# Numerical Implementations for Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR) Method: Bivariate Case

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*Abstract:* In this paper, a decomposition method Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR) is presented with numerical implementations. These method is based on bivariate square integrable functions over the rectangular hyperprism. This research is also can be considered as the extension to our works whose papers were given in the very recent international conferences. The contribution of this new paper is to apply the method not only to the functions that are as the product of two given univariate functions, but also to any bivariate analytical functions. A number of numerical examples are used to show the efficiency of the method given in this paper.

*Key–Words:* Enhanced Multivariance Products Representation (EMPR), Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR), Recursive EMPR, Decomposition Method.

## 1 Introduction

Tridiagonal Kernel Enhanced Multivariance Products Representation (TKEMPR) is based on the Enhanced Multivariance Products Representation method. For this reason, we will give the basic information of these method here.

EMPR is an expansion for a given multivariate function $f(x_1, ..., x_N)$ as

$$
\begin{aligned}
f(x_1, ..., x_N) &= f_0 \prod_{i=1}^{N} s_i(x_i) + \sum_{j=1}^{N} f_j(x_j) \prod_{\substack{i=1 \\ i \neq j}}^{N} s_i(x_i) \\
&+ \sum_{\substack{j_1, j_2=1 \\ j_1 < j_2}}^{N} f_{j_1, j_2}(x_{j_1}, x_{j_2}) \prod_{\substack{i=1 \\ i \neq j_1, j_2}}^{N} s_i(x_i) \\
&+ \sum_{\substack{j_1, j_2, j_3=1 \\ j_1 < j_2 < j_3}}^{N} f_{j_1, j_2, j_3}(x_{j_1}, x_{j_2}, x_{j_3}) \prod_{\substack{i=1 \\ i \neq j_1, j_2, j_3}}^{N} s_i(x_i) \\
&+ \cdots + f_{12...N}(x_1, ..., x_N) \quad (1)
\end{aligned}
$$

where $N$ is the number of the independent variables. Subindexed $f$s are called constant, univariate, bivariate EMPR components and so on respectively [1–4]. $s_i$s are univariate support functions. $x_j$s, are independent variables in some interval on the real line such that

$$
x_i \in [a_i, b_i], \qquad i = 1, ..., N \quad (2)
$$

The main purpose of this method is to determine the general structure of the EMPR components. In this work, we focused on bivariate functions on the unit interval $[0, 1]$. Therefore, we can rewrite the EMPR expansion for a given bivariate function $f(x, y)$ as

$$
\begin{aligned}
f(x, y) &= f_0 u(x) v(y) + f_1(x) v(y) + f_2(y) u(x) \\
&+ f_{1,2}(x, y) \quad (3)
\end{aligned}
$$

where $u$ and $v$s are the support functions. The weight functions and support functions, used in this method, are defined as the following form

$$
W_1(x) \equiv 1, \qquad \int_0^1 dx\, u(x)^2 = 1, \quad (4)
$$

$$
W_2(y) \equiv 1, \qquad \int_0^1 dy\, v(y)^2 = 1 \quad (5)
$$

For the determination of the components the vanishing conditions should be imposed as

$$
\int_0^1 dx\, f_1(x) u(x) = 0, \quad \int_0^1 dy\, f_2(y) v(y) = 0 \quad (6)
$$

$$
\int_0^1 dx\, f_{1,2}(x, y) u(x) = 0, \int_0^1 dy\, f_{1,2}(x, y) v(y) = 0 \quad (7)
$$

These equations allow us to get the EMPR components uniquely as the following form

$$f_0 = \int_0^1 dx \int_0^1 dy f(x,y) u(x) v(y) \qquad (8)$$

$$f_1(x) = \int_0^1 dy f(x,y) v(y) - f_0 u(x) \qquad (9)$$

$$f_2(y) = \int_0^1 dx f(x,y) u(x) - f_0 v(y) \qquad (10)$$

$$\begin{aligned} f_{1,2}(x,y) &= f(x,y) - f_0 u(x) v(y) \\ &\quad - f_1(x) v(y) - f_2(y) u(x) \end{aligned} \qquad (11)$$

## 2 TKEMPR Decomposition Method

In the EMPR expansion, there are $2^N$ componenets for a given $N-$variate function. Computation of them are very expensive task. This issue urges us to truncate EMPR expansion to low the multivairance and convert the expansion to an approximation method. But, in this work , we are going to choose another way to low the multivariane by vanishing the bivariate EMPR component, shown at the right side of the expansion given in (3).

Let us begin by briefly analysing TKEMPR method when applied to any bivariate square integrable functions on the interval $[\,0,1\,]$ [5–28]. For this reason, we need to start with constructing a recursion formula by using EMPR method.

The EMPR expansion for the target function $f(x,y)$ can be rewritten as the following form

$$\begin{aligned} f^{(0)}(x,y) &= \alpha_1 u_1(x) v_1(y) + \beta_1 u_2(x) v_1(y) \\ &\quad + \gamma_1 u_1(x) v_2(y) + f^{(1)}(x,y) \end{aligned} \qquad (12)$$

such that

$$f^{(0)}(x,y) = f(x,y) \qquad (13)$$

$$\alpha_1 = f_0, \qquad \beta_1 = \|f_1\|_x \qquad \gamma_1 = \|f_2\|_y \qquad (14)$$

$$u_1(x) = u(x), \qquad v_1(y) = v(y) \qquad (15)$$

$$u_2(x) = \frac{f_1(x)}{\|f_1\|_x}, \qquad v_2(y) = \frac{f_2(y)}{\|f_2\|_y} \qquad (16)$$

The norm mentioned here is explicitly given below

$$\|f_1\|_x \equiv \left( \int_0^1 dx f_1(x)^2 \right)^{1/2}$$

$$\|f_2\|_y \equiv \left( \int_0^1 dy f_2(y)^2 \right)^{1/2} \qquad (17)$$

At the begining, there are only two $u_1(x)$ and $v_1(y)$ initial support functions. After using recursion, we obtain another generated two $x$ and $y$ dependent $u_2(x)$ and $v_2(y)$ support functions. The left generated support functions, $u_1(x)$ and $u_2(x)$ are mutually orthogonal and have unit norms. Similary, the right generated support functions, $v_1(y)$ and $v_2(y)$ are also mutually orthogonal and have unit norms.

Now we can consider the following recursion which matches (12) for the $j$ step,

$$\begin{aligned} f^{(j)}(x,y) &= \alpha_{j+1} u_{j+1}(x) v_{j+1}(y) \\ \\ &\quad + \beta_{j+1} u_{j+2}(x) v_{j+1}(y) \\ \\ &\quad + \gamma_{j+1} u_{j+1}(x) v_{j+2}(y) \\ \\ &\quad + f^{(j+1)}(x,y), \qquad j = 0, 1, \dots \end{aligned} \qquad (18)$$

where

$$f^{(j)}(x,y) = f_{1,2}^{(j)}(x,y), \qquad (19)$$

$$\alpha_j = f_0^{(j)}, \qquad (20)$$

and

$$\beta_j = \left\| f_1^{(j)} \right\|_x, \quad \gamma_j = \left\| f_2^{(j)} \right\|_y, \quad j = 1, 2, 3, \dots \qquad (21)$$

and generated support functions are defined as

$$u_{j+1}(x) = \frac{f_1^{(j)}(x)}{\left\| f_1^{(j)} \right\|_x}, \qquad v_{j+1}(y) = \frac{f_2^{(j)}(y)}{\left\| f_2^{(j)} \right\|_y} \qquad (22)$$

This equality implies that $f^{(j)}(x,y)$ tends to vanish when $j$ grows unboundedly up to infinity. This can happen only when the $u_j$ functions and/or $v_j$ functions form basis sets separately. If this happens then we can arrive at the following ultimate decomposition

$$\begin{aligned} f(x,y) &= \sum_{j=1}^{\infty} \alpha_j u_j(x) v_j(y) \\ \\ &\quad + \sum_{j=1}^{\infty} \beta_j u_{j+1}(x) v_j(y) \\ \\ &\quad + \sum_{j=1}^{\infty} \gamma_j u_j(x) v_{j+1}(y). \end{aligned} \qquad (23)$$

where the univariate $u_j(x)$ functions form an infinite orthonormal function set so do the $v_j(y)$ functions separately. This equality can be rewritten in the following concise form [20–28]

$$f(x,y) = \mathbf{U}(x)^T \mathbf{K_T} \mathbf{V}(y) \qquad (24)$$

Now, we obtain the concise matrix format like three factor matrix product whose kernel is in tridiagonal matrix form, symbolized with $\mathbf{K_T}$.

# 3   Numerical Implementations

This section includes discussion convergences, computational complexities of this procedure by presenting a number of numerical examples for given bivariate functions.

As a first example, $\cos{(x + y)}$ is chosen which is not multiplicative form.

$$\mathbf{K_T} \equiv \begin{bmatrix} 0.496751 & 0.227949 \\ 0.227949 & -0.041087 \end{bmatrix}$$

such that

$$\mathbf{U} \equiv \begin{bmatrix} 1 \\ -2.17922 + 3.69149cos(x) - 2.01667sin(x) \end{bmatrix},$$

$$\mathbf{V} \equiv \begin{bmatrix} 1 \\ -2.17922 + 3.69149cos(y) - 2.01667sin(y) \end{bmatrix},$$

In this first kind of example for TKEMPR method, the approximation is constructed by using $2x2$ matrices. In other words, recursion is used for two steps. When we also do the norm analysis here, we get the norm value as 0.000499992 in working precision 20. Red surface belong to the exact function, blue one belong to the approximated function.
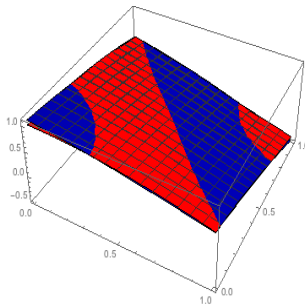


Figure 1: $cos(x + y)$, in the interval $[0, 1]$, $n = 2$

$$\mathbf{K_T} \equiv \begin{bmatrix} 0.496751 & 0.227949 & 0. \\ 0.227949 & -0.041087 & 0.012163 \\ 0. & 0.0121633 & -0.001016 \end{bmatrix}$$

such that

$$\mathbf{U} \equiv \begin{bmatrix} 1 \\ -2.17922 + 3.69149cos(x) - 2.01667sin(x) \\ -26.10199 + 23.74800cos(x) + 13.31033sin(x) \end{bmatrix}$$

$$\mathbf{V} \equiv \begin{bmatrix} 1 \\ -2.17922 + 3.69149cos(y) - 2.01667sin(y) \\ -26.10199 + 23.74800cos(y) + 13.31033sin(y) \end{bmatrix}$$

In the second kind of example, the approximation is constructed by using $3x3$ matrices. This means, the recursion for vanishing the residual component in the method is used for three steps. The norm value is determined as $2,90969.10^{-31}$ in working precision 20 [29–32].
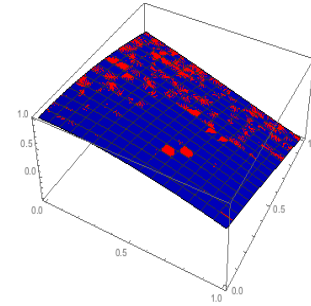


Figure 2: $cos(x + y)$, in the interval $[0, 1]$, $n = 2$

As a second example $(e^x + e^y)/(1 + x + y)$ is chosen. Similarly, we apply the method into $2x2$ matrix product form. But in comparsion with the previous example, this function has more computational complexities which depend on its structure. So the decomposed left and right support vectors will be non-apparent when they are written. However, the kernel matrix $\mathbf{K_T}$ can be written as the following form

$$\mathbf{K_T} \equiv \begin{bmatrix} 1.72098 & 0.03487 \\ 0.03487 & 0.00477 \end{bmatrix}$$

In the figure given below, it is shown the exact function and approximated function. The norm value is determined as 0.0000378336.
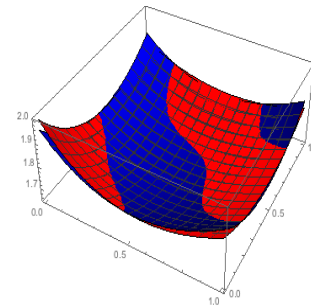


Figure 3: $(e^x + e^y)/(1 + x + y)$, in the interval $[0, 1]$, $n = 2$

# 4 Concluding Remarks

The TKEMPR method can be considered as the concise matrix format like three factor matrix product whose kernel is in tridiagonal matrix form as $\mathbf{U}(x)^T\mathbf{K_T}\mathbf{V}(y)$. The results obtained here and in previous works support that the methodcan easily incorporate into any bivariate functions to approximate well. This is also can be expand to the multivariate functions. It will be discussed in the next future works. From the results shown in figures and error analysis, we can see that increasing the dimension, or namely increasing the recursion for vanishing the residual component, reduces the error of the approximation method. It is observed that there is no need for us to enhance the recursion steps to great values. However, there can be computational difficulties because of the target functions. We obtain approximation in sufficiently good quality through the method. After constructing this method into a general algorithm for all multivariate functions, the performance of the TKEMPR method can be seen in more detail.

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