

# Mathematical Analysis of Acoustic Guitar Notes

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**Abstract:** -To understand the musical instruments in detail, mathematical models for their behavior need to be developed. This understanding helps one know acoustic principles and human perception of that instrument. The work is focused on mathematical analysis of box shaped acoustic guitar and primarily its music notes. It will help modeling the guitar string which is a distributed system. It can be excited either by plucking it or picking it. A set of 252 acoustic guitar notes are collected from each guitar player/artist by plucking all the six strings and 20 frets of each string by finger as well as pick. The aim of this work is to calculate and compare the frequencies generated by different methods such as Octave Multiplier and Linear Mass Density (LMD). Frequency analysis is carried out for different samples of music notes based on plucking style, finger or pick, plucking position and plucking expression. Understanding the different parameters contributing to generation of music notes on all frets of guitar string helps in modeling the acoustic guitar. The paper proposes a mathematical model for synthesis of Guitar notes.

**Key-Words:** - Mathematical model, acoustic guitar, plucking, picking, octave multiplier, LMD, frequency analysis

## 1 Introduction

While focusing on the mathematics of the guitar as a string instrument, we divided the instrument analysis in three parts. The set of six strings with different physical parameters is the first part and the sound-board being the second part. Third part of analysis involves the interaction of the strings and sound-board. There are different styles in which the strings can be plucked. In case of acoustic guitar, we consider two styles of plucking, *finger struck (plucked)* and *pick struck (picked)*. In [5] Gerald Schuller et al discussed 5 plucking styles finger-style (FS), picked (PK), muted (MU), slap-thumb (ST), and slap-pluck (SP) and the 5 expression styles normal (NO), vibrato (VI), bending (BE), harmonics (HA), and dead-note (DN) for feature extraction of plucking and expression styles of electric bass guitar. Migneco [4] et al proposed physical models for plucked string instruments that explore a physically-inspired signal model for plucked guitar sounds that facilitates the estimation of both string excitation and resonance parameters simultaneously. Whereas in [3] K.Lanc et al considered testing of the musical instrument string motion. The analysis of physical parameters of

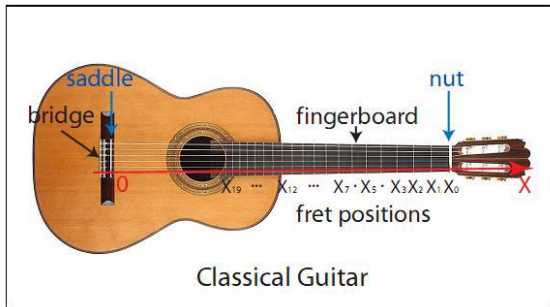
string includes tension and mass per unit length of the string. These physical parameters influence fundamental tone of string vibration. Minor deviations do occur to measurement results performed on Fender Stratocaster electric guitar by K.Lanc because of the fact that simulation model is idealized while in the experimental model, natural frequencies depend on music instrument structure as well and due to slight deviation of the exact plucking position during the measurements from position taken in simulation.

The work is divided into four main parts; section 2 covers the collection of Guitar notes, section 3 deals with mathematical analysis of plucked and picked guitar notes, section 4 covers the mathematical modelling of guitar notes and section 5 is about the discussion of the results and conclusion.

## 2 Collection of guitar notes

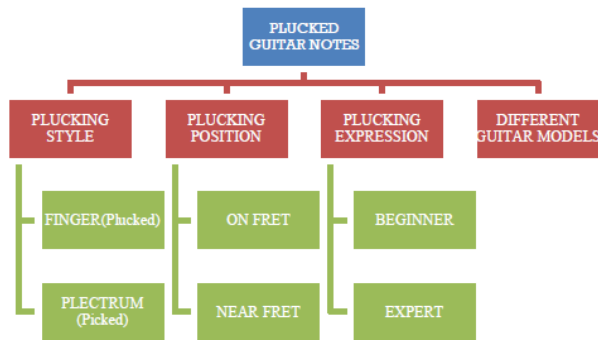
Referring to Figure 1, the fingerboard of guitar consists of 19 to 21 frets. The frets are arranged in logarithmic manner. Guitar strings produce sound by vibrating between two points, the saddle and either the nut or one of the frets. The notes have been recorded for each fret and every string

resulting 21 frets multiplied by 6 strings, i.e 126 plucked (finger struck) notes. The arrangement of frets on fingerboard is shown in figure 1. Apart from the set of 126 notes, 6 more sound notes of open strings have been recorded. So finger\_plucked and pick\_plucked give 264 notes in total. For every category, may it be plucking expression or type of guitar model; it's needed to take a set of 132 sound notes.



**Figure 1: Acoustic Guitar**

Figure 2 showcases the analysis strategy for guitar notes. The guitar sound notes are collected on the basis of four categories, plucking style, plucking position, plucking expression and different guitar types.



**Figure 2: Guitar Notes Collection**

### 3 Mathematical Analysis of guitar notes

When a guitar string is plucked, the vibration produces a standing wave between these two points, a) the saddle and b) the nut, in case of Open Strings; or the frets, in case of the note played. When the note is played, the string's effective length is changed. The fretted length changes the frequency of the note produced. In open string, it's the actual length measured between the nut and the saddle.

As mentioned earlier, the pitch or the frequency of this wave, produced by a string increases by logarithmic scale by the relation,

$$f_x = 2^{\left(\frac{x}{12}\right)} * f_o$$

where  $f_o$  stands for 'open string frequency' and 'x' represents the 'string number'. This is also known as 'Pythagoras fractions for fret spacing'.

Figure 3 shows different frets to indicate the change in effective length with change in number of fret. Frequency calculation by linear mass density is done by using the well known formula,

$$f = \left(\frac{1}{2 * L}\right) \sqrt{T / \rho}$$



**Figure 3 Arrangement of Guitar frets**

where 'L' is the effective length, 'T' is the string tension and 'ρ' is string mass density. Here the term  $\sqrt{T / \rho}$  is known as linear mass density and this is calculated separately by Dr.Helmut Herminghaus's devised formula.

In summary, pitch depends on string length and string tension. If the string's mass per unit length remains constant, the longer the string, the higher the tension required to achieve the desired pitch. If the string's length remains constant, the higher the string mass per unit length, i.e. the heavier the string, the higher the tension required to achieve the desired pitch.

The mathematical analysis involves frequency calculation of guitar strings based on 1) Pythagoras fractions for fret spacing i.e octave relationship of the frets. 2) Physical parameters such as linear mass density and length of the string. String is tied between the bridge and nut on the fingerboard. So while considering the length, it's the effective length of the string. (The effective length is sometimes also called as 'fretted' string length.)

## 4 Mathematical Modelling of Guitar notes

The work as said earlier is focused on study of mathematical relationship between the frequencies of all frets over a single string. Though mathematical relationship appears to be simple, physical parameters like string's mass density and length have their own impact on frequency. This mathematical relationship plays important role in tuning apps available on mobile phones.

Here frequency analysis was done for approximately five hundred music notes from 2-3 artists. The open string frequencies were estimated by using simple autocorrelation formula. Once the open string frequencies are estimated, all fret frequencies can be calculated by octave multiplier method.

The open string frequencies are given in Table 1.

String Number	Open string frequencies in Hz
string 1	329.764
string 2	246.6667
string 3	391.6021
string 4	293.6523
string 5	109.8861
string 6	81.9635

**Table 1 Open String Frequencies**

Frequencies on all frets with respect to these open string frequencies by Octave Multiplier method are given in Table 2.

These frequencies should match with the actual recorded notes of acoustic guitar. The frequencies, 329 Hz, 247 Hz, 196 Hz, 147 Hz, 110Hz, 82 Hz are the open string frequencies as mentioned earlier and the whole table is formed from these reference frequencies. The other highlighted frequencies, which approximately equal to 440 Hz are the tuning frequencies for the respective strings. Normally string 1 i.e thinnest string E is tuned before playing acoustic guitar. The Table 3 gives the frequency values calculated by using the autocorrelation method for finding fundamental frequency. The tuning frequencies are highlighted here too.

Another set of frequencies is prepared by the formula, in Table 4, proposed by Toyohiko Satoh, later devised by Dr. Helmut Herminghaus. Here the frequency calculations are done by considering the physical properties of string, effective length and linear mass density. Again the reference for this table is Table 1.

When we know the open string's frequency, we can calculate frequency at each fret: 1) based on octave relationship of frets 2) based on the scale length of the guitar (the length between the bottom of guitar i.e bridge/saddle and the top of guitar i.e nuts) and 3) based of mass density, effective length and tension of the string. The table 5 summarizes the frequency calculations for string 2. Similar tables have been prepared for all the strings. This helps us to monitor the frequency value variations by Octave Multiplier and LMD methods with reference to actual frequencies measured by autocorrelation method for played guitar notes.

Frequency calculations done by above mentioned three methods are verified by plotting the frequency spectrum as shown below.

		THINNEST					THICKEST
Fret Numbers		329 Hz	247 Hz	196 Hz	147 Hz	110 Hz	82 Hz
from mt		STRING E	STRING B	STRING G	STRING D	STRING A	STRING E
FRET NUMBER	OCTAVE MULTIPLIER	STRING 1	STRING 2	STRING 3	STRING 4	STRING 5	STRING 6
1	1.059463	349.199	261.5814	207.6548	155.5292	116.5409	87.29976
2	1.122462	369.9635	277.1359	220.0026	164.7774	123.4708	92.49087
3	1.189207	391.9627	293.6152	233.0846	174.5756	130.8128	97.99067
4	1.259921	415.27	311.0745	246.9445	184.9564	138.5913	103.8175
5	1.33484	439.9632	329.572	261.6286	195.9545	146.8324	109.9908
6	1.414214	466.1248	349.1693	277.1859	207.6066	155.5635	116.5312
7	1.498307	493.842	369.932	293.6682	219.9515	164.8138	123.4605
8	1.587401	523.2074	391.9293	311.1306	233.0305	174.6141	130.8018
9	1.681793	554.3189	415.2346	329.6314	246.8872	184.9972	138.5797
10	1.781797	587.2804	439.9258	349.2323	261.5679	195.9977	146.8201
11	1.887749	622.2019	466.0851	369.9987	277.1215	207.6523	155.5505
12	2	659.2	493.8	392	293.6	220	164.8
13	2.118926	698.3981	523.1629	415.3095	311.0584	233.0819	174.5995
14	2.244924	739.927	554.2718	440.0051	329.5549	246.9417	184.9817
15	2.378414	783.9253	587.2305	466.1692	349.1512	261.6256	195.9813
16	2.519842	830.54	622.149	493.8891	369.9128	277.1826	207.635
17	2.66968	879.9264	659.1439	523.2572	391.909	293.6648	219.9816
18	2.828427	932.2496	698.3387	554.3717	415.2131	311.127	233.0624
19	2.996614	987.684	739.864	587.3364	439.903	329.6276	246.921
20	3.174802	1046.415	783.8586	622.2612	466.0609	349.2282	261.6037
21	3.363586	1108.638	830.4693	659.2628	493.7744	369.9944	277.1595

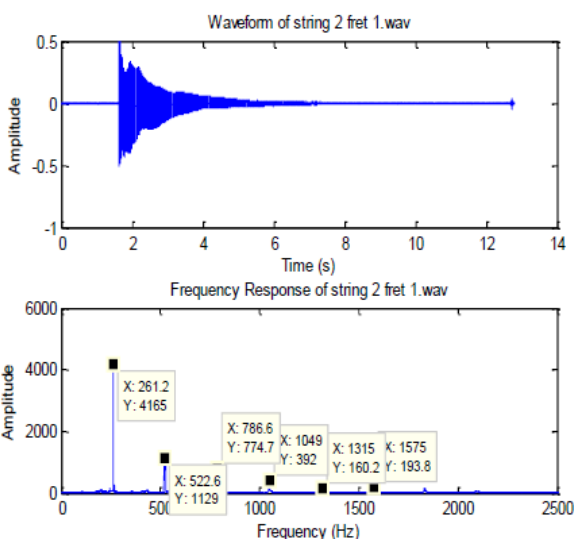
**Table 2 Frequency calculation by Octave Multiplier Method**

fret numbers	string 1	string 2	string 3	string 4	string 5	string 6
fret 1	349.9476	261.25	416.9643	155.138	116.092	175.8427
fret 2	369.8743	277.8365	664.5308	164.7293	122.7876	93.6901
fret 3	391.6539	293.3217	234.375	173.5163	129.9858	98.8782
fret 4	415.0257	329.3384	248.4657	183.3599	137.5718	104.8587
fret 5	439.7321	348.3073	264.3991	195.1505	146.0059	111.9755
fret 6	466.056	369.2701	279.0747	207.3899	154.1908	118.0254
fret 7	494.1406	390.3785	295.1695	219.5205	163.2399	125.4699
fret 8	522.9441	413.4143	312.9058	231.4912	173.3577	131.7726
fret 9	552.5286	438.8253	329.899	244.9836	182.9317	140.2062
fret 10	587.9112	464.2361	348.8742	259.8958	193.8095	148.4962
fret 11	619.2383	490.7853	371.0304	274.6198	209.4142	157.0881
fret 12	656.0172	491.0193	391.4905	580.7018	220.6801	165.2132
fret 13	694.4377	520.5853	414.4368	307.8689	232.8526	174.3304
fret 14	737.0924	550.4808	438.6489	324.867	247.1354	186.1374
fret 15	778.8694	582.3376	465.5032	344.5513	261.7188	199.5781
fret 16	822.3958	616.2961	493.5142	367.7734	277.3543	208.817
fret 17	874.7612	1320.2	524.0625	388.6792	294.2804	222.8409
fret 18	922.9667	697.672	562.0416	411.0401	310.7993	234.5785
fret 19	976.2183	735.2881	1180.9	434.2912	328.0435	247.1154
fret 20	1037.3	777.8105	622.4798	464.7887	350.8255	263.6653

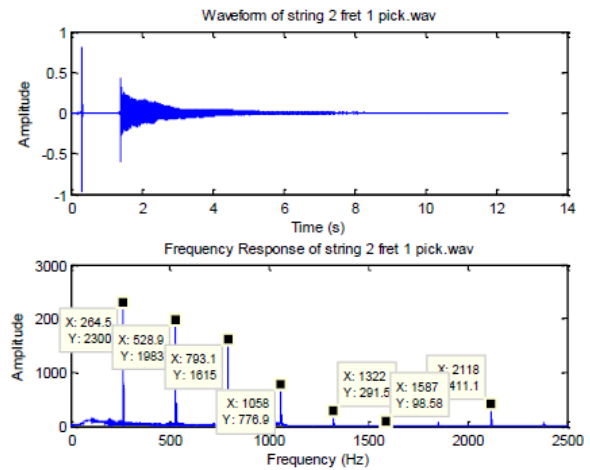
**Table 3 Frequency calculation by Autocorrelation Method for played notes**

fret number	open string length	E frets	B frets	G frets	D frets	A frets	E frets
1	0.648	352.7444	264.2372	419.526	157.1082	117.7241	88.18608
2	0.648	373.7197	279.9496	444.4723	166.4503	124.7244	93.42989
3	0.648	395.9422	296.5963	470.902	176.348	132.1409	98.98552
4	0.648	419.4862	314.2328	498.9033	186.8342	139.9984	104.8715
5	0.648	444.4301	332.9181	528.5696	197.9439	148.3231	111.1075
6	0.648	470.8573	352.7144	560	209.7143	157.1429	117.7143
7	0.648	498.856	373.6879	593.2993	222.1846	166.4871	124.7139
8	0.648	528.5195	395.9085	628.5787	235.3963	176.3869	132.1298
9	0.648	559.9469	419.4505	665.956	249.3937	186.8754	139.9867
10	0.648	593.2431	444.3923	705.5558	264.2235	197.9876	148.3107
11	0.648	628.5191	470.8172	747.5103	279.935	209.7606	157.1297
12	0.648	665.8928	498.8135	791.9596	296.5808	222.2336	166.4731
13	0.648	705.4889	528.4745	839.0519	314.2164	235.4483	176.3722
14	0.648	747.4394	559.8992	888.9446	332.9007	249.4487	186.8598
15	0.648	791.8845	593.1925	941.804	352.696	264.2817	197.971
16	0.648	838.9724	628.4656	997.8065	373.6684	279.9968	209.743
17	0.648	888.8603	665.8361	1057.139	395.8879	296.6462	222.215
18	0.648	941.7147	705.4288	1120	419.4286	314.2857	235.4286
19	0.648	997.7119	747.3758	1186.599	444.3691	332.9741	249.4279
20	0.648	1057.039	791.817	1257.157	470.7927	352.7738	264.2596
21	0.648	1119.894	838.9009	1331.912	498.7875	373.7508	279.9733

**Table 4 Frequency Calculations by Effective Length Measurement**



**Figure 4: Frequency Analysis of String 2 Fret 1, Plucked by finger**



**Figure 5: Frequency Analysis of String 2 Fret 1, Plucked by pick**

The Fourier graphs clearly show the harmonics produced. Once the harmonic frequencies are known a mathematical equation is written to generate the guitar notes. A string's mathematical equation can be put as,

$$X[n]=A1*\cos(2*\pi*fo/fs*1)+ A2*\cos(2*\pi*fo/fs*2)+.....$$

Considering as example of string 2 open plucked by pick,

$$X[n]=A1*\cos(2*\pi*244.8/48000)+A2*\cos(2*\pi*244.8*2/48000)+.....$$

### 5 Conclusions

Frequencies calculated by linear mass density have higher values than the Pythagoras Fractions' formula. Pythagoras Fractions' method also known as Octave Multiplier is purely a mathematical formula. But when we consider the physical parameters of the guitar strings, it definitely gives different values than the ideal ones. Similarly when we consider the object for plucking guitar string such as plectrum/ pick and finger, even this object has different ways of interaction with the string. Finger plucked guitar notes have considerable variations from standard values compared with plectrum/pick plucked guitar notes. In picked notes, touching area or the area of contact is smaller than finger plucked notes and this interaction introduces sharply plucked notes. Fourier analysis is helpful in estimating the number of major harmonics forming the music notes. After knowing and verifying the



frequencies, a mathematical model is proposed to generate acoustic guitar notes.

This mathematical analysis also helps in tuning the acoustic guitars.

	PICK	FINGER	OCTAVE MULTIPLIER	LMD
FRET NUMBERS	STRING 2	STRING 2	STRING 2	STRING 2
fret 1	264.6248	261.25	261.5814	264.2372
fret 2	280.6166	277.8365	277.1359	279.9496
fret 3	296.6546	293.3217	293.6152	296.5963
fret 4	314.0712	329.3384	311.0745	314.2328
fret 5	333.2819	348.3073	329.572	332.9181
fret 6	352.3584	369.2701	349.1693	352.7144
fret 7	373.2264	390.3785	369.932	373.6879
fret 8	394.9435	413.4143	391.9293	395.9085
fret 9	417.7591	438.8253	415.2346	419.4505
fret 10	443.0099	464.2361	439.9258	444.3923
fret 11	468.75	490.7853	466.0851	470.8172
fret 12	495.9432	491.0193	493.8	498.8135
fret 13	524.1316	520.5853	523.1629	528.4745
fret 14	556.3702	550.4808	554.2718	559.8992
fret 15	589.4331	582.3376	587.2305	593.1925
fret 16	623.3904	616.2961	622.149	628.4656
fret 17	661.0836	660.1	659.1439	665.8361
fret 18	704.1667	697.672	698.3387	705.4288
fret 19	743.1723	735.2881	739.864	747.3758
fret 20	787.8957	777.8105	783.8586	791.817

**Table 5 Summary of frequency calculation methods**

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