Certain Illustrative Numerical Implementations of Tridiagonal Folmat Enhanced Multivariance Products Representation (TFEMPR) for 3-Way Arrays

ZEYNEP GÜNDOĞAR
İstanbul Technical University
Informatics Institute
Maslak, 34469, İstanbul
TURKEY (TÜRKİYE)
gundogarzeynep@gmail.com.tr

METİN DEMİRALP
İstanbul Technical University
Informatics Institute
Maslak, 34469, İstanbul
TURKEY (TÜRKİYE)
metin.demiralp@be.itu.edu.tr

Abstract: In this work we focus on a recently developed method we called Tridiagonal Folmat Enhanced Multivariance Products Representation (TFEMPR) using folded matrices (folmat) in order to decompose multiway arrays. We give results of some numerical implementations for 3-way arrays. To avoid complexity of multiway array decomposition on each direction (way, dimension) we use folmats which provides binary decomposition.

Key–Words: Enhanced Multivariance Products Representation (EMPR), Tridiagonal Matrix Enhanced Multivariance Products Representation (TMEMPR),Tridiagonal Folmat Enhanced Multivariance Products Representation (TFEMPR), Folded Matrices, Multiway arrays, Decomposition

1 Introduction

Multivariate analysis and decomposition have always been considered important in the nature of scientific problems [1–8]. Decomposing of multivariance and representing multivariance in terms of less variate enable us to reduce computational complexity and facilitates the analysis. Enhanced Multivariance Products Representation (EMPR) [9–13] is a decomposition method and represents multiway arrays in terms of less variate arrays. EMPR of a matrix, which is in fact a 2-way array, can be given as follows.

$$\mathbf{A} = \alpha\mathbf{u}\mathbf{v}^T + \mathbf{a_1}\mathbf{v}^T + \mathbf{u}\mathbf{a_2}^T + \mathbf{A_{12}} \quad (1)$$

This representation consists of four term respectively a constant, a first-way, a second-way and finally a two-way term. Here $\mathbf{u}$ and $\mathbf{v}$ stand for the support vectors. The additive terms except the two-way component are composed of outer products. It is possible to construct a representation consisting only outer products by using EMPR consecutively on the target function and then the two-way components. After the first four term representation construction for a chosen support vectors, EMPR is applied to the two-way component by using different support vectors which are constructed from the univariate vectors of the first EMPR accordingly. Then a new EMPR is applied on the newly constructed two-way array and so on. Each step where four term EMPR is used to construct new constant, univariate and two-way components employs the existing components obtained one step before. This method produces a representation whose core matrix between the orthogonal matrices whose columns are left and right support vectors is tridiagonal. Hence this recur sive construction is called “Tridiagonal Matrix Enhanced Multivariance Products Representation(TMEMPR) and has been developed by Demiralp and his group [14–17]. Compact form of TMEMPR is as follows

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \quad (2)$$

The columns of $\mathbf{U}$ and $\mathbf{V}$ matrices are support vectors and $\mathbf{\Sigma}$ is a tridiagonal matrix consisting of contributions coming from outer products at each recursion step. Although the structure of TMEMPR may seem to be similar to Singular Value Decomposition (SVD), TMEMPR differs with the feautere of being a recursive method and the structure of $\mathbf{\Sigma}$ matrix which has just main diagonal for the singular value decomposition.

In order to decompose multiway arrays via folded matrices (folmats) defined by Demiralp [18, 19], a brand new method called “Tridiagonal Folmat Enhanced Multivariance Products Representation” is constructed [20]. This paper focuses on TFEMPR and organised as follows. Section 2 introduces folded matrices. In section 3 we give a brief explanation about TFEMPR especially for 3-way arrays. Numerical implementations of the method is given in section 4 and consequently the results are given in section 5.

2 Folded Matrices (Folmats)

Vectors and matrices in ordinary linear algebra are considered respectively as one-way and two-way arrays in literature. Matrices have two space (row and column space) and also have feature of mapping. Folded matrices (Folmats) defined for making analogy between matrices and multiway arrays and adapt the features of matrix to multiway arrays such as inner product, norm, outer product etc. Folmat is shown as,

$$\mathbf{A}_{i_1,i_2,\cdots,i_n;j_1,j_2,\cdots,j_n} \equiv \mathbf{A}_{G_L;G_R} \tag{3}$$

where semicolon separates indices of multiway array into two groups. Left side indices are correlated to somehow row space and the right-side indices are somehow correlated with the column space. We use folmats to get rid of computational complexity of decomposition of multiway arrays on each way. Semicolon enables binary decomposition by separating indices into two groups.

3 Tridiagonal Folmat Enhanced Multivariance Products Representation (TFEMPR) for 3-Way Arrays

Tridiagonal Folmat Enhanced Multivariance Products Representation (TFEMPR) which is one of the decomposition methods for multiway arrays has the all features of binary decompositions due to the usage of folmat concept as given in section 2. TFEMPR method can be considered as higher order analogues of matrix decomposition. In this section we are going to give details about the TFEMPR for 3-way arrays.

Now we will show the details of the method's step by step construction. First we will apply EMPR to a folmat. We can write

$$\begin{aligned} \mathbf{A}_{G_L;G_R} &= a_0\mathbf{U}_{G_L;}\mathbf{V}^T_{G_R;} + \mathbf{a}^{(1)}_{G_L;}\mathbf{V}^T_{G_R;} \\ &+ \mathbf{U}_{G_L;}{\mathbf{a}^{(2)}_{G_R;}}^T + \mathbf{A}^{(1,2)}_{G_L;G_R} \end{aligned} \tag{4}$$

where $\mathbf{A}_{G_L;G_R}$ stands for a folmat and G_L is called as left grid which includes row space indices, G_R is called as right grid includes column space indices. This representation consists of four terms respectively, the constant term, the term towards left grid, the term towards right grid and remainder term ($\mathbf{A}^{(1,2)}_{G_L;G_R}$). $\mathbf{U}_{G_L;}$ and $\mathbf{V}_{G_R;}$ are preselected support folded vectors(folvec) respectively on left and right grid.

The first step of TFEMPR for 3-way arrays is given explicitly as follows:

$$\begin{aligned} \mathbf{A}_{ij;k} &= a_0\mathbf{U}_{ij;}\mathbf{V}^T_{k;} + \mathbf{a}^{(1)}_{ij;}\mathbf{V}^T_{k;} \\ &+ \mathbf{U}_{ij;}{\mathbf{a}^{(2)}_{k;}}^T + \mathbf{A}^{(1,2)}_{ij;k} \end{aligned} \tag{5}$$

Herein the location of semicolon is important. Because left and right grids can be changed depending on how to locate the semicolon. This is a flexibility for the method. There is 3 situations given below for 3-way arrays.

$$\mathbf{A}_{ij;k} \quad \mathbf{A}_{i;jk} \quad \mathbf{A}_{ik;j} \tag{6}$$

Here we will take $\mathbf{A}_{ij;k}$ into the consideration .

In (5) there are four components to be determined $a_0, \mathbf{a}^{(1)}_{ij;}, \mathbf{a}^{(2)}_{k;}, \mathbf{A}^{(1,2)}_{ij;k}$. It is necessary to provide the two preconditions for determining the components. First condition is the unit norm standardization and can be given as follows for the left and right grid support folvecs

$$\mathbf{U}^T_{ij;}\mathbf{U}_{ij;} = \sum_{i=1}^{I}\sum_{j=1}^{J} u^2_{i,j} = 1 \tag{7}$$

$$\mathbf{V}^T_{k;}\mathbf{V}_{k;} = \sum_{k=1}^{K} v^2_k = 1 \tag{8}$$

Support folded vectors should satisfy these unit norm standardization. The second constraints on the components are the vanishing conditions:

$$\mathbf{U}^T_{ij;}\mathbf{a}^{(1)}_{ij;} = 0 \tag{9}$$

$${\mathbf{a}^{(2)}_{k;}}^T\mathbf{V}_{k;} = 0 \tag{10}$$

These conditions point out the orthogonality of support folvec and the relevant component through the same grid. And also $\mathbf{U}_{ij;}$ should be in the left null space of $\mathbf{A}^{(1,2)}_{ij;k}$ and $\mathbf{V}_{k;}$ is in the right null space of $\mathbf{A}^{(1,2)}_{ij;k}$

$$\mathbf{U}^T_{ij;}\mathbf{A}^{(1,2)}_{ij;k} = \mathbf{0}_{;k} \tag{11}$$

$$\mathbf{A}^{(1,2)}_{ij;k}\mathbf{V}_{k;} = \mathbf{0}_{ij;} \tag{12}$$

Under these conditions by multiplying (5) with $\mathbf{U}^T_{ij;}$from left and with $\mathbf{V}_{k;}$ from right we obtain a_0 as follows

$$a_0 = \mathbf{U}^T_{ij;}\mathbf{A}_{ij;k}\mathbf{V}_{k;} \tag{13}$$

In order to get $\mathbf{a}^{(1)}_{ij;}$ we multiply (5) with $\mathbf{V}_{k;}$ from right

$$\mathbf{A}_{ij;k}\mathbf{V}_{k;} = a_0\mathbf{U}_{ij;} + \mathbf{a}^{(1)}_{ij;} \tag{14}$$

$$\mathbf{a}^{(1)}_{ij;} = \mathbf{A}_{ij;k}\mathbf{V}_{k;} - a_0\mathbf{U}_{ij;} \tag{15}$$

and combine (13) with (14) we obtain $\mathbf{a}^{(1)}_{ij;}$.

$$\mathbf{a}^{(1)}_{ij;} = \left(\mathbf{I}_{ij;ij} - \mathbf{U}_{ij;}\mathbf{U}^T_{ij;}\right)\mathbf{A}_{ij;k}\mathbf{V}_{k;} \tag{16}$$

Here $\left(\mathbf{I}_{ij;ij} - \mathbf{U}_{ij;}\mathbf{U}_{ij;}^T\right)$ is a projection operator and projects onto the complement of the space spanned by $\mathbf{U}_{ij;}$.

In the same manner, if we multiply (5) with $\mathbf{U}_{ij;}^T$ from left,

$$\mathbf{U}_{ij;}^T\mathbf{A}_{ij;k} = a_0\mathbf{V}_{k;}^T + {\mathbf{a}_{k;}^{(2)}}^T \tag{17}$$

and reorganise (17) as follows

$$\mathbf{a}_{k;}^{(2)} = \mathbf{A}_{ij;k}^T\mathbf{U}_{ij;} - a_0\mathbf{V}_{k;} \tag{18}$$

then $\mathbf{a}_{k;}^{(2)}$ is obtained as

$$\mathbf{a}_{k;}^{(2)} = \left(\mathbf{I}_{k;k} - \mathbf{V}_{k;}\mathbf{V}_{k;}^T\right)\mathbf{A}_{ij;k}^T\mathbf{U}_{ij;} \tag{19}$$

Here $\left(\mathbf{I}_{k;k} - \mathbf{V}_{k;}\mathbf{V}_{k;}^T\right)$ is a matrix projecting onto the complement of the space spanned by$\mathbf{V}_{k;}$.

Remainder term $\mathbf{A}_{ij;k}^{(1,2)}$ is obtained by extracting, first three terms in (5) from target folmat.

$$\begin{aligned}\mathbf{A}_{ij;k}^{(1,2)} &= \mathbf{A}_{ij;k} - a_0\mathbf{U}_{ij;}\mathbf{V}_{k;}^T \\ &- \mathbf{a}_{ij;}^{(1)}\mathbf{V}_{k;}^T - \mathbf{U}_{ij;}{\mathbf{a}_{k;}^{(2)}}^T\end{aligned} \tag{20}$$

By reorganising this equation compact form of $\mathbf{A}_{ij;k}^{(1,2)}$ can be obtained as follows

$$\mathbf{A}_{ij;k}^{(1,2)} = \left(\mathbf{I}_{ij;ij} - \mathbf{U}_{ij;}\mathbf{U}_{ij;}^T\right)\mathbf{A}_{ij;k}\left(\mathbf{I}_{k;k} - \mathbf{V}_{k;}\mathbf{V}_{k;}^T\right) \tag{21}$$

$\mathbf{U}_{ij;}$ is in the left null space and $\mathbf{V}_{k;}$ is in the right null space of $\mathbf{A}_{ij;k}^{(1,2)}$. This equality indicates that support folvecs generated from target folmat enter into the null space of remainder term. In other words, rank of $\mathbf{A}_{ij;k}^{(1,2)}$ decreases by 1.

So far we determined the components of the representation (5). From now on operations to be taken is for building recursive structure to give TFEMPR.

For simplicity, we define

$$\begin{aligned}\alpha^{(1)} &= a_0 \\ \beta^{(1)} &= \|\mathbf{a}_{ij;}^{(1)}\| \\ \gamma^{(1)} &= \|\mathbf{a}_{k;}^{(2)}\|\end{aligned} \tag{22}$$

and for begining recursion step

$$\begin{aligned}\alpha^{(1)} \equiv a_0, \qquad & {\mathbf{A}_{ij;k}}^{(0)} \equiv \mathbf{A}_{ij;k}, \\ & {\mathbf{A}_{ij;k}}^{(1)} \equiv \mathbf{A}_{ij;k}^{(1,2)} \\ \mathbf{u}_{G_L;}^{(1)} = \mathbf{u}_{G_L;} \qquad & \mathbf{v}_{G_R;}^{(1)} = \mathbf{v}_{G_R;}\end{aligned} \tag{23}$$

(5) is rewritten by using these definitions

$$\begin{aligned}{\mathbf{A}_{ij;k}}^{(0)} &= \alpha^{(1)}\mathbf{U}_{ij;}{}^{(1)}{\mathbf{V}_{k;}^{(1)}}^T \\ &+ \beta^{(1)}\mathbf{U}_{ij;}{}^{(2)}{\mathbf{V}_{k;}^{(1)}}^T \\ &+ \gamma^{(1)}\mathbf{U}_{ij;}{}^{(1)}{\mathbf{V}_{k;}^{(2)}}^T + \mathbf{A}_{ij;k}^{(1)}\end{aligned} \tag{24}$$

For each recursion step we need new support folvecs. Because of this we define new support folvecs orthogonal to old ones as follows

$$\mathbf{u}_{ij;}^{(2)} = \frac{1}{\|\mathbf{a}_{ij;}^{(1)}\|}\mathbf{a}_{ij;}^{(1)} \qquad \mathbf{v}_{k;}^{(2)} = \frac{1}{\left\|\mathbf{a}_{k;}^{(2)}\right\|}\mathbf{a}_{k;}^{(2)} \tag{25}$$

The goal of the recursion is to get rid of remainder term. To this end on each step we apply EMPR to the remainder term.

Recursive structure for jth step is:

$$\begin{aligned}{\mathbf{A}_{ij;k}}^{(j)} &= \alpha^{(j+1)}\mathbf{U}_{ij;}{}^{(j+1)}{\mathbf{V}_{k;}^{(j+1)}}^T \\ &+ \beta^{(j+1)}\mathbf{U}_{ij;}{}^{(j+2)}{\mathbf{V}_{k;}^{(j+1)}}^T \\ &+ \gamma^{(j+1)}\mathbf{U}_{ij;}{}^{(j+1)}{\mathbf{V}_{k;}^{(j+2)}}^T \\ &+ \mathbf{A}_{ij;k}^{(j+1)}\end{aligned} \tag{26}$$

The rank of remainder term decreases by 1 on each step so the remainder term vanishes when one of its null spaces becomes full rank. Then (26) turns out to be the equality given below.

$$\begin{aligned}\mathbf{A}_{ij;k} &= \sum_{\{i\}\in G_L}\alpha_i\mathbf{U}_{ij;}{}^{(j+1)}{\mathbf{V}_{k;}^{(j+1)}}^T \\ &+ \sum_{\{i\}\in G_L}\beta_i\mathbf{U}_{ij;}{}^{(j+2)}{\mathbf{V}_{k;}^{(j+1)}}^T \\ &+ \sum_{\{i\}\in G_L}\gamma_i\mathbf{U}_{ij;}{}^{(j+1)}{\mathbf{V}_{k;}^{(j+2)}}^T \\ &= \overline{\mathbf{U}}_{ij;ij}\mathbf{\Sigma_{ij;k}}{\overline{\mathbf{V}}_{k;k}}^T,\end{aligned} \tag{27}$$

Here the entities of $\overline{\mathbf{U}}_{ij;ij}$ and $\overline{\mathbf{V}}_{k;k}$ are orthonormal folmats. Because the entities of these folmats are normalized support folvecs and orthogonal to each other we can write

$$\overline{\mathbf{U}}_{ij;ij} \equiv \left[\mathbf{U}_{ij;}^{(1)} \;\dots\; \mathbf{U}_{ij;}^{(m)}\right] \tag{28}$$

$$\overline{\mathbf{V}}_{k;k} \equiv \left[\mathbf{V}_{k;}^{(1)} \;\dots\; \mathbf{V}_{k;}^{(n)}\right] \tag{29}$$

where the superscripts(m) and (n) over $\overline{\mathbf{U}}_{ij;ij}$ and $\overline{\mathbf{V}}_{k;k}$ respectively represents the sizes of the left and right grids respectively and have the values through $m = ij$ and $n = k$.

$\mathbf{\Sigma_{m;n}}$ consists of αs, βs and γs calculated at each step of recursion. These parameters represent the contribution of outer products. Under assumption of $m < n$ we can write

$$\mathbf{\Sigma_{m;n}} = \begin{bmatrix} \alpha_1 & \gamma_1 & 0 & 0 & 0 & \cdots \\ \beta_1 & \alpha_2 & \gamma_2 & 0 & 0 & \cdots \\ 0 & \beta_2 & \alpha_3 & \gamma_3 & 0 & \cdots \\ 0 & 0 & \beta_3 & \alpha_4 & \gamma_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & \beta_{m-1} & \alpha_m \end{bmatrix} \quad (30)$$

By using α, β and γ parameters we can define quality measurers at each recursion step to measure how good the approximation is. Quality measurer definition is given as follows for the kth step of the recursion

$$\sigma_k = \sum_{i=1}^{k} \frac{\alpha_i^2 + \beta_i^2 + \gamma_i^2}{\|\mathbf{A}\|} \quad (31)$$

4 Numerical Implementations

In this section we will give implementations to test the method for certain multiway arrays. In the implemantations, tables contain α, β and γ parameters calculated at each step and quality measurer.

There are two remarkable points in the implementations for this method. One of the those points is the choice of $\mathbf{A}_{ij;k}$ which is flexible for the method. We give only results for the $\mathbf{A}_{ij;k}$ type folmats. The other point is the choice of support folvecs at the beginning step. Here we use support folvecs whose all elements are equal.

First implementation: For a given $4 \times 4 \times 4$ multiway array as follow:

$$\mathbf{A}_{ijk} = ijk \qquad i, j, k = 1, 2, 3, 4 \quad (32)$$

We apply TFEMPR to this multiway array whose structure is important due to the fact that it consists of outer products. We expect to get exact result before four recursion step. For this multiway array the results are given in the below table:

Adm	α	β	γ	σ
i=1	125	82.92	55.90	0.95
i=2	37.08	0	0	1

Second implementation: The target multiway array is in $5 \times 4 \times 5$ type and given as

$$\mathbf{A}_{ijk} = \sin(100i)^3 \cos(20(j+k))^3 \qquad i, k = 1, 2, 3, 4, 5 \quad j = 1, 2, 3, 4 \quad (33)$$

Quality measurer of TFEMPR for this multiway array gives exactly 1 at 5th recursion step.

Adm	α	β	γ	σ
i=1	-0.1509	0.1861	0.8155	0.0634
i=2	-1.5446	0.4794	0.5790	0.3224
i=3	-0.1761	0.8182	0.8281	0.4441
i=4	2.1859	0.0972	1.1033	0.9712
i=5	0.5731	0.0000	0.0000	1.0000

Third implementation: The purpose of this implementation is to test the different support folvecs in TFEMPR. Multiway array ($3 \times 3 \times 3$ type) is given as follows explicitly.

$$\mathbf{X}(:,:,\mathbf{1}) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
$$\mathbf{X}(:,:,\mathbf{2}) = \begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix}$$
$$\mathbf{X}(:,:,\mathbf{3}) = \begin{bmatrix} 19 & 20 & 21 \\ 22 & 23 & 24 \\ 25 & 26 & 27 \end{bmatrix} \quad (34)$$

Here we choose normalised support folvec as having elements which are all same (Unit support) as below:

$$\mathbf{U}_{i,j} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{V}_k = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (35)$$

The other type support folvecs are obtained by directional averaging as follows :

$$\mathbf{U}_{i,j} = \frac{1}{\sqrt{1824}}\begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix} \quad \mathbf{V}_k = \frac{1}{\sqrt{750}}\begin{bmatrix} 5 \\ 14 \\ 23 \end{bmatrix} \quad (36)$$

For this implementation unit support folvec gives better result than the average support. Quality measurers for two support folvecs are located in the following table

Step number	Unit Support	Average Support
i=1	1.0000	0.9946
i=2	1.0000	1.0000
i=3	1.0000	1.0000

Fourth implementation: In this implementation we consider multiway array discussed in second implemetation again but this time for $5 \times 5 \times 5$

$$\mathbf{X}_{ijk} = \sin(100i)^3 \cos(20(j+k))^3 \qquad i, k = 1, 2, 3, 4, 5 \quad (37)$$

In this implementation usage of average support gives better than unit support folvec

Step number	Unit Support	Average support
$i = 1$	0.0634	0.3097
$i = 2$	0.3224	0.4223
$i = 3$	0.4441	0.9654
$i = 4$	0.9712	1.0000
$i = 5$	1.0000	1.0000

5 Concluding Remarks

Concluding remarks has been itemized as follows:

- In this paper we have emphasized on TFEMPR method for three way arrays. We have restricted our implementations only for three way arrays We intend to increase dimension of multiway arrays in our future work implementations.

- TFEMPR method ends up with three factor $\mathbf{U\Sigma V}^T$. In literature there are analogues of this decomposition and factorization method such as commonly known method singular value decomposition. Bench marking of these two methods are left to the future works.

- The two flexibilities of this method are the choice of support folvecs at the begining of the recursion and how to choose left and right grids which we mentioned in the implementations. Implementations were done on this issue but results did not given here Since the determination how to choose the left and right grids has not been generalized yet.

- In this paper we have only focused on synthetic data. This method is implementable to problems in many areas. We also applied TFEMPR to real data sets obtained from videos. We left the results of video datas which are three way arrays to the extended paper.

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