Duality Between Augmented Complex Kalman Filters and Bivariate Real Kalman Filters

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Abstract: - The covariance and the pseudo-covariance matrices characterize the complex signals. The augmented or widely linear model, which takes into account these matrices, leads to the Augmented Complex Kalman Filter and its variations. The duality between the augmented complex Kalman filters and the corresponding dual bivariate real Kalman filters is addressed. The complex Kalman filters and the dual real Kalman filters compute the same estimates. Stability and convergence analysis for real-valued Kalman Filter also apply to the proposed Kalman filters. The computational requirements of the complex and dual real Kalman filters are presented. The ability to determine the fastest filter depends on the model dimensions, namely the state and measurement dimensions.

Key-Words: - Augmented or Widely Linear Model, Augmented Complex Kalman Filter, Augmented Complex Information Kalman Filter, Augmented Complex Kalman Filter Gain Elimination, Duality, Calculation Burden

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1 Introduction

Kalman filter [1], [2] is the best-known estimation and prediction algorithm and has been used with success in a wide range of applications: temperature prediction [3], object detection and tracking [4], electric load estimation [5], short-term temperature forecasts [6], autonomous orbit determination of BeiDou Navigation Satellite System [7], vehicle movement estimation [8], GPS position estimation and prediction [9], cases prediction of Covid-19 [10], multi-observation fusion applications related to timescale [11], structural parameter tracking [12], applications with time-correlated measurement errors [13], control effectiveness estimation on airplanes [14], applications in aircraft state estimation [15], vehicle location estimation [16], estimation with unlimited sensing measurements [17], multi-target localization [18], Kalman filter-based tracking-by-detection (KFTBD) tracker [19], ECG signal de-noising [20].

The conventional and the augmented complex Kalman filters [21] are the most well-known estimation algorithms that have been successfully used in various applications where complex signals are involved, such as applications in tracking, oceanography, array processing, communications, biomedicine [22], distribution state estimation [23], two-dimensional local navigation systems [24], tracking for Global Navigation Satellite System meta-signals [25].

Complex signals have two fundamental statistical properties: the covariance matrix that has to do with the total power of the signal and the pseudo-covariance matrix that has to do with the correlation between the real part and the imaginary part of the signal [22]. The augmented model or widely linear model [26] takes into account both the covariance as well as the pseudo-covariance matrices.

Using the augmented model or widely linear model, the Augmented Complex Kalman Filter is derived [21], [26]. In addition, it has been shown [26], [27] that the Augmented Complex Kalman Filter has a dual bivariate Real Kalman Filter. This dual filter is faster than the Augmented Complex Kalman Filter.

Furthermore, two variations of the Augmented Complex Kalman Filter have been proposed: the Augmented Complex Information Kalman Filter [28] that uses the information matrices (the inverses of the covariance and pseudo-covariance matrices) and the Augmented Complex Kalman Filter Gain Elimination [29] that eliminates the Kalman filter gain. These variations may be faster than the Augmented Complex Kalman Filter.

Motivated by reducing the computational complexity and consequently minimizing the computational time, in this paper we address the duality of these variations of the Augmented Complex Kalman Filter with dual bivariate real-valued Kalman filters. In fact, using the two variations of Augmented Complex Kalman Filter we prove the derivation of the corresponding dual Kalman filters. Simulation results confirm that the three complex Kalman filters and the corresponding dual Kalman filters are equivalent to each other since they compute the same estimates. Furthermore, we determine the computational requirements of the complex and dual Kalman filters. Finally, we detect the fastest filter by only taking into account the state and measurement dimensions.

2 Problem Formulation

Consider the **augmented or widely linear model** [26] described by the state space equations:

$$x^a(k) = F^a(k)x^a(k-1) + w^a(k) \quad (1)$$

$$z^a(k) = H^a(k)x^a(k) + v^a(k) \quad (2)$$

In this model, $x^a(k) = \begin{bmatrix} x(k) \\ \bar{x}(k) \end{bmatrix}$ is the $2n \times 1$ augmented state vector, $z^a(k) = \begin{bmatrix} z(k) \\ \bar{z}(k) \end{bmatrix}$ is the $2m \times 1$ augmented measurement vector, $w^a(k) = \begin{bmatrix} w(k) \\ \bar{w}(k) \end{bmatrix}$ is the $2n \times 1$ augmented state noise vector, $v^a(k) = \begin{bmatrix} v(k) \\ \bar{v}(k) \end{bmatrix}$ is the $2m \times 1$ augmented measurement noise vector; note that $\bar{x}$ denotes the complex conjugate of the complex variable x. Also, the augmented initial state $x^a(0)$ is non-circular Gaussian with known mean ${x_0}^a = \begin{bmatrix} x_0 \\ \bar{x}_0 \end{bmatrix}$ and known covariance ${P_0}^a = \begin{bmatrix} P_0 & \Pi_0 \\ \bar{\Pi}_0 & \bar{P}_0 \end{bmatrix}$.

The model parameters, which are assumed to be known, are: the $2n \times 2n$ augmented transition matrix $F^a(k) = \begin{bmatrix} F(k) & A(k) \\ \bar{A}(k) & \bar{F}(k) \end{bmatrix}$, the $2m \times 2n$ augmented output matrix $H^a(k) = \begin{bmatrix} H(k) & B(k) \\ \bar{B}(k) & \bar{H}(k) \end{bmatrix}$, the augmented covariance matrix $Q^a(k) = \begin{bmatrix} Q(k) & U(k) \\ \bar{U}(k) & \bar{Q}(k) \end{bmatrix}$ of the non-circular Gaussian zero mean state noise process, the augmented covariance matrix $R^a(k) = \begin{bmatrix} R(k) & V(k) \\ \bar{V}(k) & \bar{R}(k) \end{bmatrix}$ of the non-circular Gaussian zero mean measurement noise process; note that $Q(k), R(k)$ are Hermitian covariance matrices (M is a Hermitian matrix when it is equal to its conjugate transpose $M^* = M$), while $U(k), V(k)$ are symmetric pseudo-covariance matrices (M is a symmetric matrix when it is equal to its transpose $M^T = M$). The model becomes time invariant in the special case where all the model parameters are constant in time: $F^a(k) = F^a, H^a(k) = H^a, Q^a(k) = Q^a, R^a(k) = R^a$.

The pair (F^a, H^a) is observable if the associated observability matrix is full rank [27]. There is no discussion about controllability as there is no input matrix in this model.

The Augmented Complex Kalman Filters have been derived from the above augmented or widely linear model. Given the measurements till time k, the Augmented Complex Kalman Filters compute iteratively the augmented state estimation $x^a(k|k) = \begin{bmatrix} x(k|k) \\ \bar{x}(k|k) \end{bmatrix}$ with the corresponding augmented estimation error covariance matrix $P^a(k|k) = \begin{bmatrix} P(k|k) & \Pi(k|k) \\ \bar{\Pi}(k|k) & \bar{P}(k|k) \end{bmatrix}$ as well as the augmented state prediction $x^a(k+1|k) = \begin{bmatrix} x(k+1|k) \\ \bar{x}(k+1|k) \end{bmatrix}$ with the corresponding augmented prediction error covariance matrix $P^a(k+1|k) = \begin{bmatrix} P(k+1|k) & \Pi(k+1|k) \\ \bar{\Pi}(k+1|k) & \bar{P}(k+1|k) \end{bmatrix}$.

The **Augmented Complex Kalman Filter (ACKF)** [21], [26] uses the augmented Kalman filter gain $K^a(k) = \begin{bmatrix} K(k) & G(k) \\ \bar{G}(k) & \bar{K}(k) \end{bmatrix}$. The time varying ACKF (ACKFtv) has the form:

ACKFtv

initial conditions

$x^a(0|-1) = {x_0}^a$

$P^a(0|-1) = {P_0}^a$

iterations $k = 0,1, \ldots$

$K^a(k) = P^a(k|k-1)H^{a*}(k)[H^a(k)P^a(k|k-1)H^{a*}(k) + R^a(k)]^{-1}$

$x^a(k|k) = x^a(k|k-1) + K^a(k)[z^a(k) - H^a(k)x^a(k|k-1)]$

$P^a(k|k) = P^a(k|k-1) - K^a(k)H^a(k)P^a(k|k-1)$

$x^a(k+1|k) = F^a(k)x^a(k|k)$

$P^a(k+1|k) = Q^a(k) + F^a(k)P^a(k|k)F^{a*}(k)$

For time invariant model, the time invariant ACKF (ACKFti) is derived.

The augmented steady-state complex Kalman filter is derived for complex augmented or widely linear systems [29]; then the solution of the augmented complex Riccati equation is required as a necessary prerequisite in order to determine the steady-state parameters of the augmented steady-state complex Kalman filter before observing any measurements.

It is worth to note that the use of the pseudo-covariance matrix in ACKF can improve the performance of CCKF (Conventional Complex Kalman Filter) [27]. In fact, the analysis in [26] has shown that the ACKF offers significant performance gains over the CCKF for noncircular signals, and the same performance as the CCKF for circular signals.

The performances of ACKF and CCKF were compared in [27] and the basic results were: a) the mean squared error (MSE) of the ACKF is significantly smaller than the MSE of a CCKF that does not exploit non-zero complementary covariance, b) the MSE of the ACKF converges in the general case of improper noises.

The effect of signal non-circularity on the mean square behavior of the CCKF was analyzed in [26] and the Cramer–Rao lower bound (CRLB) for the ACKF was established.

The **Augmented Complex Information Kalman Filter (ACIKF)** [28] uses the augmented information state estimation $y^a(k|k) = P^{a^{-1}}(k|k)x^a(k|k)$ and the corresponding augmented information estimation error covariance matrix $S^a(k|k) = P^{a^{-1}}(k|k)$ as well as the augmented information state prediction $y^a(k+1|k) = P^{a^{-1}}(k+1|k)x^a(k+1|k)$ and the corresponding augmented information prediction error covariance matrix $S^a(k+1|k) = P^{a^{-1}}(k+1|k)$. The time varying ACIKF (ACIKFtv) has the form:

ACIKFtv
initial conditions
$x^a(0|-1) = x_0{}^a$
$P^a(0|-1) = P_0{}^a$
$y^a(0|-1) = P^{a^{-1}}(0|-1)x^a(0|-1) = P_0{}^{a^{-1}}x_0{}^a$
$S^a(0|-1) = P^{a^{-1}}(0|-1) = P_0{}^{a^{-1}}$
iterations $k = 0,1,\ldots$
$y^a(k|k) = y^a(k|k-1) + H^{a^*}(k)R^{a^{-1}}(k)z^a(k)$
$S^a(k|k) = S^a(k|k-1) + H^{a^*}(k)R^{a^{-1}}(k)H^a(k)$
$P^a(k|k) = S^{a^{-1}}(k|k)$
$x^a(k|k) = P^a(k|k)y^a(k|k)$
$K^a(k) = P^a(k|k)H^{a^*}(k)R^{a^{-1}}(k)$
$P^a(k+1|k) = Q^a(k) + F^a(k)P^a(k|k)F^{a^*}(k)$
$S^a(k+1|k) = P^{a^{-1}}(k+1|k)$
$y^a(k+1|k) = S^a(k+1|k)F^a(k)P^a(k|k)y^a(k|k)$
$x^a(k+1|k) = P^a(k|k)y^a(k+1|k)$

For time invariant model, the time invariant ACIKF (ACIKFti) is derived; then $R^{a^{-1}}, H^{a^*}R^{a^{-1}}, H^{a^*}R^{a^{-1}}H^a$ are computed off-line.

It is worth to note that ACKF and ACIKF are equivalent with respect to a) the derivation of the state estimations and predictions and the corresponding error covariances, b) their stability [28].

The **Augmented Complex Kalman Filter Gain Elimination (ACKFGE)** [29] substitutes the augmented Kalman filter gain by the matrix $\Lambda^a(k) = P^a(k|k-1)H^{a^*}(k)R^{a^{-1}}(k)$. The time varying ACKFGE (ACKFGEtv) has the form:

ACKFGEtv
initial conditions
$x^a(0|-1) = x_0{}^a$
$P^a(0|-1) = P_0{}^a$
iterations $k = 0,1,\ldots$
$\Lambda^a(k) = P^a(k|k-1)H^{a^*}(k)R^{a^{-1}}(k)$
$x^a(k|k) = [I^a + \Lambda^a(k)H^a(k)]^{-1}\{x^a(k|k-1) + \Lambda^a(k)z^a(k)\}$
$P^a(k|k) = [I^a + \Lambda^a(k)H^a(k)]^{-1}P^a(k|k-1)$
$x^a(k+1|k) = F^a(k)x^a(k|k)$
$P^a(k+1|k) = Q^a(k) + F^a(k)P^a(k|k)F^{a^*}(k)$

For time invariant model, the time invariant ACKFGE (ACKFGEti) is derived; then $R^{a^{-1}}, H^{a^*}R^{a^{-1}}$ are computed off-line.

It is worth to note that ACKF and ACKFGE are equivalent with respect to the derivation of the state estimations and predictions and the corresponding error covariances.

3 Dual Augmented Complex Kalman Filters

Consider the duality concept used in [26], [27], where for a $n \times 1$ complex vector $x = x^R + jx^I$ (where x^R denotes its real part and x^I denotes its imaginary part), its $2n \times 1$ augmented vector $x^a = \begin{bmatrix} x \\ \bar{x} \end{bmatrix}$ is related to its $2n \times 1$ dual vector $x^d = \begin{bmatrix} x^R \\ x^I \end{bmatrix}$

by the relation $x^a = J_n x^d$, where $J_n = \begin{bmatrix} I_n & jI_n \\ I_n & -jI_n \end{bmatrix}$ is of dimension $2n \times 2n$ and I_n is the $n \times n$ identity matrix. Note that the following property holds: $J_n^* = 2J_n^{-1}$, with $J_n^{-1} = \frac{1}{2}\begin{bmatrix} I_n & I_n \\ -jI_n & jI_n \end{bmatrix}$. Also, the $2m \times 2n$ augmented matrix M^a is related to the $2m \times 2n$ dual matrix M^d by the relation $M^a = J_m M^d J_n^{-1}$ and the $2n \times 2n$ augmented covariance matrix P^a is related to the $2n \times 2n$ dual matrix P^d by the relation $P^a = J_n P^d J_n^*$.

Then, for the augmented or widely linear model we have:

$x^a(k) = J_n x^d(k), z^a(k) = J_m z^d(k), w^a(k) = J_n w^d(k), v^a(k) = J_m v^d(k)$ and

$F^a(k) = J_n F^d(k) J_n^{-1}, H^a(k) = J_m H^d(k) J_n^{-1}, Q^a(k) = J_n Q^d(k) J_n^*, R^a(k) = J_m R^d(k) J_m^*$.

Note that $F^d(k), H^d(k), Q^d(k), R^d(k)$ are real matrices and that $Q^d(k), R^d(k)$ symmetric.

In fact,

$$F^d(k) = J_n^{-1} F^a(k) J_n = \begin{bmatrix} F^R(k) + A^R(k) & -F^I(k) + A^I(k) \\ F^I(k) + A^I(k) & F^R(k) - A^R(k) \end{bmatrix}$$

$$H^d(k) = J_m^{-1} H^a(k) J_n = \begin{bmatrix} H^R(k) + B^R(k) & -H^I(k) + B^I(k) \\ H^I(k) + B^I(k) & H^R(k) - B^R(k) \end{bmatrix}$$

$$Q^d(k) = J_n^{-1} Q^a(k) {J_n^*}^{-1} = \frac{1}{2} J_n^{-1} Q^a(k) J_n = \frac{1}{2}\begin{bmatrix} Q^R(k) + U^R(k) & -Q^I(k) + U^I(k) \\ Q^I(k) + U^I(k) & Q^R(k) - U^R(k) \end{bmatrix}$$

$$R^d(k) = J_m^{-1} R^a(k) {J_m^*}^{-1} = \frac{1}{2} J_n^{-1} R^a(k) J_n = \frac{1}{2}\begin{bmatrix} R^R(k) + V^R(k) & -R^I(k) + V^I(k) \\ R^I(k) + V^I(k) & R^R(k) - V^R(k) \end{bmatrix}$$

Furthermore, we have:

$x^a(k|k) = J_n x^d(k|k), x^a(k+1|k) = J_n x^d(k+1|k)$ and

$P^a(k|k) = J_n P^d(k|k) J_n^*, P^a(k+1|k) = J_n P^d(k+1|k) J_n^*$.

Note that $x^d(k|k), x^d(k+1|k)$ are real vectors and that $P^d(k|k), P^d(k+1|k)$ are real symmetric matrices.

In fact,

$$x^d(k|k) = J_n^{-1} x^a(k|k) = \begin{bmatrix} x^R(k|k) \\ x^I(k|k) \end{bmatrix}$$

$$x^d(k+1|k) = J_n^{-1} x^a(k+1|k) = \begin{bmatrix} x^R(k+1|k) \\ x^I(k+1|k) \end{bmatrix}$$

$$P^d(k|k) = J_n^{-1} P^a(k|k) {J_n^*}^{-1} = \frac{1}{2} J_n^{-1} P^a(k|k) J_n = \frac{1}{2}\begin{bmatrix} P^R(k|k) + \Pi^R(k|k) & -P^I(k|k) + \Pi^I(k|k) \\ P^I(k|k) + \Pi^I(k|k) & P^R(k|k) - \Pi^R(k|k) \end{bmatrix}$$

$$P^d(k+1|k) = J_n^{-1} P^a(k+1|k) {J_n^*}^{-1} = \frac{1}{2} J_n^{-1} P^a(k+1|k) J_n = \frac{1}{2}\begin{bmatrix} P^R(k+1|k) + \Pi^R(k+1|k) & -P^I(k+1|k) + \Pi^I(k+1|k) \\ P^I(k+1|k) + \Pi^I(k+1|k) & P^R(k+1|k) - \Pi^R(k+1|k) \end{bmatrix}$$

Also, we have:

$x_0{}^a = x^a(0|-1) = J_n x^d(0|-1) = J_n x_0{}^d$ and

$P_0{}^a = P^a(0|-1) = J_n P^d(0|-1) J_n^* = J_n P_0{}^d J_n^* = 2 J_n P_0{}^d J_n^{-1}$

Finally,

$I^a = J_n I^d J_n^{-1}$

The **Dual Augmented Complex Kalman Filter (DACKF)** [26] is derived by the Augmented Complex Kalman Filter using

$K^a(k) = J_n K^d(k) J_m^{-1}$

Note that $K^d(k)$ is real. In fact

$$K^d(k) = J_n^{-1} K^a(k) J_m = \begin{bmatrix} K^R(k) + G^R(k) & -K^I(k) + G^I(k) \\ K^I(k) + G^I(k) & K^R(k) - G^R(k) \end{bmatrix}$$

The time varying Dual Augmented Complex Kalman Filter (DACKFtv) has the form:

DACKFtv

initial conditions

$x^a(0|-1) = x_0{}^a$

$P^a(0|-1) = P_0{}^a$

$x^d(0|-1) = x_0{}^d = J_n^{-1} x^a(0|-1) = J_n^{-1} x_0{}^a$

$P^d(0|-1) = P_0{}^d = J_n^{-1} P^d(0|-1) {J_n^*}^{-1} = J_n^{-1} P_0{}^a {J_n^*}^{-1} = \frac{1}{2} J_n^{-1} P_0{}^a J_n$

iterations $k = 0,1,\ldots$

$F^d(k) = J_n^{-1} F^a(k) J_n$

$H^d(k) = J_m^{-1} H^a(k) J_n$

$Q^d(k) = J_n^{-1} Q^a(k) {J_n^*}^{-1}$

$R^d(k) = J_m^{-1} R^a(k) {J_m^*}^{-1}$

$K^d(k) = P^d(k|k-1) {H^d}^T(k) \left[H^d(k) P^d(k|k-1) {H^d}^T(k) + R^d(k)\right]^{-1}$

$x^d(k|k) = x^d(k|k-1) + K^d(k)\left[z^d(k) - H^d(k) x^d(k|k-1)\right]$

$P^d(k|k) = P^d(k|k-1) - K^d(k) H^d(k) P^d(k|k-1)$

$x^d(k+1|k) = F^d(k) x^d(k|k)$

$P^d(k+1|k) = Q^d(k) + F^d(k) P^d(k|k) {F^d}^T(k)$

For time invariant model, the time invariant DACKF (DACKFti) is derived; then F^d, H^d, Q^d, R^d are computed off-line and once.

It is worth to note that the time varying Dual Augmented Complex Kalman Filter and the time invariant Dual Augmented Complex Kalman Filter have the same structure as the real time varying and time invariant Kalman filters.

A theoretical bound for the performance advantage of ACKF over CCKF was provided in [26]; the analysis also has addressed the duality with bivariate real-valued Kalman Filter. Moreover, due to the duality between the bivariate real-valued Kalman Filter and ACKF, the stability and convergence analysis for real-valued Kalman Filter also apply to the ACKF [26].

In the following, the **Dual Augmented Complex Information Kalman Filter (DAICKF)** is derived by the Augmented Complex Information Kalman Filter.

Proof.

$$x^a(0|-1) = {x_0}^a = J_n x^d(0|-1)$$
$$\Rightarrow x^d(0|-1) = {x_0}^d = J_n^{-1} x^a(0|-1) = J_n^{-1}{x_0}^a$$

$$P^a(0|-1) = {P_0}^a = J_n P^d(0|-1) J_n^*$$
$$\Rightarrow P^d(0|-1) = {P_0}^d = J_n^{-1} P^d(0|-1) {J_n^*}^{-1} = J_n^{-1}{P_0}^a {J_n^*}^{-1} = \tfrac{1}{2} J_n^{-1}{P_0}^a J_n$$

$${S_0}^d = \left({P_0}^d\right)^{-1} = \left(P^d(0|-1)\right)^{-1}$$
$$S^a(0|-1) = {S_0}^a = \left(P^a(0|-1)\right)^{-1} = {P_0}^{a^{-1}} = \left(J_n P^d(0|-1) J_n^*\right)^{-1} = \left(J_n {P_0}^d J_n^*\right)^{-1} = {J_n^*}^{-1}\left({P_0}^d\right)^{-1} J_n^{-1} = {J_n^*}^{-1}{S_0}^d J_n^{-1} = \tfrac{1}{2} J_n {S_0}^d J_n^{-1}$$
$$\Rightarrow S^d(0|-1) = {S_0}^d = 2J_n^{-1}{S_0}^a J_n = \left(\tfrac{1}{2}J_n^{-1}{P_0}^a J_n\right)^{-1} = {P_0}^{d^{-1}}$$

Then

$$S^a(k|k-1) = \tfrac{1}{2} J_n S^d(k|k-1) J_n^{-1}$$

$${y_0}^d = {S_0}^d {x_0}^d$$
$$y^a(0|-1) = \left(P^a(0|-1)\right)^{-1} x^a(0|-1) = {P_0}^{a^{-1}}{x_0}^a = \left(J_n P^d(0|-1) J_n^*\right)^{-1} J_n x^d(0|-1) = \left(J_n {P_0}^d J_n^*\right)^{-1} J_n {x_0}^d = {J_n^*}^{-1}\left({P_0}^d\right)^{-1} J_n^{-1} J_n {x_0}^d = {J_n^*}^{-1}{S_0}^d {x_0}^d = {J_n^*}^{-1}{y_0}^d$$
$$\Rightarrow y^d(0|-1) = {y_0}^d = J_n^* y^a(0|-1) = J_n^* {y_0}^a = J_n^* {S_0}^a {x_0}^a = J_n^* {P_0}^{a^{-1}} {x_0}^a$$
$$= J_n^* {J_n^*}^{-1} {S_0}^d J_n^{-1} {x_0}^a = {S_0}^d J_n^{-1} {x_0}^a = {S_0}^d {x_0}^d = {P_0}^{d^{-1}} {x_0}^d$$

Then

$$y^a(k|k-1) = \tfrac{1}{2} J_n y^d(k|k-1)$$

$$y^a(k|k) = \tfrac{1}{2} J_n y^d(k|k)$$
$$y^a(k|k) = y^a(k|k-1) + H^{a^*}(k) R^{a^{-1}}(k) z^a(k) = \tfrac{1}{2} J_n y^d(k|k-1) + \left\{J_m H^d(k) J_n^{-1}\right\}^* \left\{J_m R^d(k) J_m^*\right\}^{-1} J_m z^d(k)$$

$$= \tfrac{1}{2} J_n y^d(k|k-1) + {J_n^{-1}}^* H^{d^T}(k) J_m^* {J_m^*}^{-1} R^{d^{-1}}(k) {J_m}^{-1} J_m z^d(k)$$

$$= \tfrac{1}{2} J_n y^d(k|k-1) + {J_n^*}^{-1} H^{d^T}(k) J_m^* {J_m^*}^{-1} R^{d^{-1}}(k) {J_m}^{-1} J_m z^d(k)$$
$$= \tfrac{1}{2} J_n y^d(k|k-1) + \tfrac{1}{2} J_n H^{d^T}(k) J_m^* {J_m^*}^{-1} R^{d^{-1}}(k) {J_m}^{-1} J_m z^d(k)$$
$$= \tfrac{1}{2} J_n y^d(k|k-1) + \tfrac{1}{2} J_n H^{d^T}(k) R^{d^{-1}}(k) z^d(k)$$
$$= \tfrac{1}{2} J_n \left[y^d(k|k-1) + H^{d^T}(k) R^{d^{-1}}(k) z^d(k)\right]$$
$$= \tfrac{1}{2} J_n \left[y^d(k|k-1) + H^{d^T}(k) R^{d^{-1}}(k) z^d(k)\right]$$
$$= \tfrac{1}{2} J_n y^d(k|k)$$
$$\Rightarrow y^d(k|k) = y^d(k|k-1) + H^{d^T}(k) R^{d^{-1}}(k) z^d(k)$$

$$S^a(k|k) = \tfrac{1}{2} J_n S^d(k|k) J_n^{-1}$$
$$S^a(k|k) = S^a(k|k-1) + H^{a^*}(k) R^{a^{-1}}(k) H^a(k) = \tfrac{1}{2} J_n S^d(k|k-1) J_n^{-1} + \left\{J_m H^d(k) J_n^{-1}\right\}^* \left\{J_m R^d(k) J_m^*\right\}^{-1} \left\{J_m H^d(k) J_n^{-1}\right\}$$

$$= \tfrac{1}{2} J_n S^d(k|k-1) J_n^{-1} + {J_n^{-1}}^* H^{d^T}(k) J_m^* {J_m^*}^{-1} R^{d^{-1}}(k) {J_m}^{-1} J_m H^d(k) J_n^{-1}$$
$$= \tfrac{1}{2} J_n S^d(k|k-1) J_n^{-1} + {J_n^*}^{-1} H^{d^T}(k) R^{d^{-1}}(k) H^d(k) J_n^{-1}$$
$$= \tfrac{1}{2} J_n S^d(k|k-1) J_n^{-1} + \tfrac{1}{2} J_n H^{d^T}(k) R^{d^{-1}}(k) H^d(k) J_n^{-1}$$
$$= \tfrac{1}{2} J_n \left[S^d(k|k-1) + H^{d^T}(k) R^{d^{-1}}(k) H^d(k)\right] J_n^{-1}$$
$$\Rightarrow S^d(k|k) = S^d(k|k-1) + H^{d^T}(k) R^{d^{-1}}(k) H^d(k)$$

$$P^a(k|k) = S^{a^{-1}}(k|k) = \left(\tfrac{1}{2} J_n S^d(k|k) J_n^{-1}\right)^{-1} = 2 J_n S^{d^{-1}}(k|k) J_n^{-1}$$
$$P^a(k|k) = S^{a^{-1}}(k|k)$$

$$\Rightarrow J_n P^d(k|k) J_n^* = \left(\tfrac{1}{2} J_n S^d(k|k) J_n^{-1}\right)^{-1}$$
$$= 2 J_n S^{d^{-1}}(k|k) J_n^{-1}$$
$$= 2 J_n S^{d^{-1}}(k|k) \tfrac{1}{2} J_n^*$$
$$= J_n S^{d^{-1}}(k|k) J_n^*$$
$$\Rightarrow P^d(k|k) = S^{d^{-1}}(k|k)$$

$$x^a(k|k) = P^a(k|k) y^a(k|k)$$
$$\Rightarrow J_n x^d(k|k) = J_n P^d(k|k) J_n^* \tfrac{1}{2} J_n y^d(k|k)$$
$$= J_n P^d(k|k) 2 J_n^{-1} \tfrac{1}{2} J_n y^d(k|k)$$
$$\Rightarrow x^d(k|k) = P^d(k|k) y^d(k|k)$$

$$K^a(k) = P^a(k|k) H^{a^*}(k) R^{a^{-1}}(k)$$
$$\Rightarrow J_n K^d(k) J_m^{-1}$$
$$= J_n P^d(k+1|k) J_n^* \left\{J_m H^d(k) J_n^{-1}\right\}^* \left\{J_m R^d(k) J_m^*\right\}^{-1}$$

$$= J_n P^d(k+1|k) J_n^* J_n^{-1^*} H^{d^T}(k) J_m^* J_m^{*^{-1}} R^{d^{-1}}(k) J_m^{-1}$$

$$= J_n P^d(k+1|k) J_n^* J_n^{*^{-1}} H^{d^T}(k) J_m^* J_m^{*^{-1}} R^{d^{-1}}(k) J_m^{-1}$$
$$\Rightarrow K^d(k) = P^d(k|k) H^{d^T}(k) R^{d^{-1}}(k)$$

$$P^a(k+1|k) = Q^a(k) + F^a(k) P^a(k|k) F^{a^*}(k)$$
$$\Rightarrow J_n P^d(k+1|k) J_n^*$$
$$= J_n Q^d(k) J_n^*$$
$$+ J_n F^d(k) J_n^{-1} J_n P^d(k|k) J_n^* \left\{J_n F^d(k) J_n^{-1}\right\}^*$$

$$= J_n Q^d(k) J_n^*$$
$$+ J_n F^d(k) J_n^{-1} J_n P^d(k|k) J_n^* J_n^{-1^*} F^d(k)^* J_n^*$$

$$= J_n Q^d(k) J_n^*$$
$$+ J_n F^d(k) J_n^{-1} J_n P^d(k|k) J_n^* J_n^{*^{-1}} F^{d^T}(k) J_n^*$$
$$\Rightarrow P^d(k+1|k) = Q^d(k) + F^d(k) P^d(k|k) F^{d^T}(k)$$

$$S^a(k+1|k) = \tfrac{1}{2} J_n S^d(k|k-1) J_n^{-1}$$
$$S^a(k+1|k) = P^{a^{-1}}(k+1|k)$$
$$\Rightarrow \tfrac{1}{2} J_n S^d(k|k-1) J_n^{-1} = \left\{J_n P^d(k+1|k) J_n^*\right\}^{-1}$$
$$= J_n^{*^{-1}} P^{d^{-1}}(k+1|k) J_n^{-1}$$
$$= \tfrac{1}{2} J_n P^{d^{-1}}(k+1|k) J_n^{-1}$$
$$\Rightarrow S^d(k+1|k) = P^{d^{-1}}(k+1|k)$$

$$y^a(k+1|k) = \tfrac{1}{2} J_n y^d(k+1|k)$$
$$y^a(k+1|k) = S^a(k+1|k) F^a(k) P^a(k|k) y^a(k|k)$$
$$\Rightarrow \tfrac{1}{2} J_n y^d(k+1|k) = \tfrac{1}{2} J_n S^d(k+1|k)$$
$$J_n^{-1} J_n F^d(k) J_n^{-1} J_n P^d(k+1|k) J_n^* \tfrac{1}{2} J_n y^d(k|k)$$
$$= \tfrac{1}{2} J_n S^d(k+1|k)$$
$$J_n^{-1} J_n F^d(k) J_n^{-1} J_n P^d(k+1|k) 2 J_n^{-1} \tfrac{1}{2} J_n y^d(k|k)$$

$$= \tfrac{1}{2} J_n S^d(k+1|k) F^d(k) P^d(k+1|k) y^d(k|k)$$
$$\Rightarrow y^d(k+1|k)$$
$$= S^d(k+1|k) F^d(k) P^d(k|k) y^d(k|k)$$

$$x^a(k+1|k) = P^a(k|k) y^a(k+1|k)$$
$$\Rightarrow J_n x^d(k+1|k) = J_n P^d(k|k) J_n^* \tfrac{1}{2} J_n y^d(k+1|k)$$
$$= J_n P^d(k|k) 2 J_n^{-1} \tfrac{1}{2} J_n y^d(k+1|k)$$
$$= J_n P^d(k|k) y^d(k+1|k)$$
$$\Rightarrow x^d(k+1|k) = P^d(k|k) y^d(k+1|k)$$

Thus, the time varying Dual Augmented Complex Information Kalman Filter (DACIKFtv) has the form:

DACIKFtv

initial conditions

$x^a(0|-1) = x_0{}^a$

$P^a(0|-1) = P_0{}^a$

$x^d(0|-1) = x_0{}^d = J_n^{-1} x^a(0|-1) = J_n^{-1} x_0{}^a$

$P^d(0|-1) = P_0{}^d = J_n^{-1} P^d(0|-1) J_n^{*^{-1}} = J_n^{-1} P_0{}^a J_n^{*^{-1}}$
$= \tfrac{1}{2} J_n^{-1} P_0{}^a J_n$

$y^d(0|-1) = y_0{}^d = S_0{}^d x_0{}^d = P_0{}^{d^{-1}} x_0{}^d$

$S^d(0|-1) = S_0{}^d = P_0{}^{d^{-1}}$

iterations $k = 0,1,\ldots$

$F^d(k) = J_n^{-1} F^a(k) J_n$

$H^d(k) = J_m^{-1} H^a(k) J_n$

$Q^d(k) = J_n^{-1} Q^a(k) J_n^{*^{-1}}$

$R^d(k) = J_m^{-1} R^a(k) J_m^{*^{-1}}$

$y^d(k|k) = y^d(k|k-1) + H^{d^T}(k) R^{d^{-1}}(k) z^d(k)$

$S^d(k|k) = S^d(k|k-1) + H^{d^T}(k) R^{d^{-1}}(k) H^d(k)$

$P^d(k|k) = S^{d^{-1}}(k|k)$

$x^d(k|k) = P^d(k|k) y^d(k|k)$

$K^d(k) = P^d(k|k) H^{d^T}(k) R^{d^{-1}}(k)$

$P^d(k+1|k) = Q^d(k) + F^d(k) P^d(k|k) F^{d^T}(k)$

$S^d(k+1|k) = P^{d^{-1}}(k+1|k)$

$y^d(k+1|k) = S^d(k+1|k) F^d(k) P^d(k|k) y^d(k|k)$

$x^d(k+1|k) = P^d(k|k) y^d(k+1|k)$

For time invariant model, the time invariant DACIKF (DACIKFti) is derived; then F^d, H^d, Q^d, R^d are computed off-line and once and $R^{d^{-1}}, H^{d^T} R^{d^{-1}}, H^{d^T} R^{d^{-1}} H^d$ are computed off-line.

It is worth to note that DACKF and DACIKF are equivalent with respect to the derivation of the state estimations and predictions and the corresponding error covariances.

In the following, the **Dual Augmented Complex Kalman Filter Gain Elimination (DACKFGE)** is derived by the Augmented Complex Kalman Filter Gain Elimination.

Proof.

$x^a(0|-1) = x_0{}^a = J_n x^d(0|-1)$
$\Rightarrow x^d(0|-1) = x_0{}^d = J_n^{-1} x^a(0|-1) = J_n^{-1} x_0{}^a$

$P^a(0|-1) = P_0{}^a = J_n P^d(0|-1) J_n^*$
$\Rightarrow P^d(0|-1) = P_0{}^d = J_n^{-1} P^d(0|-1) J_n^{*\,-1}$
$= J_n^{-1} P_0{}^a J_n^{*\,-1} = \frac{1}{2} J_n^{-1} P_0{}^a J_n$

$\Lambda^a(k) = P^a(k|k-1) H^{a*}(k) R^{a^{-1}}(k)$
$\Rightarrow J_n \Lambda^d(k) J_m^{-1}$
$= J_n P^d(k|k-1) J_n^* \{J_m H^d(k) J_n^{-1}\}^* \{J_m R^d(k) J_m^*\}^{-1}$

$= J_n P^d(k|k-1) J_n^* J_n^{-1^*} H^{d^T}(k) J_m^* J_m^{*\,-1} R^{d^{-1}}(k) J_m{}^{-1}$

$= J_n P^d(k|k-1) J_n^* J_n^{*\,-1} H^{d^T}(k) J_m^* J_m^{*\,-1} R^{d^{-1}}(k) J_m{}^{-1}$
$\Rightarrow \Lambda^d(k) = P^d(k|k-1) H^{d^T}(k) R^{d^{-1}}(k)$

$x^a(k|k) = [I^a + \Lambda^a(k) H^a(k)]^{-1} \{x^a(k|k-1)$
$+ \Lambda^a(k) z^a(k)\}$
$\Rightarrow J_n x^d(k|k)$
$= \left[J_n I^d J_n^{-1}\right.$
$\left. + J_n \Lambda^d(k) J_m^{-1} J_m H^d(k) J_n^{-1}\right]^{-1} \{J_n x^a(k|k-1)$
$+ J_n \Lambda^d(k) J_m^{-1} J_m z^d(k)\}$

$= \left[J_n \left(I^d + \Lambda^d(k) H^d(k)\right) J_n^{-1}\right]^{-1} \{J_n x^a(k|k-1)$
$+ J_n \Lambda^d(k) z^d(k)\}$

$= J_n \left[I^d + \Lambda^d(k) H^d(k)\right]^{-1} J_n^{-1} J_n \{x^a(k|k-1)$
$+ \Lambda^d(k) z^d(k)\}$
$\Rightarrow x^d(k|k) = \left[I^d + \Lambda^d(k) H^d(k)\right]^{-1} \{x^d(k|k-1)$
$+ \Lambda^d(k) z^d(k)\}$

$P^a(k|k) = [I^a + \Lambda^a(k) H^a(k)]^{-1} P^a(k|k-1)$
$\Rightarrow J_n P^d(k|k) J_n^*$
$= \left[J_n I^d J_n^{-1}\right.$
$\left. + J_n \Lambda^d(k) J_m^{-1} J_m H^d(k) J_n^{-1}\right]^{-1} J_n P^d(k|k-1) J_n^*$

$= \left[J_n \left(I^d + \Lambda^d(k) H^d(k)\right) J_n^{-1}\right]^{-1} J_n P^d(k|k-1) J_n^*$

$= J_n \left[I^d + \Lambda^d(k) H^d(k)\right]^{-1} J_n^{-1} J_n P^d(k|k-1) J_n^*$

$\Rightarrow P^d(k|k) = \left[I^d + \Lambda^d(k) H^d(k)\right]^{-1} P^d(k|k-1)$

$x^a(k+1|k) = F^a(k+1|k) x^a(k|k)$
$\Rightarrow J_n x^d(k+1|k) = J_n F^d(k) J_n^{-1} J_n x^d(k|k)$
$\Rightarrow x^d(k+1|k) = F^d(k) x^d(k|k)$

$P^a(k+1|k) = Q^a(k) + F^a(k) P^a(k|k) F^{a^*}(k)$
$\Rightarrow J_n P^d(k+1|k) J_n^*$
$= J_n Q^d(k) J_n^*$
$+ J_n F^d(k) J_n^{-1} J_n P^d(k|k) J_n^* \{J_n F^d(k) J_n^{-1}\}^*$

$= J_n Q^d(k) J_n^*$
$+ J_n F^d(k) J_n^{-1} J_n P^d(k|k) J_n^* J_n^{-1^*} F^d(k)^* J_n^*$

$= J_n Q^d(k) J_n^*$
$+ J_n F^d(k) J_n^{-1} J_n P^d(k|k) J_n^* J_n^{*\,-1} F^{d^T}(k) J_n^*$
$\Rightarrow P^d(k+1|k) = Q^d(k) + F^d(k) P^d(k|k) F^{d^T}(k)$

Thus, the time varying Dual Augmented Complex Kalman Filter Gain Elimination (DACKFGEtv) has the form:

DACKFGEtv
initial conditions
$x^a(0|-1) = x_0{}^a$
$P^a(0|-1) = P_0{}^a$
$x^d(0|-1) = x_0{}^d = J_n^{-1} x^a(0|-1) = J_n^{-1} x_0{}^a$
$P^d(0|-1) = P_0{}^d = J_n^{-1} P^d(0|-1) J_n^{*\,-1} = J_n^{-1} P_0{}^a J_n^{*\,-1}$
$= \frac{1}{2} J_n^{-1} P_0{}^a J_n$
iterations $k = 0,1,\ldots$
$F^d(k) = J_n^{-1} F^a(k) J_n$
$H^d(k) = J_m^{-1} H^a(k) J_n$
$Q^d(k) = J_n^{-1} Q^a(k) J_n^{*\,-1}$
$R^d(k) = J_m^{-1} R^a(k) J_m^{*\,-1}$
$\Lambda^d(k) = P^d(k|k-1) H^{d^T}(k) R^{d^{-1}}(k)$
$x^d(k|k) = \left[I^d + \Lambda^d(k) H^d(k)\right]^{-1} \{x^d(k|k-1)$
$+ \Lambda^d(k) z^d(k)\}$
$P^d(k|k) = \left[I^d + \Lambda^d(k) H^d(k)\right]^{-1} P^d(k|k-1)$
$x^d(k+1|k) = F^d(k) x^d(k|k)$
$P^d(k+1|k) = Q^d(k) + F^d(k) P^d(k|k) F^{d^T}(k)$

For time invariant model, the time invariant DACKFGE (DACKFGEti) is derived; then F^d, H^d, Q^d, R^d are computed off-line and once and $R^{d^{-1}}, H^{d^T} R^{d^{-1}}$ are computed off-line.

It is worth to note that ACKF and ACKFGE are equivalent with respect to the derivation of the state estimations and predictions and the corresponding error covariances.

4 Simulation Example

Consider the constant velocity movement with the position and the velocity assumed to be complex (concerning the movement in two dimensions), which is described by an augmented or widely linear model with model parameters [30]:

$$F^a(k) = \begin{bmatrix} F(k) & 0 \\ 0 & \bar{F}(k) \end{bmatrix} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(where T is the sampling interval)

$$H^a(k) = \begin{bmatrix} H(k) & B(k) \\ \bar{B}(k) & \bar{H}(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$Q^a(k) = \begin{bmatrix} Q(k) & U(k) \\ \bar{U}(k) & \bar{Q}(k) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$R^a(k) = \begin{bmatrix} R(k) & V(k) \\ \bar{V}(k) & \bar{R}(k) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 + 0.1j \\ 0.1 - 0.1j & 0.2 \end{bmatrix}$$

We assumed a movement with constant velocity (2, 3) m/s. We have implemented the augmented as well the dual filters taking sampling interval T = 1 s and initial conditions

$$x_0{}^a = \begin{bmatrix} x_0 \\ \bar{x}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2+3j \\ 0 \\ 2-3j \end{bmatrix},$$

$$P_0{}^a = \begin{bmatrix} P_0 & \Pi_0 \\ \bar{\Pi}_0 & \bar{P}_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The simulation results confirm that the three complex Kalman filters and the corresponding dual Kalman filters are equivalent to each other since they compute the same position estimates, as it is shown in Fig. 1. All the algorithms present Mean Absolute Error (MAE) 0.3777 and % Root Mean Square Error (%RMSE) 0.4209.

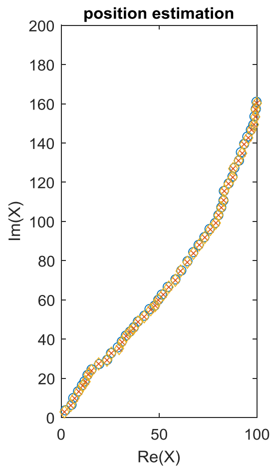

Fig. 1. Position estimation

A subject of future research is to investigate the applicability of the presented filters to estimation applications where complex signals are involved, especially for multidimensional cases.

5 Selection of the Faster Filter

It is obvious that all the augmented as well as the dual Kalman filters are iterative algorithms. Hence, their computational time depends on their per-iteration calculation burden required for the on-line calculations; the calculation burden of the off-line calculations (initialization process) is not taken into account.

The calculation burdens of the augmented and dual filters are given in the Appendix and summarized in Table 1.

Table 1. Calculation burdens of complex Kalman filters

ACKFtv	$64n^3 - 4n^2 + 2n + 64n^2m + 8nm + 64nm^2 + \frac{1}{6}(208m^3 - 120m^2 + 20m)$
ACKFti	$64n^3 - 4n^2 + 2n + 64n^2m + 8nm + 64nm^2 + \frac{1}{6}(208m^3 - 120m^2 + 20m)$
ACIKFtv	$\frac{1}{6}(800n^3 + 72n^2 - 80n) + 64n^2m - 8nm + 32nm^2 + \frac{1}{6}(208m^3 - 96m^2 + 8m)$
ACIKFti	$\frac{1}{6}(800n^3 + 108n^2 - 86n) + 32n^2m + 4nm$
ACKFGEtv	$\frac{1}{6}(784n^3 - 108n^2 + 8n) + 64n^2m - 8nm + 32nm^2 + \frac{1}{6}(208m^3 - 96m^2 + 8m)$
ACKFGEti	$\frac{1}{6}(784n^3 - 108n^2 + 8n) + 64n^2m + 4nm$
DACKFtv	$\frac{1}{6}(144n^3 + 87n^2 - 9n) + 24n^2m + 20nm + 24nm^2 + \frac{1}{6}(56m^3 + 15m^2 + m)$
DACKFti	$24n^3 + 8n^2 - 2n + 24n^2m + 16nm + 24nm^2 + \frac{1}{6}(56m^3 - 2m)$
DACIKFtv	$\frac{1}{6}(256n^3 + 231n^2 - 37n) + 24n^2m + 8nm + 16nm^2 + \frac{1}{6}(56m^3 + 15m^2 - 11m)$
DACIKFti	$\frac{1}{6}(256n^3 + 204n^2 - 34n) + 16n^2m + 4nm - 2m$
DACKFGEtv	$\frac{1}{6}(248n^3 + 123n^2 - 17n) + 32n^2m + 4nm + 16nm^2 + \frac{1}{6}(56m^3 + 15m^2 + m)$
DACKFGEti	$\frac{1}{6}(248n^3 + 84n^2 - 20n) + 32n^2m + 4nm$

Fig. 2-5 depict the faster filter with respect to the model dimensions. The faster filter depends on the model dimensions, namely the state vector dimension n and the measurement vector dimension m.

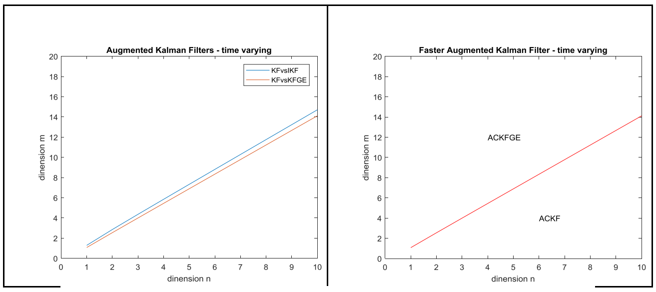

Fig. 2. The faster augmented filter – tv model

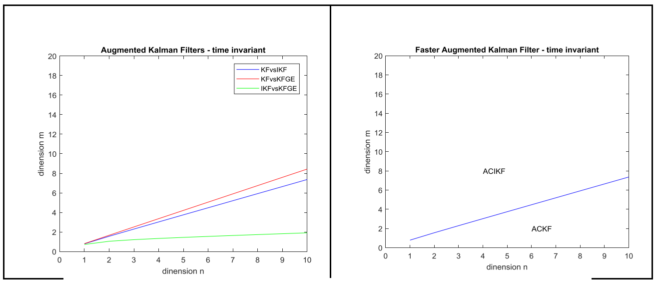

Fig. 3. The faster augmented filter – ti model

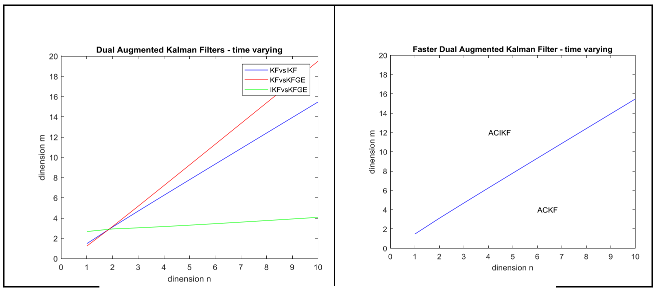

Fig. 4. The faster dual filter – tv model

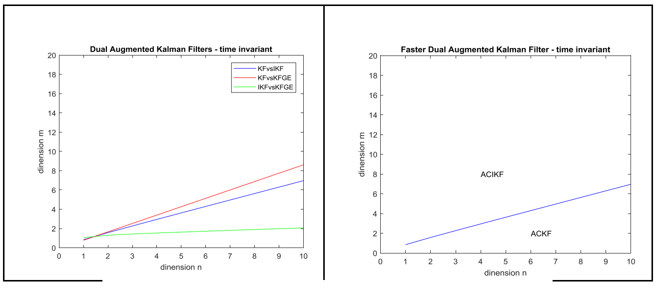

Fig. 5. The faster dual filter – ti model

The following rules of thumb are derived:

Augmented Complex Kalman Filters

time varying	ACIKF faster than ACKF	m/n>1.5
	ACKFGE faster than ACKF	m/n>1.4
	ACKFGE faster than ACIKF	**always**
	fastest filter	
	ACKF when m/n<1.4	
	ACKFGE when 1.4<m/n	
time invariant	ACIKF faster than ACKF	m/n>0.75
	ACKFGE faster than ACKF	m/n>0.85
	ACIKF faster than ACKFGE	m/n>0.2
	fastest filter	
	ACKF when m/n<0.75	
	ACIKF when 0.75<m/n	

Dual Augmented Complex Kalman Filters

time varying	DACIKF faster than DACKF	m/n>1.55
	DACKFGE faster than DACKF	m/n>1.9
	DACIKF faster than DACKFGE	m/n>0.4
	fastest filter	
	DACKF when m/n<1.55	
	DACIKF when 1.55<m/n	
time invariant	DACIKF faster than DACKF	m/n>0.7
	DACKFGE faster than DACKF	m/n>0.85
	DACIKF faster than DACKFGE	m/n>0.2
	fastest filter	
	DACKF when m/n<0.7	
	DACIKF when 0.7<m/n	

Table 2 presents the % cases of fastest filter for model dimensions n=1:10 and m=1:10.

Table 2. % cases of fastest filter for model dimensions n=1:10 and m=1:10

Augmented Complex Kalman Filters	ACKF	ACIKF	ACKFGE	AKF/ ACKFGE
time varying	64	0	35	1
time invariant	36	64	0	-
Dual Augmented Complex Kalman Filters	DACKF	DACIKF	DAKFGE	
time varying	70	29	1	
time invariant	33	66	1	

Furthermore, the Dual Augmented Complex Kalman Filters are faster than the Augmented Complex Kalman Filters for both time varying as well as for time invariant models.

Fig. 6-8 depict the speedup form the augmented filters to the dual filters. The following rule of thumb arises: dual filters are 3 times faster than augmented filters.

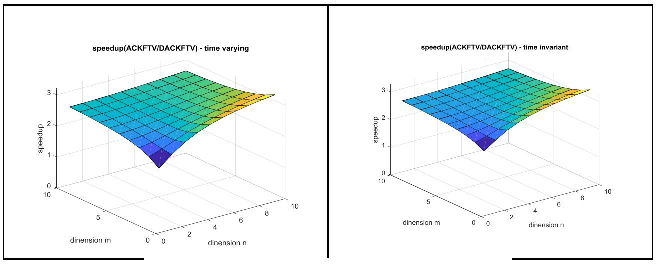

Fig. 6. Speedup – ACKF filters

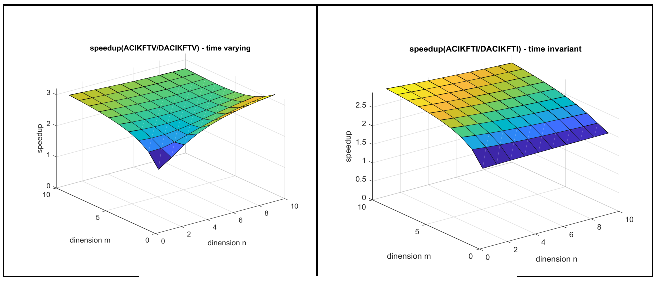

Fig. 7. Speedup – ACIKF filters

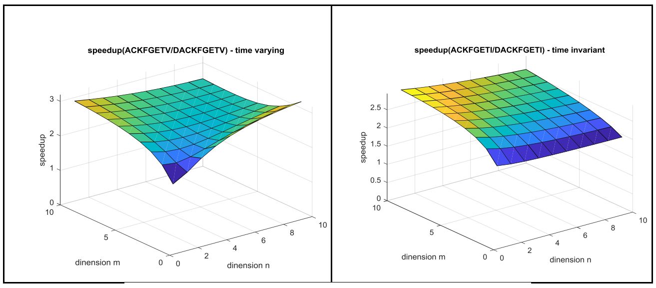

Fig. 8. Speedup – ACKFGE filters

6 Conclusion

Complex signals, which arise in many applications, are characterizes by the covariance and the pseudo-covariance matrices. These matrices are taken into account in the augmented or widely linear model, form where the Augmented Complex Kalman Filter is derived. There are also derived its variations, namely the Augmented Complex Information Kalman Filter that uses the information matrices (the inverses of the covariance and pseudo-covariance matrices) and the Augmented Complex Kalman Filter Gain Elimination that eliminates the Kalman filter gain. All these complex filters compute the state estimation using noised measurements.

The duality between the augmented complex Kalman filters and the corresponding dual bivariate real Kalman filters is addressed. The complex Kalman filters and the dual real Kalman filters compute the same estimates. This results from the proofs of the dual filters and is confirmed by a simulation example.

It is important to note that, due to the duality between the bivariate real-valued Kalman Filter and ACKF, the stability and convergence analysis for real-valued Kalman Filter also apply to the ACKF [26]; then, ACKF, like the bivariate real-valued Kalman Filter, achieves the Cramer–Rao lower bound (CRLB) [26]. Also, the mean squared error (MSE) of ACKF converges in the general case of improper noises [27]. In addition, the three augmented complex Kalman filters and the corresponding dual augmented Kalman filters are equivalent to each other since they compute the same estimations and predictions and the corresponding error covariances at every time iteration. Also, the Dual Augmented Complex Kalman Filters have the same structure as the real Kalman filters. Hence, stability and convergence analysis for real-valued Kalman Filter also apply to the proposed algorithms.

The computational requirements of the complex and dual real Kalman filters are presented. The knowledge of the computational requirements of the filters is the basis of determining the fastest complex filter and the fastest dual real filter. The dual real filters are 3 times faster than the complex filters. The ability to determine the fastest filter depends on the state and measurement dimensions. This is a useful result, since the fastest filter can be a-priori selected, i.e. before its implementation.

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Appendix

Calculation Burdens of the Augmented and Dual Filters

The calculation burdens of complex matrices operations are given in [28].

ACKFtv and ACKFti

Matrix Operation	Calculation Burden
$H^a(k)P^a(k \mid k-1)$	$32n^2m - 12nm$
$H^a(k)P^a(k \mid k-1)H^{a*}(k)$	$32nm^2 - 6m^2 + m$
$H^a(k)P^a(k \mid k-1)H^{a*}(k) + R^a(k)$	$2m^2 + m$
$[H^a(k)P^a(k \mid k-1)H^{a*}(k) + R^a(k)]^{-1}$	$(208m^3 - 96m^2 + 8m)/6$
$K^a(k) = P^a(k \mid k-1)H^{a*}(k)[H^a(k)P^a(k \mid k-1)H^{a*}(k) + R^a(k)]^{-1}$	$32nm^2 - 12nm$
$H^a(k)x^a(k \mid k-1)$	$16nm - 2m$
$z^a(k) - H^a(k)x^a(k \mid k-1)$	$2m$
$K^a(k)[z^a(k) - H^a(k)x^a(k \mid k-1)]$	$16nm - 2n$
$x^a(k \mid k) = x^a(k \mid k-1) + K^a(k)[z^a(k) - H^a(k)x^a(k \mid k-1)]$	$2n$
$K^a(k)H^a(k)P^a(k \mid k-1)$	$32n^2m - 6n^2 + n$
$P^a(k \mid k) = P^a(k \mid k-1) - K^a(k)H^a(k)P^a(k \mid k-1)$	$2n^2 + n$
$x^a(k+1 \mid k) = F^a(k)x^a(k \mid k)$	$16n^2 - 2n$
$F^a(k)P^a(k \mid k)$	$32n^3 - 12n^2$
$F^a(k)P^a(k \mid k)F^{a*}(k)$	$32n^3 - 6n^2 + n$
$P^a(k+1 \mid k) = Q^a(k) + F^a(k)P^a(k \mid k)F^{a*}(k)$	$2n^2 + n$

ACIKFtv and ACIKFti

Matrix Operation	Calculation Burden
$R^{a-1}(k)$	$\frac{1}{6}(208m^3 - 96m^2 + 8m)$
$H^{a*}(k)R^{a-1}(k)$	$32nm^2 - 12nm$
$H^{a*}(k)R^{a-1}(k)H^a(k)$	$32n^2m - 6n^2 + n$
$H^{a*}(k)R^{a-1}(k)z^a(k)$	$16nm - 2n$
$y^a(k \mid k) = y^a(k \mid k-1) + H^{a*}(k)R^{a-1}(k)z^a(k)$	$2n$
$S^a(k \mid k) = S^a(k \mid k-1) + H^{a*}(k)R^{a-1}(k)H^a(k)$	$2n^2 + n$
$P^a(k \mid k) = S^{a-1}(k \mid k)$	$\frac{1}{6}(208n^3 - 96n^2 + 8n)$
$x^a(k \mid k) = P^a(k \mid k)y^a(k \mid k)$	$16n^2 - 6n$
$K^a(k) = P^a(k \mid k)H^{a*}(k)R^{a-1}(k)$	$32n^2m - 12nm$
$F^a(k)P^a(k \mid k)$	$32n^3 - 12n^2$
$F^a(k)P^a(k \mid k)F^{a*}(k)$	$32n^3 - 6n^2 + n$
$P^a(k+1 \mid k) = Q^a(k) + F^a(k)P^a(k \mid k)F^{a*}(k)$	$2n^2 + n$
$S^a(k+1 \mid k) = P^{a-1}(k+1 \mid k)$	$\frac{1}{6}(208n^3 - 96n^2 + 8n)$
$F^a(k)P^a(k \mid k)y^a(k \mid k)$	$16n^2 - 2n$
$y^a(k+1 \mid k) = S^a(k+1 \mid k)F^a(k)P^a(k \mid k)y^a(k \mid k)$	$16n^2 - 6n$
$x^a(k+1 \mid k) = P^a(k \mid k)y^a(k+1 \mid k)$	$16n^2 - 6n$

ACKFGEtv and ACKFGEti

Matrix Operation	Calculation Burden
$R^{a-1}(k)$	$(208m^3 - 96m^2 + 8m)/6$
$H^{a*}(k)R^{a-1}(k)$	$32nm^2 - 12nm$
$\Lambda^a(k) = P^a(k \mid k-1)H^{a*}(k)R^{a-1}(k)$	$32n^2m - 12nm$
$\Lambda^a(k)H^a(k)$	$32n^2m - 4n^2$
$I^a + \Lambda^a(k)H^a(k)$	n
$[I^a + \Lambda^a(k)H^a(k)]^{-1}$	$(208n^3 - 96n^2 + 8n)/6$
$\Lambda^a(k)z^a(k)$	$16nm - 2n$
$x^a(k \mid k-1) + \Lambda^a(k)z^a(k)$	$2n$
$x^a(k \mid k) = [I^a + \Lambda^a(k)H^a(k)]^{-1}\{x^a(k \mid k-1) + \Lambda^a(k)z^a(k)\}$	$16n^2 - 2n$
$P^a(k \mid k) = [I^a + \Lambda^a(k)H^a(k)]^{-1}P^a(k \mid k-1)$	$32n^3 - 14n^2 + n$
$x^a(k+1 \mid k) = F^a(k)x^a(k \mid k)$	$16n^2 - 2n$
$F^a(k)P^a(k \mid k)$	$32n^3 - 12n^2$
$F^a(k)P^a(k \mid k)F^{a*}(k)$	$32n^3 - 6n^2 + n$
$P^a(k+1 \mid k) = Q^a(k) + F^a(k)P^a(k \mid k)F^{a*}(k)$	$2n^2 + n$

The calculation burdens of real matrix operations are summarized in Table 3. The calculation burden of the inverse of a real symmetric matrix is given in [31].

Table 3. Calculation Burden of real matrices operations

Matrices Operation	Matrices Dimensions	Calculation Burden
$M1 + M2 = M$	$(d_1 \times d_2) + (d_1 \times d_2)$	d_1d_2
$M1 + M2 = S$ symmetric	$(d \times d) + (d \times d)$	$\frac{1}{2}d^2 + \frac{1}{2}d$
$I + M1 = M$	$(d \times d) + (d \times d)$	d
$M1 \cdot M2 = M$	$(d_1 \times d_2) \cdot (d_2 \times d_3)$	$2d_1d_2d_3 - d_1d_3$
$M1 \cdot M2 = S$ symmetric	$(d_1 \times d_2) \cdot (d_2 \times d_1)$	${d_1}^2d_2 + d_1d_2 - \frac{1}{2}{d_1}^2 - \frac{1}{2}d_1$
M^{-1} symmetric	$(d \times d)$	$(7d^3 - d)/6$

The calculation burden of the dual model parameters derivation, which is required in each iteration of the dual algorithms, is given in Table 4.

Table 4. Calculation Burden of dual model parameters derivation

Matrices Operation	scalar operations	Calculation Burden
$F(k) + A(k)$	n^2 complex+complex adds	$2n^2$
$F(k) - A(k)$	n^2 complex+complex adds	$2n^2$
$F^d(k) = J_n^{-1}F^a(k)J_n = \begin{bmatrix} Real(F(k)) + A(k)) & -Imag(F(k)) - A(k)) \\ Imag(F(k)) + A(k)) & Real(F(k)) - A(k)) \end{bmatrix}$		$4n^2$
$H(k) + B(k)$	nm complex+complex adds	$2nm$
$H(k) - B(k)$	nm complex+complex adds	$2nm$
$H^d(k) = J_m^{-1}H^a(k)J_n = \begin{bmatrix} Real(H(k)) + B(k)) & -Imag(H(k)) - B(k)) \\ Imag(H(k)) + B(k)) & Real(H(k)) - B(k)) \end{bmatrix}$		$4nm$
$Q(k) + U(k)$	$\frac{1}{2}n^2 - \frac{1}{2}n$ complex+complex adds n real+complex adds	n^2
$Q(k) - U(k)$	$\frac{1}{2}n^2 - \frac{1}{2}n$ complex+complex adds n real+complex adds	n^2
$\frac{1}{2}J_n^{-1}Q^a(k)J_n$	$\frac{1}{2}n^2 + \frac{1}{2}n$ real*real mults	$n^2/2 + n/2$
$Q^d(k) = J_n^{-1}Q^a(k)J_n^{*-1} = \frac{1}{2}J_n^{-1}Q^a(k)J_n = \frac{1}{2}\begin{bmatrix} Real(Q(k)) + U(k)) & -Imag(Q(k)) - U(k)) \\ Imag(Q(k)) + U(k)) & Real(Q(k)) - U(k)) \end{bmatrix}$		$5n^2/2 + n/2$
$R(k) + V(k)$	$\frac{1}{2}m^2 - \frac{1}{2}m$ complex+complex adds m real+complex adds	m^2
$R(k) - V(k)$	$\frac{1}{2}m^2 - \frac{1}{2}m$ complex+complex adds m real+complex adds	m^2

$\frac{1}{2}J_m^{-1}R^a(k)J_m$	$\frac{1}{2}m^2 + \frac{1}{2}m$ real*real mults	$m^2/2 + m/2$
$R^d(k) = J_m^{-1}R^a(k){J_m^*}^{-1} = \frac{1}{2}J_m^{-1}R^a(k)J_m = \frac{1}{2}\begin{bmatrix} Real(R(k)) + V(k)) & -Imag(R(k)) - V(k)) \\ Imag(R(k)) + V(k)) & Real(R(k)) - V(k)) \end{bmatrix}$		$5m^2/2 + m/2$
$F^d(k), H^d(k), Q^d(k), R^d(k)$	$13n^2/2 + n/2 + 4nm + 5m^2/2 + m/2$	

The calculation burdens of complex matrices operations are given in [28]. Dual algorithms use the matrices dimensions $N = 2n$ and $M = 2m$.

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Matrix Operation	Calculation Burden
$F^d(k), H^d(k), Q^d(k), R^d(k)$	$13n^2/2 + n/2 + 4nm + 5m^2/2 + m/2$
$H^d(k)P^d(k\|k-1)$	$2N^2M - NM$
$H^d(k)P^d(k\|k-1){H^d}^T(k)$	$NM^2 + NM - M^2/2 - M/2$
$H^d(k)P^d(k\|k-1){H^d}^T(k) + R^d(k)$	$M^2/2 + M/2$
$\left[H^d(k)P^d(k\|k-1){H^d}^T(k) + R^d(k)\right]^{-1}$	$(7M^3 - M)/6$
$K^d(k) = P^d(k\|k-1){H^d}^T(k) \left[H^d(k)P^d(k\|k-1){H^d}^T(k) + R^d(k)\right]^{-1}$	$2NM^2 - NM$
$H^d(k)x^d(k\|k-1)$	$2NM - M$
$z^d(k) - H^d(k)x^d(k\|k-1)$	M
$K^d(k)[z^d(k) - H^d(k)x^d(k\|k-1)]$	$2NM - N$
$x^d(k\|k) = x^d(k\|k-1) + K^d(k) [z^d(k) - H^d(k)x^d(k\|k-1)]$	N
$K^d(k)H^d(k)P^d(k\|k-1)$	$N^2M + NM - N^2/2 - N/2$
$P^d(k\|k) = P^d(k\|k-1) - K^d(k)H^d(k)P^d(k\|k-1)$	$N^2/2 + N/2$
$x^d(k+1\|k) = F^d(k)x^d(k\|k)$	$2N^2 - N$
$F^d(k)P^d(k\|k)$	$2N^3 - N^2$
$F^d(k)P^d(k\|k){F^d}^T(k)$	$N^3 + N^2/2 - N/2$
$P^d(k+1\|k) = Q^d(k) + F^d(k)P^d(k\|k){F^d}^T(k)$	$N^2/2 + N/2$

DACIKFtv and DACIKFti

Matrix Operation	Calculation Burden
$F^d(k), H^d(k), Q^d(k), R^d(k)$	$13n^2/2 + n/2 + 4nm + 5m^2/2 + m/2$
${R^d}^{-1}(k)$	$(7M^3 - M)/6$
${H^d}^T(k){R^d}^{-1}$	$2NM^2 - NM$
${H^d}^T(k){R^d}^{-1}(k)H^d(k)$	$N^2M + NM - N^2/2 - N/2$
${H^d}^T(k){R^d}^{-1}(k)z^d(k)$	$2NM - M$
$y^d(k\|k) = y^d(k\|k-1) + {H^d}^T(k){R^d}^{-1}(k)z^d(k)$	N
$S^d(k\|k) = S^d(k\|k-1) + {H^d}^T(k){R^d}^{-1}(k)H^d(k)$	$N^2/2 + N/2$
$P^d(k\|k) = {S^d}^{-1}(k\|k)$	$(7N^3 - N)/6$
$x^d(k\|k) = P^d(k\|k)y^d(k\|k)$	$2N^2 - N$
$K^d(k) = P^d(k\|k){H^d}^T(k){R^d}^{-1}(k)$	$2N^2M - NM$
$F^d(k)P^d(k\|k)$	$2N^3 - N^2$
$F^d(k)P^d(k\|k){F^d}^T(k)$	$N^3 + N^2/2 - N/2$
$P^d(k+1\|k) = Q^d(k) + F^d(k)P^d(k\|k){F^d}^T(k)$	$N^2/2 + N/2$
$S^d(k+1\|k) = {P^d}^{-1}(k+1\|k)$	$(7N^3 - N)/6$
$F^d(k)P^d(k\|k)y^d(k\|k)$	$2N^2 - N$
$y^d(k+1\|k) = S^d(k+1\|k)F^d(k)P^d(k\|k)y^d(k\|k)$	$2N^2 - N$
$x^d(k+1\|k) = P^d(k\|k)y^d(k+1\|k)$	$2N^2 - N$

DACKFGEtv and DACKGEFti

Matrix Operation	Calculation Burden
$F^d(k), H^d(k), Q^d(k), R^d(k)$	$13n^2/2 + n/2 + 4nm + 5m^2/2 + m/2$
${R^d}^{-1}(k)$	$(7M^3 - M)/6$
${H^d}^T(k){R^d}^{-1}$	$2NM^2 - NM$
$\Lambda^d(k) = P^d(k\|k-1){H^d}^T(k){R^d}^{-1}(k)$	$2N^2M - NM$
$\Lambda^d(k)H^d(k)$	$2N^2M - N^2$
$I^d + \Lambda^d(k)H^d(k)$	N
$[I^d + \Lambda^d(k)H^d(k)]^{-1}$	$(7N^3 - N)/6$
$\Lambda^d(k)z^d(k)$	$2NM - N$
$x^d(k\|k-1) + \Lambda^d(k)z^d(k)$	N
$x^d(k\|k) = [I^d + \Lambda^d(k)H^d(k)]^{-1} \{x^d(k\|k-1) + \Lambda^d(k)z^d(k)\}$	$2N^2 - N$
$P^d(k\|k) = [I^d + \Lambda^d(k)H^d(k)]^{-1}P^d(k\|k-1)$	$N^3 + N^2/2 - N/2$
$x^d(k+1\|k) = F^d(k)x^d(k\|k)$	$2N^2 - N$
$F^d(k)P^d(k\|k)$	$2N^3 - N^2$
$F^d(k)P^d(k\|k){F^d}^T(k)$	$N^3 + N^2/2 - N/2$
$P^d(k+1\|k) = Q^d(k) + F^d(k)P^d(k\|k){F^d}^T(k)$	$N^2/2 + N/2$