it is assumed that the target rotation outside of the plane between the radar and the target is small, and the standard range alignment has been applied to the data so that the coarse translation motion has been removed. In Section 3, and using Bessel series expansion, we show that equation (8) will be enough and used to find all the reflector parameters; x, y, z, Φ , and Θ .

2.1 Some Approximations and the Estimation of z_i and $y_i\Phi$:

An approximate formula for $f_{Di}(t)$ of equation (7) is obtained using Taylor series expansion:

$$f_{Di}(t) \approx \frac{4\pi}{\lambda}\left[\begin{array}{l}x_i\Phi\Theta t^2+y_i\left(1-\frac{\Theta^2t^2}{2}\right)\Phi t \\ +z_i\end{array}\right] \quad (10)$$

Collecting terms as power of “t”, we get for the compensated range:

$$f_{Di}(t) \approx \frac{4\pi}{\lambda}\left[z_i+(y_i\Phi)t+x_i\Phi\Theta t^2\right] \quad (11)$$

Using the least square estimation method, this equation could be used to find an estimate for z_i and $y_i\Phi$. The coefficient of “t^2” is relatively small to be detected. The least square method, if not given good initial estimates, will yield poor estimates. This will be clear in the results Section. Even if we have good initial estimates, equation (11) is not enough to find all the parameters of the reflectors. Thus, we need different views of the target to get the desired parameters. We also need the Doppler shift equation to find the unknowns.

2.2 The Doppler shift equation:

The Doppler shift $f'_{Di}(t)$, which is the derivative of the phase of equation (6), is defined as:

$$f'_{Di}(t) = \frac{1}{2\pi}\frac{d\phi_i(t)}{dt} \quad (12)$$

i.e.

$$\begin{aligned}\frac{\lambda}{2}f'_{Di}(t) &= x_i\left(\sin\alpha(t)\frac{d\kappa(t)}{dt}\cos\kappa(t)+\sin\kappa(t)\frac{d\alpha(t)}{dt}\cos\alpha(t)\right) \\ &+ y_i\left(\cos\alpha(t)\frac{d\kappa(t)}{dt}\cos\kappa(t)-\sin\kappa(t)\frac{d\alpha(t)}{dt}\sin\alpha(t)\right) \\ &- z_i\frac{d\kappa(t)}{dt}\sin\kappa(t)\end{aligned} \quad (13)$$

Where λ is the wavelength of the carrier frequency, $\kappa(t) = \Phi t$, and $\alpha(t) = \Theta t$.

For each reflector we have five unknowns; the x, y, z coordinates and the yaw and rotation angles. The yaw and rotation angles, however, are common for all the reflectors. Thus, we need at least five independent measurements to find the estimates of the unknowns. For constant values of Θ, the rotational (yaw) velocity and constant Φ, we get the Doppler shift, f'_{Di}, as:

$$\frac{\lambda}{2} f'_{Di}$$
$$= x_i\left(\sin\alpha(t)\Phi\cos\kappa(t) + \sin\kappa(t)\Theta\cos\alpha(t)\right)$$
$$+ y_i\left(\cos\alpha(t)\Phi\cos\kappa(t) - \sin\kappa(t)\Theta\sin\alpha(t)\right)$$
$$- z_i\Phi\sin\kappa(t)$$

Which is approximated as:

$$\frac{\lambda}{2} f'_{Di} \approx y_i\Phi + \left(2x_i\Theta - z_i\Phi\right)\Phi t$$
$$- y_i\Phi\left(\frac{\Phi^2}{2} + \frac{3\Theta^2}{2}\right)t^2 \qquad (14)$$

2.3 Least squares Method for the Estimation of z_i and $y_i\Phi$ the reflector coordinates:

The conventional methods use the compensated range equation (10) to find an estimate for z_i. Using least squares estimation we use $f_{Di}(t) \approx \frac{4\pi}{\lambda}\left[y_i\left(\sin\Phi t\right) + z_i\right]$ to find an estimate for z_i and $y_i\Phi$.

If the observation time is relatively long (several cycles and several seconds), one could obtain an estimate for both y_i and Φ. Unfortunately, and in many realistic situations, we do not have this luxury. Another point is that the accuracy of the estimates depends, on a large extent, on the additive noise level.

The Doppler shift equation (13) is made of three independent equations of the powers of "t". They are: $y_i\Phi$, $\left(2x_i\Theta - z_i\Phi\right)\Phi$, and $y_i\Phi\left(\frac{\Phi^2}{2} + \frac{3\Theta^2}{2}\right)$.

The coefficient of quadratic term of t^2 is relatively small to render it of any value. Thus, effectively we only have two independent equations; the constant coefficient and the coefficient of 't'. We need another two independent equations to find a unique estimate for the unknowns. In conventional methods, these are obtained through multiple views of the target.

3. The Proposed Solution:

As we have seen in Section 2, a unique solution for the reflector coordinates is unattainable. One must have several views and/or several reflectors to get the desired estimates. Instead, we propose to use Bessel functions expansion to get other independent equations. This will enable us to find, from a single view, a unique solution for all the unknowns.

3.1 The estimation of y_i, z_i and Φ:

Assume that we observe the signal "$\beta\sin(w_m t) + \Psi$" where $\beta = \frac{4\pi}{\lambda} y_i$, $w_m = \Phi$, and $\Psi = \frac{4\pi}{\lambda} z_i$. For small values of w_m <<1, we could not separate the amplitude from the frequency. Through the method of the sum of least squares, we could find an estimate for the product "$\beta w_m = \beta 2\pi f_m$" and an estimate for $\Psi = \frac{4\pi}{\lambda} z_i$.

If we take the cosine of the signal, we obtain $u(t) = \cos(\beta\sin(w_m t) + \Psi)$. If we take the Fourier transform of the signal (using maximum entropy method or others), we find the peak value at the location $|n_{\max}| f_m \approx (2 + \beta - \ln(1 + 2\beta)) f_m$. This is another independent equation in the unknowns β and w_m. Thus, a unique estimate of the unknowns is obtained. To improve the estimates, we use Bessel function expansion with the initial guesses obtained from least square.

We start by the explanation of phase modulation (PM) and the use of Bessel functions to extract the phase.

3.2 Bessel Functions Expansion and PM [6]:

As derived in the appendix, the cosine of the phase (compensated range) is actually a phase modulated signal. This signal could be expanded in terms of the Bessel functions as (see the appendix):

$$\hat{u}(t) = J_0(\hat{\beta})$$
$$+ 2\left[\sum_{n=1}^{n=\infty} J_{2n}(\hat{\beta})\cos\left(2\pi\left(2n\hat{f}_m\right)t + \hat{\Psi}\right)\right] \qquad (15)$$

Where $\hat{u}(t)$ is the estimated signal, $\hat{\beta}$ is the estimated index, $2\pi\hat{f}_m$ is the estimated w_m, and $J_n(\beta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{\beta}{2}\right)^{n+2k}$ is the Bessel function of order n.

By reducing the sum of squared error between the observed, u(t), and the estimated $\hat{u}(t)$, we would be able to find an estimate for β and an estimate for w_m. The initial guess for $\hat{\beta}$ and for $2\pi\hat{f}_m$ are used in the minimization process to get more accurate estimates.

3.3 The estimation of x_i and Θ :

As we did with the estimation of y_i and Φ, we use the approximate compensated observations:

$$f_{Di}(t) - \frac{4\pi}{\lambda}\left(\hat{y}_i\left(\sin\hat{\Phi}t\right)+\hat{z}_i\right) \approx$$

$$\frac{4\pi}{\lambda}\left[\frac{x_i}{2}\left(1-\sin((\Phi+\Theta)t+\frac{\pi}{2})\right)\right]$$

$$= \frac{4\pi}{\lambda}\left[\frac{x_i}{2}-\frac{x_i}{2}\sin((\Phi+\Theta)t+\frac{\pi}{2})\right] \quad (16)$$

where "$\wedge$" over the symbol means its estimated value. Using the method of least squares , we get an estimate for x_i and an estimate for $\frac{x_i}{2}(\Phi + \Theta)$. Taking the Fourier transform (or using maximum entropy), the location of the peak of the spectrum is obtained as $(2+\beta-\ln(1+2\beta))f_m$ where

$$\beta = \frac{4\pi}{\lambda}\frac{x_i}{2} \text{ and } f_m = (\Phi+\Theta)/(2\pi)$$

This will yield an estimate for "$\Phi+\Theta$" and an estimate for x_i. These estimates represent initial guesses for the minimization using the method of least squares as:

$$\min_{f_m,\beta,\Psi}\sum_t\left\{\begin{matrix}\left(J_0(\beta)+ \\ 2\left[\sum_{n=1}^{n=\infty}J_{2n}(\beta)\cos(2\pi(2nf_m)t+\Psi)\right]\right) \\ -u(t)\end{matrix}\right\}^2$$

(17)

where $u(t) = f_{Di}(t) - \frac{4\pi}{\lambda}\left(\hat{y}_i\left(\sin\hat{\Phi}t\right)+\hat{z}_i\right)$

3.4 More than one modulated signal:

We could also estimate all the unknowns in one minimization step. This is computationally expensive. Here we use the estimated parameters as initial guesses and proceed with the minimization w.r.t all the unknowns. This is explained next.

The compensated echo is represented by the sum of two sinusoids as:

$$f_{Di}(t) \approx \frac{4\pi}{\lambda}\left[\begin{matrix}\frac{x_i}{2}\left(1-\sin((\Phi+\Theta)t+\frac{\pi}{2})\right) \\ +y_i\left(\sin\Phi t\right)+z_i\end{matrix}\right]$$

Using the Bessel function expansion we will be able to get a series approximation as follows.

If $u(t)=\cos\left(\sum_{k=1}^{K}\beta_k\sin(2\pi f_k t+\theta_k)+\Psi\right)$, K=2

(18)

where $\beta_1 = \frac{4\pi}{\lambda}\frac{x_i}{2}$, $\beta_2 = \frac{4\pi}{\lambda}y_i$, $2\pi f_1 = -(\Phi+\Theta)$, $2\pi f_2 = \Phi$, $\theta_1 = \pi/2$, $\theta_2 = 0$, $\Psi = (\frac{x_i}{2}+z_i)$

then u(t) has the expansion:

$$u(t) = \sum_{k_K=-\infty}^{\infty}\cdots\sum_{k_1=-\infty}^{\infty}\left[\prod_{i=1}^{K}J_{k_i}(\beta_i)\right] \quad (19)$$

$$\cos\left(\sum_{i=1}^{K}k_i(2\pi f_i t+\theta_i)+\Psi\right)$$

For K=2 we get:

$$u(t) = \sum_{k_2=-\infty}^{\infty}\sum_{k_1=-\infty}^{\infty}\left[\prod_{i=1}^{2}J_{k_i}(\beta_i)\right]$$

$$\cos\left(\sum_{i=1}^{2}k_i(2\pi f_i t+\theta_i)+\Psi\right)$$

$$= \sum_{k_2=-\infty}^{\infty}\sum_{k_1=-\infty}^{\infty}\left[J_{k_1}(\beta_1)J_{k_2}(\beta_2)\right]$$

$$\cos\left(k_1(2\pi f_1 t+\theta_1)+k_2(2\pi f_2 t+\theta_2)+\Psi\right)$$

(20)

After some manipulations (see the appendix) we get:

$$u(t) = 2J_0(\beta_1)\sum_{k_2=1}^{\infty}\left[J_{2k_2}(\beta_2)\right]$$

$$\cos\left((k_2 2\pi f_2)t+k_2\theta_2+\Psi\right)$$

$$+4\sum_{k_2=1}^{\infty}J_{2k_2}(\beta_2)\sum_{k_1=1}^{\infty}\left[J_{2k_1}(\beta_1)\right]$$

$$\cos\left((k_1 2\pi f_1+k_2 2\pi f_2)t+k_1\theta_1+k_2\theta_2+\Psi\right)$$

$$+2J_0(\beta_2)\sum_{k_1=1}^{\infty}\left[J_{2k_1}(\beta_1)\right]$$

$$\cos\left((k_1 2\pi f_1)t+k_1\theta_1+\Psi\right) \quad (21)$$

$$+\left[J_0(\beta_1)J_0(\beta_2)\right]\cos(\Psi)$$

It is this equation we shall use to find more accurate estimates for all the unknowns.

3.5 Summary of the Proposed Algorithm:

(1) Use the equation $f_{Di}(t) \approx \frac{4\pi}{\lambda}\left[y_i\left(\sin\Phi t\right)+z_i\right]$ to find an estimate for $y_i \Phi$,and z_i

(2) Use the location of the peak of the power spectral density $|n_{\max}|\, f_m \approx (2+\beta-\ln(1+2\beta))\, f_m$ to find another independent equation for y_i and Φ and find their estimates

(3) Use Bessel function expansion to improve the estimates of y_i, z_i and Φ

$$\min_{f_m,\beta,\Psi}\sum_t\left\{\begin{pmatrix} J_0(\beta)+ \\ 2\left[\sum_{n=1}^{n=\infty} J_{2n}(\beta)\cos(2\pi(2nf_m)t+\Psi)\right] \\ -u(t) \end{pmatrix}\right\}^2$$

where $u(t) = f_{Di}(t)-\frac{4\pi}{\lambda}\left(\hat{y}_i\left(\sin\hat{\Phi}t\right)+\hat{z}_i\right)$,

$\beta=\frac{4\pi}{\lambda}y_i$, $2\pi f_m=\Phi$, and $\Psi=\frac{4\pi}{\lambda}z_i$.

(4) Repeat the above steps to find an estimate for x_i and Θ

4. Simulation and Conclusions:

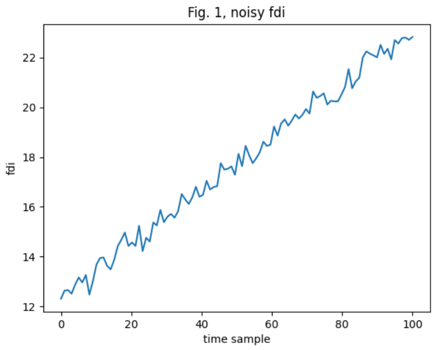
In this section we apply the proposed approach to a simulated signal.

The compensated range is given as:

$$f_{Di}(t) = \frac{4\pi f}{C}\left|\begin{matrix}\left(x_i\sin\Phi t\right)\sin\Theta t \\ +y_i\left(\sin\Phi t\cos\Theta t\right) \\ +z_i\cos\Phi t\end{matrix}\right|, C = f\lambda$$

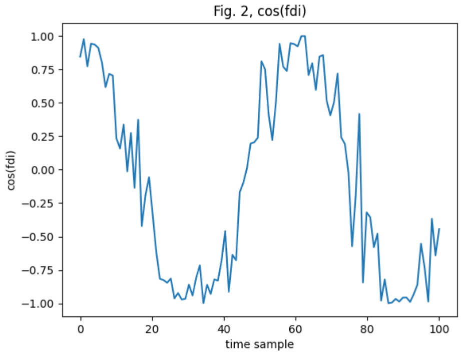
$z_i = 1$, $x_i = 5$, $y_i = 10$, $\Theta = 0.12$, and $\Phi = 0.08$, $\lambda = 1$

In Figure 1, we show noisy fdi

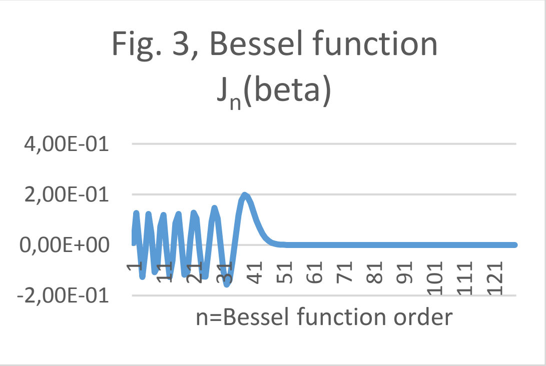


Using least square estimation, the estimated values are: $\frac{4\pi}{\lambda}\hat{z}_i = 12.44$, $\frac{4\pi}{\lambda}\hat{y}_i\hat{\Theta} = 0.1054$

In Figure 2, we show cos(fdi).



In Figure 3, we show $J_n(\beta)$ for different values of n and for $\beta = 40$



Here the maximum value of $J_n(40)$ occurs at n=41, and the max value of $J_{41}(40)=0.2$

In Fig. 4, we show the Bessel function of order 40, $J_{40}(\beta)$, for different values of β

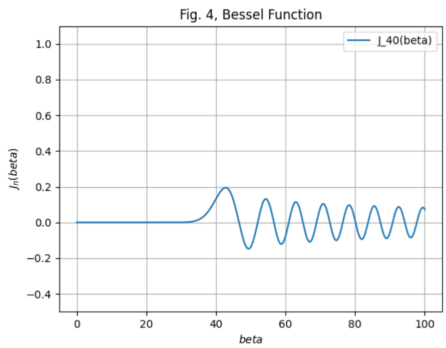


Fig. 4, Bessel Function

In Fig. 5, we show the spectrum of the cos(fdi). The peak is located at 1.8

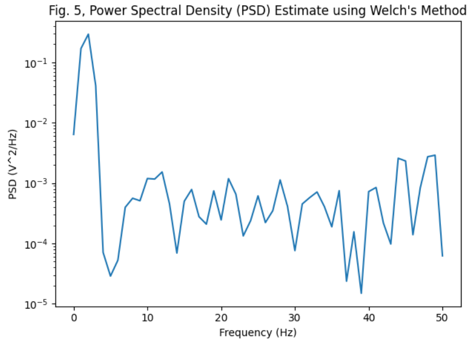


Fig. 5, Power Spectral Density (PSD) Estimate using Welch's Method

Since $u(t)=J_0(\beta)+2\left[\sum_{n=1}^{n=\infty} J_{2n}(\beta)\cos\left(2\pi(2nf_m)t+\Psi\right)\right]$, we need to find β and $2\pi f_m$ that minimizes the sum of squared error between the observed signal and the Bessel-function-expansion based signal. We do this by the minimization of the sum of squared error; Viz:

$$\min_{f_m,\beta,\Psi}\sum_t\left\{\begin{pmatrix} J_0(\beta)+ \\ 2\left[\sum_{n=1}^{n=\infty} J_{2n}(\beta)\cos\left(2\pi(2nf_m)t+\Psi\right)\right] \\ -u(t)\end{pmatrix}\right\}^2$$

Subject to the constraints $0.8\hat{\Phi}<\Phi<1.2\hat{\Phi}$ and $0.8\hat{y}_i<y_i<1.2\hat{y}_i$

The minimization results in the estimated $\hat{\beta}=37.6$, the estimated $\hat{w}_m=\hat{\Phi}=0.088$, and the estimated $\hat{\Psi}=\hat{z}_i=1.006$. The true simulated signal and the estimated signal (using Bessel function expansion) are shown in Fig. 6. The true values are $z_i=1.0$, $y_i=10.0$ and $\Phi=0.08$. The true signal is $u(t)=\cos\left(f_{Di}(t)\right)$. The Bessel function expansion is used for the approximate signal $\cos\left(\frac{4\pi}{\lambda}\left[y_i\left(\sin\Phi t\right)+z_i\right]\right)$

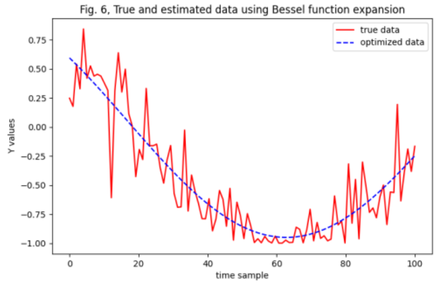


Fig. 6, True and estimated data using Bessel function expansion

4.1 Better Estimate for β :

Since $f_{Di}(t)\approx\frac{4\pi}{\lambda}\left[y_i\left(\sin\Phi t\right)+z_i\right]$ then $\beta\approx\frac{4\pi}{\lambda}y_i$. The location of the peak of Fourier transform is $|n_{\max}|f_m\approx\left(2+\beta-\ln(1+2\beta)\right)f_m$. Then we could always change the wavelength λ to obtain different values of $\beta\approx\frac{4\pi}{\lambda}y_i$ and consequently different values of the peak of the Fourier transform. This will give us a set of equations in the unknowns y_i and f_m . Using regression, we get a new and more accurate estimate for y_i and f_m.

We could also, simultaneously estimate the x,y and z coordinates of the different reflectors since they have the same angles Θ and Φ . This will improve the accuracy of the estimates of the angles as well as the estimates of the coordinates of the reflectors. In this case, the cost function will be the summation of the errors resulting from the different Bessel expansion

4.2 Estimation of x_i and Φ :

We have already found an estimate for $\beta_2=\frac{4\pi}{\lambda}y_i$, $2\pi f_2=\Phi$, and $\Psi=z_i$, we need an estimate for $\beta_1=\frac{4\pi}{\lambda}\frac{x_i}{2}$, $2\pi f_1=\Phi+\Theta$. We could use adaptive noise canceling to remove, from $f_{Di}(t)$, the term $y_i\left(\sin\Phi t\right)+z_i$ and retain only the term $\frac{x_i}{2}\left(-\sin((\Phi+\Theta)t+\frac{\pi}{2})\right)$. In this case, we apply Bessel function expansion to the term $\cos\left(\frac{4\pi}{\lambda}\left[\frac{x_i}{2}\left(-\sin((\Phi+\Theta)t+\frac{\pi}{2})\right)\right]\right)$. The adaptive noise canceling, however, will introduce noise and reduce the accuracy of the estimates.

Instead we use the Bessel function expansion for

$$f_{Di}(t)-\frac{4\pi}{\lambda}\left(\hat{y}_i\left(\sin\hat{\Phi}t\right)+\hat{z}_i\right)$$

$$\approx\frac{4\pi}{\lambda}\left[\frac{x_i}{2}\left(1-\sin((\Phi+\Theta)t+\frac{\pi}{2})\right)\right]$$

the equation:

$$\approx\frac{4\pi}{\lambda}\left[\left(x_i\sin\hat{\Phi}t\right)\sin\Theta t\right]$$

where "$\wedge$" over the symbol means its estimated value.

Through least square estimation, we are able to find an estimate for the product $\hat{x}_i\hat{\Theta}=0.69$. We need another equation to find an estimate for x_i and an estimate for Θ. It was noticed that, for small $\hat{\Phi}<<1$, and for small observation time around 1 second, Bessel function expansion uses the approximation

$$f_{Di}(t)-\frac{4\pi}{\lambda}\left(\hat{y}_i\left(\sin\hat{\Phi}t\right)+\hat{z}_i\right)$$

$$\approx\frac{4\pi}{\lambda}\hat{\Phi}x_i\sin\Theta t$$

i.e. $$\beta\approx\frac{4\pi}{\lambda}\hat{\Phi}x_i$$

The location of the peak of Fourier transform is approximately $(2+\beta-\ln(1+2\beta))f_m$, where $f_m=\Theta/2\pi$. This will give the second independent equation.

Unfortunately the location of the peak is <<1 and the spectrum analysis method (Fourier, Welch, …) could not find. Thus, we multiply the equation of the compensated $f_{Di}(t)$ to obtain:

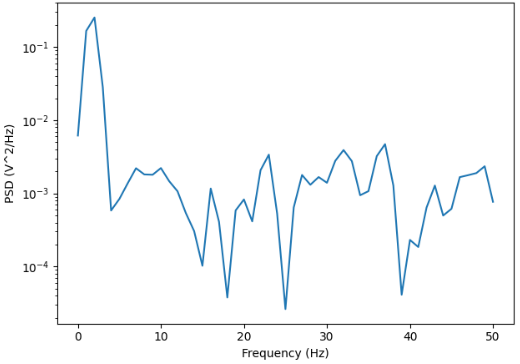
$$scale\left[f_{Di}(t)-\frac{4\pi}{\lambda}\left(\hat{y}_i\left(\sin\hat{\Phi}t\right)+\hat{z}_i\right)\right]$$

$$\approx scale\frac{4\pi}{\lambda}\hat{\Phi}x_i\sin\Theta t$$

And the new $\beta=scale\frac{4\pi}{\lambda}\hat{\Phi}x_i$

Using the scaled version, with scale=10, the spectral density is shown in Fig. 7

Fig. 7, Power Spectral Density (PSD) Estimate using Welch's Method



The location of the peak is $(2+\beta-\ln(1+2\beta))f_m$=1.0. This is the second equation in the unknowns x_i and Θ. Using the two equations, we get the estimates $\hat{x}_i=5.2$ and $\hat{\Theta}=0.14$

4.3 An Improved Estimates for x_i and Θ:

Using the Bessell function expansion for the scaled and compensated fdi, we minimize the sum of squared errors w.r.t x_i and Θ

$$\min_{x_i,\Theta}\left\{\begin{matrix}u(t)-J_0(\beta)-\\2\left[\sum_{n=1}^{n=\infty}J_{2n}(\beta)\cos(2n\Theta t)\right]\end{matrix}\right\}^2,$$

$$\beta=scale\frac{4\pi}{\lambda}\hat{\Phi}x_i$$

Subject to the constraints $0.8\hat{\Theta}<\Theta<1.2\hat{\Theta}$ and $0.8\hat{x}_i<x_i<1.2\hat{x}_i$

Where

$$u(t)=scale\left[f_{Di}(t)-\frac{4\pi}{\lambda}\left(\hat{y}_i\sin(\hat{\Phi}t)+\hat{z}_i\right)\right],$$

scale=10

The constraints ensures that we obtain a unique solution.

The constrained minimization yielded the improved estimates:

$$\hat{x}_i = 4.98 \text{ and } \hat{\Theta} = 0.13$$

The true noisy scaled and compensated data and the Bessel function expansion estimate are shown in Fig. 8

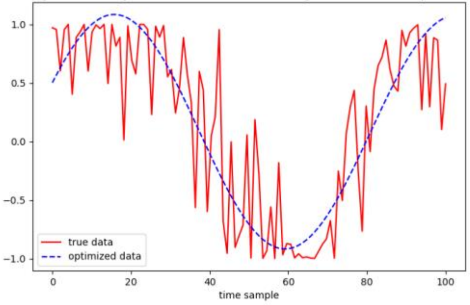


Fig. 8, True noisy data and estimated data using Bessel function expansion

4.4 The Effect of the Noise:

As expected, as the noise level increases the accuracy of the estimates decreases. As long as we are able to locate the peak of the power spectral density, we have reasonable estimates.

4.5 Summary and Future Work:

It was assumed that the target is made of several reflectors and we are able to separate each reflector.

In this paper we were able to find estimates for all the three coordinates of the reflector from only the range data. There is no need to cross range data. The Bessel function expansion of the compensated range data acted as frequency amplifier. Thus, the small frequency, related to the reflector coordinates, was amplified and estimated. Future work will use fractional Fourier transform to better isolate the chirp signal included in the compensated range.

References:

[1]. Sidney 1. Borison, Stephen B. Bowling, and Kevin M. Cuomo, 1992, “Super-Resolution Methods for Wideband Radar”, THE LINCOLN LABORATORY JOURNAL , VOLUME 5, NUMBER 3, pp. 441-461

[2]. Wang, R.; Zhu,W.; Li, C.; Zhu, B.; Pang, H. PP-ISEA: An Efficient Algorithm for High-Resolution Three-Dimensional Geometry Reconstruction of Space Targets using Limited Inverse Synthetic Aperture Radar Images. Sensors 2024, 24, 3550.https://doi.org/10.3390/s24113550

[3]. Mcfadden, F.E. Three-dimensional reconstruction from ISAR sequences. In Proceedings of the SPIE 4744 Sensor Technology and Data Visualization, Orlando, FL, USA, 1–5 April 2002.

[4]. S. Sun, and Y. Jiang, 2017,” Three-dimensional shipborne inverse synthetic aperture radar imaging based on single receiver”, Remote Sensing Letters, Vol. 8, #4, pp. 320-329.

A. Tufan, 2012, COMPARATIVE EVALUATION OF ISAR PROCESSING ALGORITHMS, Ph. D. thesis, Turkey

[5]. Hund, 1942, Frequency Modulation, McGraw Hill, New York.

Appendix

In this appendix we state some of the properties of Bessel functions $J_n(\beta)$.

Bessel Functions and Phase Modulation [6]:
The phase modulated signal could be expressed as expansion of Bessel functions

$$\cos(\beta \sin(w_m t + \theta) + \varphi(t)) = J_0(\beta)$$

$$+ \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(n(w_m t + \theta) + \varphi(t))$$

$$\sin(\omega_c t + B_1 \sin \omega_{m1} t + B_2 \sin \omega_{m2} t) = \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J$$

$$\cos\left(\omega_c t + \left(\sum_{i=1}^{k} B_i \sin(\omega_i t + \theta_i)\right) + \phi\right) = \sum_{k_k} \cdots \sum_{k_1} \left(\prod_{i=1}^{k}\right.$$

The phase modulated signal, u(t), is defined as:

$$u(t) = A_c \cos(w_c t + \beta \sin(w_m t))$$

i.e. $u(t) = \mathrm{Re}\left\{A_c e^{j\beta \sin w_m t} e^{j w_c t}\right\}$

where A_c is the amplitude, w_c is the carrier frequency, β is the amplitude of the signal to be modulated, and w_m is the frequency of the modulated signal.

In order to get the signal of interest "$\beta \sin(w_m t)$" from the measured u(t), one could use $ar\cos u(t)/A_c - w_c t$ which is impractical and has many discontinuities. Instead we use Bessel function expansion.

$e^{j\beta \sin w_m t}$ is periodic and it can be expanded in the Fourier series:

$$e^{j\beta \sin w_m t} = \sum_{n=-\infty}^{\infty} C_n e^{j n w_m t}$$

Where $C_n = \dfrac{w_m}{2\pi} \displaystyle\int_{-\pi/w_m}^{\pi/w_m} e^{j\beta \sin w_m t} e^{-j n w_m t} dt = J_n(\beta)$

Thus, $e^{j\beta \sin w_m t} = \displaystyle\sum_{n=-\infty}^{\infty} J_n(\beta) e^{j n w_m t}$

where $J_n(\beta)$ is the Bessel function,

$$J_n(\beta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{\beta}{2}\right)^{n+2k}$$

For $w_c = 0$ and $A_c = 1$,

$u(t) = \mathrm{Re}\left\{e^{j\beta \sin w_m t}\right\}$, $w_c = 0$ and $A_c = 1$

$\varphi(t)$ could represent the noise

Recall that $\cos(-A) = \cos(A)$ and $J_{-n}(\beta) = (-1)^n J_n(\beta)$, then:

$$\cos(\beta \sin(w_m t + \theta) + \varphi) = J_0(\beta)$$

$$+ 2\sum_{n=1}^{\infty} J_{2n}(\beta) \cos(2n(w_m t + \theta) + \varphi)$$

$$\sin(\omega_c t + B_1 \sin(\omega_{m1} t + B_2 \sin(\omega_{m2} t))) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_n(B_1) J_k(n B_2) \sin(\omega_c + n\omega_{m1} + k\omega_{m2})t$$

Expanding equation in terms of real and imaginary parts we get:

$$\cos(\beta \sin(w_m t)) = \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(w_m t)$$

For small values of w_m, we get:

$$\cos(\beta w_m t) \approx \sum_{n=-\infty}^{\infty} J_n(\beta)$$

Thus, $u(t) = \mathrm{Re}\left\{A_c e^{j\beta \sin w_m t} e^{j w_c t}\right\}$

$$= \mathrm{Re}\left\{ A_c e^{j w_c t} \sum_{n=-\infty}^{\infty} C_n e^{j n w_m t} \right\}$$

$$= \mathrm{Re}\left\{ A_c e^{j w_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j n w_m t} \right\}$$

$$= \mathrm{Re}\left\{ A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j(w_c + n w_m)t} \right\}$$

$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left((w_c + n w_m)t\right)$$

If the carrier frequency is zero and using $J_{-n}(\beta) = (-1)^n J_n(\beta)$, we get:

$$u(t) = J_0(\beta) + 2\left[\sum_{n=1}^{n=\infty} J_{2n}(\beta) \cos\left(2\pi(2n f_m)t\right)\right], \; w_c = 0$$

Where $J_n(\beta) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{\beta}{2}\right)^{n+2k}$

Estimation of β:

If we take the Fourier transform of $\cos(\beta \sin w_m t)$, the peak of the Fourier transform will be located at the peak of $J_n(\beta)$. The magnitude of $J_n(\beta)$ becomes small for $|n| > |\beta|$. The maximum (peak) value of $J_n(\beta)$ is attained at $n_{\max} = |n| \approx 2+\beta+\ln(1+2\beta)$. The value of $J_n(\beta)$ decreases rapidly after the maximum value. Notice that the maximum value of $J_n(\beta)$ is not at the frequency f_m but at the frequency $2n_{\max} f_m, |n_{\max}| \approx 2+\beta-\ln(1+2\beta)$. This is another useful information. If we take the Fourier transform of the signal, we find the peak value at the location $|n_{\max}| f_m \approx (2+\beta-\ln(1+2\beta)) f_m$ and the peak value = $J_{n_{\max}}(\beta)$. For $|\beta| >> 10$, this will give an estimate for 2 $f_m \beta$.

$$u(t) = J_0(\beta) + 2\left[\sum_{n=1}^{n=\infty} J_{2n}(\beta)\cos(2\pi(2nf_m)t)\right]$$

The Bessel functions are nearly 0 until the index (β) equals the order (2n). Then they have a bump and tail off as a sort of damped sinusoid. Thus, the u(t) could be approximated as:

$$u(t) \approx J_0(\beta) + 2J_{2n_{\max}}(\beta)\cos(2\pi(2n_{\max} f_m)t)$$

Where $J_{2n_{\max}}(\beta)$ is the maximum value of $J_{2n}(\beta), n = 1,2,...\infty$

Thus, u(t) is a sum of cosines at frequencies $2nf_m$ with amplitudes $J_{2n}(\beta)$, n=1,2, ….

Some Properties of the Bessel Function:

$$J_{-n}(\beta) = (-1)^n J_n(\beta)$$

$$J_n(\beta) \approx \frac{\beta^n}{2^n n!} \quad \text{for small } \beta$$

For large values of β, we have the approximation:

$$J_n(\beta) \approx \sqrt{\frac{2}{\pi\beta}}\cos\left(\beta - \frac{2n+1}{4}\pi\right), \ \beta >> 1$$

$$J_{2n}(\beta) \approx (-1)^n \sqrt{\frac{1}{\pi\beta}}(\cos\beta + \sin\beta), \ \beta >> 1$$

$$J_{2n+1}(\beta) \approx (-1)^{n+1} \sqrt{\frac{1}{\pi\beta}}(\cos\beta - \sin\beta), \ \beta >> 1$$

$$J_{n+1}(\beta) = \frac{2n}{\beta} J_n(\beta) - J_{n-1}(\beta), \ \forall\beta$$

As n increases, $\lim_{n\to\infty} J_n(\beta) = 0 \ \ \forall\beta$

$$J_n(\beta) = \frac{\beta^n}{2^n (n!)}[1+\alpha_1+\alpha_2+...] \ \forall\beta$$

$$\alpha_1 = -\frac{\beta^2}{2(2n+2)}, \qquad \alpha_2 = -\frac{\beta^2}{4(2n+4)}\alpha_1,$$

$$\alpha_3 = -\frac{\beta^2}{6(2n+6)}\alpha_2, \ \alpha_4 = -\frac{\beta^2}{8(2n+6)}\alpha_3 \ldots \text{etc}$$

Discrete Hankel transform:

For f(t) a stationary or non stationary process we have:

$$F_v(u) = \int_0^\infty f(t) J_v(ut) t dt \ , \ v \ge -1/2$$

$$f(t) = \int_0^\infty F_v(u) J_v(ut) u du$$

Where $J_v(ut)$ is the Bessel function.

$\int_0^\infty J_v(wt) J_v(ut) t dt = \frac{\delta(w-u)}{w}$, **Orthogonality:**

In some situations we have a chirp or a linear-in-time equations. We could use the Hankel transform to separate the components.

If $f(t) = \begin{cases} t^v, t \le T \\ 0, t > T \end{cases}$, then its Hankel transform is given by

$$H_v[f(t);\rho] = \frac{T^{v+1}}{\rho} J_{v+1}(\rho t)$$

Also we use the linearity property:

$$H_v[c_1 f(t) + c_2 g(t)] = c_1 H_v[f(t);\rho] + c_2 H_v[g(t);\rho]$$

The scaling property:

$$H_v[f(at);\rho] = \frac{1}{a^2} F\left(\frac{\rho}{a}\right)$$

Where $H_v[f(t);\rho] = F(\rho) = \int_0^\infty t f(t) J_v(\rho t) dt$

Fourier-Bessel transform:

$f(t) = \sum_{n=1}^{\infty} C_n J_o(\lambda_n t)$, 0<t<T

Where λ_n is the nth zero of $J_0(x)$

$$C_n = \frac{2\int_0^T tf(t)J_0(\lambda_n t)dt}{T^2[J_1(\lambda_n T)]^2}$$

$$\sum_{v=-\infty}^{\infty} J_v(x)J_{n-v}(y) = J_n(x+y)$$

$$\sum_{v=-\infty}^{\infty} J_v(x)J_{n+v}(y) = J_n(y-x)$$

$$\sum_{v=-\infty}^{\infty} J_v(x)J_{n+v}(x) = \delta_{n,0}$$

$$\sum_{v=-\infty}^{\infty} vJ_v(x)J_{n+v}(x) = \frac{x}{2}\left(\delta_{n,1} + \delta_{n,-1}\right)$$

$$\sum_{v=-\infty}^{\infty} v^2 J_v(x)J_{n+v}(x) = \frac{x}{2}\left(\delta_{n,-1} - \delta_{n,1}\right) + \frac{x^2}{4}\left(\delta_{n,-2} + 2\delta_{n,0} + \delta_{n,2}\right)$$

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta)$$

Differentiate twice w.r.t. θ and setting $\theta = \pi/2$ we get:

$$x\sin x = 2\{2^2 J_2(z) - 4^2 J_4(z) + \ldots\}$$

$$\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin((2n-1)\theta)$$

Differentiate $\sin(x\sin\theta)$ w.r.t. θ we get:

$$\cos(x\sin\theta)\left(x\cos\theta\right) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)(2n-1)\cos((2n-1)\theta)$$

Setting $\theta = 0$ we get:

$$x = 2\sum_{n=1}^{\infty}(2n-1)J_{2n-1}(x)$$

Differentiate $\sin(x\sin\theta)$ twice w.r.t. θ and put $\theta = \pi/2$ we get:

$$x\cos x = 2\{1^2 J_1(z) - 3^2 J_3(z) + \ldots\}$$

$$\cos(x) = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$$

$$\sin(x) = +2\sum_{n=1}^{\infty}(-1)^{n+1} J_{2n+1}(x)$$

$$\sin(\phi - x\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(x)\sin(n\theta + \phi)$$

$$e^{x\cos\theta} = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x)\cos(n\theta),$$

where $I_n(x) = (i)^{-n} J_n(ix)$ is a modified Bessel function of the first kind

$$e^{ix\cos\theta} = J_0(x) + 2\sum_{n=1}^{\infty}(i)^n J_n(x)\cos(n\theta)$$

$$J_0(x) = \frac{2}{\pi^2}\int_0^{\pi} t\cos(x\sin t)dt$$

$$J_0(x) = \frac{2}{\pi}\int_0^{\infty}\sin(x\cosh t)dt$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin t - nt)dt$$

$$\frac{\partial J_n(x)}{\partial x} = \frac{1}{\pi}\int_0^{\pi} -\sin t\sin(x\sin t - nt)dt$$

For large x

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

$$\frac{\partial J_n(x)}{\partial x} \approx -\left(\frac{2}{\pi x}\right)^{1/2}\sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

$$-\frac{1}{\pi x^2}\left(\frac{2}{\pi x}\right)^{-1/2}\cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

Derivative w.r.t. x:

$$\frac{\partial J_n(x)}{\partial x} = 0.5\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

For large x

$$\frac{\partial J_n(x)}{\partial x} \approx 0.5\sqrt{\frac{2}{\pi x}}\begin{bmatrix}\cos\left(x - \frac{\pi}{4} - \frac{(n-1)\pi}{2}\right) \\ -\cos\left(x - \frac{\pi}{4} - \frac{(n+1)\pi}{2}\right)\end{bmatrix}$$

$$\approx 0.5\sqrt{\frac{2}{\pi x}}\begin{bmatrix}\cos(x - \frac{\pi}{4})\cos(-\frac{(n-1)\pi}{2}) \\ -\sin(x - \frac{\pi}{4})\sin(-\frac{(n-1)\pi}{2}) \\ -\cos(x - \frac{\pi}{4})\cos(-\frac{(n+1)\pi}{2}) \\ +\sin(x - \frac{\pi}{4})\sin(-\frac{(n+1)\pi}{2})\end{bmatrix}$$

$$\approx 0.5\sqrt{\frac{2}{\pi x}}\left[\begin{matrix}\cos(x-\frac{\pi}{4})\begin{pmatrix}\cos(-\frac{(n-1)\pi}{2})\\ -\cos(-\frac{(n+1)\pi}{2})\end{pmatrix}\\ +\sin(x-\frac{\pi}{4})\begin{pmatrix}-\sin(-\frac{(n-1)\pi}{2})\\ +\sin(-\frac{(n+1)\pi}{2})\end{pmatrix}\end{matrix}\right]$$

$$\approx 0.5\sqrt{\frac{2}{\pi x}}\left[\begin{matrix}\cos(x-\frac{\pi}{4})\begin{pmatrix}\cos(\frac{(n-1)\pi}{2})\\ -\cos(\frac{(n+1)\pi}{2})\end{pmatrix}\\ +\sin(x-\frac{\pi}{4})\begin{pmatrix}\sin(\frac{(n-1)\pi}{2})\\ -\sin(\frac{(n+1)\pi}{2})\end{pmatrix}\end{matrix}\right]$$

$$\frac{\partial J_0(x)}{\partial x}=-J_1(x)$$