Curved Linear Antennas as Quantum Traps

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Abstract: - Curved linear antennas of arbitrary shape can be considered as a special case of quantum thin conducting wire traps of their free electrons and a numerical method is proposed for the calculation of their current distribution. In a case of low energy excitation with a proper frequency, only the fundamental mode with its lowest Eigen-value appears and the respective Eigen-function is its current distribution along it. In a previous paper Papageorgiou et al., proposed a numerical method of calculating the radiation pattern of any arbitrary shape linear thin antenna, with a known current distribution of its free electrons along with it. Also in a recent paper Papageorgiou et al., introduced the Resonant Transmission Line (RTL) method for numerically tackling the problem of linear quantum wires with arbitrary curvature. This method is also applied here in order to calculate the fundamental Eigenvalue and its respective Eigenfunction of any arbitrary shape curved linear thin antenna. The analysis reveals a strict dependency of the energy Eigen-values to the curvature magnitude with significant lowering of the first harmonic beyond a threshold value which severely affects excitability of the respective Eigen-function. The proposed method is applied in the study of a very sensitive antenna made of constant curvature circular arcs.

Key-Words: - Schrödinger equation, linear antenna, circular arc antenna

1 Introduction

In a recent series of publications Stockhofe and Schmelcher [3] as well as Zambetaki *et al.*, [4] and J. K. Pedersen *et al.*, [5] proposed a treatment of the Schrodinger equation in a curvilinear coordinate system for one dimensional quantum waveguides. Linear thin conducting antennae can be considered as one dimensional quantum traps acting as electromagnetic waveguide resonator, i.e. their free electrons form standing waves trapped inside their one dimensional curved linear space. This consideration can be used in order to define the fundamental mode of distribution of the standing wave currents, of any arbitrary shape linear antennae under a proper frequency excitation.

Given the currents on the linear antennae their radiation patterns can be calculated according to [1] Papageorgiou *et al.*, already used in [2], this recently introduced case of the Schrodinger equation on a curved one dimensional path, in the light of the proposed Resonant Transmission Line (RTL) simulation and the related numerical method. By a segmentation process it was shown how to apply this method in the case of any linear curved quantum wire of any arbitrary varying curvature. The numerical calculation of Eigen-values is simplified using the resonance condition of the equivalent RTL and for each Eigen-value the respective Eigen-function can be also calculated numerically. We will show that for special shaped linear antennas made of a series of circular arcs of the same curvature, the fundamental "Energy" of excitation can be reduced significantly increasing their sensitivity with possible applications in mobile communication.

2 Description of the resonance RTL method

A generic equivalence of the Schrodinger problem or the general Sturm-Liouville problem in one dimension has been established in [2], and [6], which is valid not only for ODE problems but also for PDEs in separable coordinate systems.

To this aim, we consider the representation of a non homogeneous lossless transmission line defined along its geometric length s, with $V(s)$ and $I(s)$ its voltage and current values respectively, and $X(s)$,

Y(s) its "reactance" and "admittance" per length unit. The general PDE representation of such a line is given as

$$\begin{cases} \partial \; V(s) / \partial s = -jX(s) I(s) \\ \partial \; I(s) / \partial s = -jY(s) V(s) \end{cases} \tag{1}$$

It is very easy to show that the set (1) is equivalent to the generic Sturm-Liouville equation

$$\partial\left(j\frac{1}{Y(s)}\partial y(s)/\partial s\right)/\partial s = -jX(s)\cdot y(s) \Rightarrow \partial V(s)/\partial s = -jX(s)\cdot I(s) \tag{2}$$

This is the exact same form of the corresponding Schrödinger operator under the identification of the current $I(s)$ with the wave-function $y(s)$ and the voltage $V(s)$ with the expression $j\frac{1}{Y(s)}\partial y(s)/\partial s$.

Considering an infinitesimal length transmission line δs where both admittance and reactance can be taken as constant, the description becomes identical to a homogeneous transmission line of length δs, which is equivalent with the so called T-circuit shown in fig(1). The respective impedances of the T-circuit are then given as

$$\begin{cases} Z_B = Z(s)\tanh(\gamma(s)\cdot\delta s/2) \\ Z_P = Z(s)/\sinh(\gamma(s)\cdot\delta s) \end{cases} \tag{3}$$

In (3) we identify the local transmission factor as $\gamma(s) = j\sqrt{X(s)Y(s)}$ and the characteristic impedance as $Z(s) = \sqrt{X(s)/Y(s)} = -j\gamma(s)/Y(s)$. For $\gamma(s)\cdot\delta s \ll 1$ we can always approximate this, with a proper choice of the step δs, as

$$\begin{cases} Z_B(s) = Z(s)\gamma(s)\delta s/2 = -j\gamma(s)^2\cdot\delta s/(2Y(s)) \\ Z_P(s) = Z(s)/(\gamma(s)\cdot\delta s) = 1/(jY(s)\cdot\delta s) \end{cases} \tag{4}$$

A successive set of such T-circuits can be used to approximate a transmission line with continuously varying parameters of reactance and admittance. In any real non homogeneous transmission line, both $\gamma(s)$ as well as $Y(s)$ are functions of the excitation frequency associated with the energy parameter E.

The energy parameter values, for which the whole line becomes tuned so as to achieve maximal power transmission, are the resonant values which stand for the RTL Eigen-values and the corresponding current values along the line are the RTL's Eigen-functions. From the well known properties of transmission lines, for any such resonant line, the total reactance's calculated from the left and right terminals towards any intermediate point must equal each other with opposite signs. Hence, the resonant values of frequencies or energies can be found from the roots of the function (5) with L_1, L_2 the total lengths towards any central point, i.e. $L_1+L_2=L$ and L the overall length of the equivalent transmission line.

$$\begin{gathered} Z_{left}(0 \to L_1) + Z_{right}(L \to L_2) = 0 \\ Z_n = (Z_{n-1} + Z_B)\cdot Z_P/(Z_{n-1} + Z_B + Z_P) + Z_B \end{gathered} \tag{5}$$

Given the terminal impedances, the left and right the overall reactance can be calculated, for any E value. Having found the Eigen-values from the roots of (5), it is equally possible to extract the exact shape of the Eigen-functions from the current values. From the general theory of the transmission line equation we know that a solution via a transfer matrix can always be written in the form of a dynamical system $\mathbf{x}_n = \hat{T}_n\mathbf{x}_{n-1}$, where $\mathbf{x}_n = [V_n, I_n]$, the voltage-current vector and T_n a matrix of the form

$$\hat{T}_n = \begin{pmatrix} \cosh(\gamma(s)\cdot\delta s) & Z(s)\cdot\sinh(\gamma(s)\cdot\delta s) \\ \sinh(\gamma(s)\cdot\delta s)/Z(s) & \cosh(\gamma(s)\cdot\delta s) \end{pmatrix} \approx \tag{6}$$
$$\approx \begin{pmatrix} 1 & Z(s)\cdot(\gamma(s)\cdot\delta s) \\ (\gamma(s)\cdot\delta s)/Z(s) & 1 \end{pmatrix}$$

In (6) we used $\delta s = s_{n+1} - s_n \quad and \quad s = (s_{n+1} + s_n)/2$ Thus starting from any terminal point $\mathbf{x}_0 = [V_0, I_0]$ and using the (6) the $\mathbf{x}_n = [V_n, I_n] \quad for \quad n = 1,2,...N$ can be calculated, i.e. the values of $y_n = I_n \quad and \quad \partial y_n/\partial s = -j\cdot Y_n\cdot V_n$ are calculated.

3 Thin curved linear antennae as quantum traps

Under any external electric field acting on a quantum or conducting thin wire, free electrons will be affected by the wire's curvature and some deviation from the original values of the Eigen-values of the straight line of equal length thin wire and from its respective set of its Eigen-functions should be expected. To analyze the situation we assume a parametric representation of the curve on which the wire lies given by three scalar functions of an abstract parameter *s* as $x(t)$, $y(t)$ and $z(t)$ which can be split into *N* sections by an arbitrary choice of t_1, …, t_N (n=1,2,…. n,… N) such that $\sqrt{\Delta x_n^{\,2} + \Delta y_n^{\,2} + \Delta z_n^{\,2}} < \Delta l$ with Δl a sufficiently small length with $\Delta x_n = x\,(t_n) - x\,(t_{n-1})$ and similarly for Δy_n, Δz_n.

Any individual section will then have its own curvature σ given by the relation (7)

$$\sigma^2 = \frac{(\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 + (\dot{z}\ddot{x} - \dot{x}\ddot{z})^2 + (\dot{x}\ddot{y} - \dot{y}\ddot{x})^2}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^3} \quad (7)$$

In (7), simple and double dots stand for the 1st and 2nd derivatives with respect to the parameter *t* and their evaluation is taking place in the middle point of each section $(t_n + t_{n-1})/2$. We can easily prove that the parameter *t* can always be replaced by the length *s* along the curved thin wire given by

$$s = \int_0^t \sqrt{x'^2 + y'^2 + z'^2} \cdot dt$$

This means that for any set of consecutive parameter values t_n the respective values of s_n and σ_n can be numerically calculated. Thus at the present paper the curvature σ can be considered as function of the length parameter s.

The Schrödinger equation for a curved one dimensional (thin) wire developed along its parametric length *s* with $0 \le s \le L$ is given by:

$$\begin{aligned} &\partial^2 y(s) / \partial^2 s = -\left(\frac{\sigma^2(s)}{4} + \mathrm{E} \cdot (m / 2\hbar^2) \right) \cdot y(s), \\ &or, \;\; \partial^2 y(s) / \partial^2 s = -\left(\frac{\sigma^2(s)}{4} + \varepsilon \right) \cdot y(s) \end{aligned} \quad (8)$$

In (8), the standard curvature can be given by the local radius *R(s)* via $\sigma(s) = 1 / R(s)$ This, homogeneous, linear 2nd order ODE can be solved with the aid of the Resonant Transmission Line (RTL) technique previously introduced by Papageorgiou et.al in [7] and used already in a variety of other 2nd order ODEs and PDEs. Following the analysis there, equation (1) reduces to the case of a transmission line with $\mathrm{Y}(s) = 1,\;\; \mathrm{X}(s) = \varepsilon + \sigma^2(s)/4$ which give the local propagation constant at each point as

$$\gamma^2(s) = -\left(\varepsilon + \sigma^2(s)/4\right) \quad (9)$$

We take every small part of the curved wire of length *δs* as equivalent to a T-circuit of impedances Z_B and Z_P with reference to figure 1, being given as

$$\begin{cases} Z_B(s) = -j\gamma(s)^2 \delta s / 2 \\ Z_P(s) = -j / \delta s \end{cases} \quad (10)$$

We also take the terminal impedances of the equivalent line at the boundaries *s* =0 and *L* to be infinite or zero so as to make *y(s)=I(s)* or $\partial y(s) / \partial s = -jV(s)$ zero at these points.

We now consider a varying curvature which introduces an equivalent effective potential of geometric origin and we compute the resulting Eigenvalues and eigenfunctions. To this purpose, we divide the wire in small parts of very small length *δs* along which we may take the curvature to be practically constant. Each such element is then equivalent to the T-circuit parameterized as in (2) and (3). The whole wire is then equivalent to a lossless non homogeneous transmission line made by the succession of T-circuits terminated at zero or infinite impedances.

We then have the freedom to choose any arbitrary intermediate point and calculate the respective "left" and "right" impedances as functions of the energy *ε*. Eigen-values will correspond to the roots of the function $Z_{left}(\varepsilon) + Z_{right}(\varepsilon)$, where the subscripts stand for the left and right boundary points of the linear wire where the calculation of successive impedances starts. The respective Eigen-functions are numerically obtained in any set of successive points on the linear wire $\{s_1, s_2, \ldots, s_N\}$ for each and any Eigen-value from the values of the respective currents of the T-circuits through the application of a Transfer Matrix on a set of initial conditions in the form

$$\begin{bmatrix} U_{n+1} \\ I_{n+1} \end{bmatrix} = \begin{pmatrix} \cosh(\delta s\ \gamma_n) & Z_n \sinh(\delta s\ \gamma_n) \\ Z_n^{-1}\sinh(\delta s\ \gamma_n) & \cosh(\delta s\ \gamma_n) \end{pmatrix} \begin{bmatrix} U_n \\ I_n \end{bmatrix}$$
$$Z_n = -j\gamma_n \quad \delta s = s_{n+1} - s_n \tag{11}$$

Initial values are taken for the point s=0 as $V_0 = 1$ and $I_0 = 0$ for $y(0) = 0$, or $V_0 = 0$ and $I_0 = 1$ for $[\partial y(s)/\partial s]_{s=0} = 0$. For infinitesimal displacements, (11) can always be approximated as

$$\begin{bmatrix} U_{n+1} \\ I_{n+1} \end{bmatrix} = \begin{pmatrix} 1 & -j\delta s\ \gamma_n^2 \\ j\delta s & 1 \end{pmatrix} \begin{bmatrix} U_n \\ I_n \end{bmatrix},$$
$$\gamma_n^{\ 2} = -\left(\varepsilon + \sigma^2(s\ )/4\right) \tag{12}$$
$$\delta s = s_{n+1} - s_n \quad and \quad s = (s_{n+1} + s_n)/2$$

The trajectory obtained this way contains the representation of the Eigen-function from the current values $[I_1, I_2, \ldots I_N]$ and the Voltage values $[V_1, V_2, \ldots V_N]$ at the chosen points of the curved wire.

From quantum mechanical properties it is known that the square of $y(s_n) = I_n$, represents the expected probability of the free electron to be placed at the point s_n. Thus for a large number of free electrons of the curved wire the squared values of the set $[I_1, I_2, \ldots I_N]$ are giving the electric charge at the set of points $[s_1, s_2, \ldots s_N]$ and the imaginary part of the set of the product values $[I_1V_1, I_2V_2, \ldots I_NV_N]$ are giving the electric current values of the free electrons inside the conducting linear wire.

The Boundary conditions for the linear wires is that the electric currents on the boundary points s=0 and s= L are zero, thus at the boundaries $y(s)\cdot[\partial y(s)/\partial s]_{s=L,R} = 0$. This means that either $[\partial y(s)/\partial s]_{s=R.L} = 0$ or $[y(s)]_{s=R,L} = 0$. We then naturally anticipate that under an external excitation there will be a tendency of (trillions) of free electrons to be present at these points proportionally to their electric current mode i.e. these are representing the Eigen-functions of the real electric currents in the linear curved wire considered that as an electromagnetic waveguide.

We performed a numerical exploration of the effects of curvature in one dimensional wire model of a curved planar wire of a circular arc of an angle ξ and length $L=1$, i.e. of constant curvature $\sigma = \xi$. Since the first harmonic with the lower energy Eigen-value appears to be the most important for any energy transfer mechanism, as well as for the maximal concentration of the free electron density, we concentrate on this case. For the first electric current harmonic we expect to have $[\partial y(s)/\partial s]_{s=0} = 0$ and $[y(s)]_{s=1} = 0$ at the terminal points (0, *1*) and a single maximum in the middle point (for the symmetric curved linear antenna).

According to the previous theory for this linear wire trap (or linear antenna) $\gamma^2 = -\left(\varepsilon + \sigma^2/4\right) = -(\varepsilon + \xi^2/4)$, thus we can use the procedures and the equation $Z_{left}(0 \to 1/2) + Z_{right}(L \to 1/2) = 0$ in order to define the Eigen-value $\varepsilon(\xi)$. For each defined value of ε we can generate the Eigen-function of the electric currents. Results of our simulations are shown in figure 2 where the Eigen-value of the first harmonic is plotted as a function of the arc ξ.

The fundamental Eigen-functions of the electric currents of linear antennae for $\xi_1=0.99\pi$, and $\xi_2=1.1\pi$, where the respective fundamental Eigen-values are $\varepsilon_1=0.049101155060641$ and $\varepsilon_2=19.221044297508335$ are shown in figures 3 and 4. We notice that for $\xi<\pi$, the fundamental Eigen-function is representing a standing wave of wave length $\lambda=2$, while for $\xi>\pi$ the standing wave has a wave length $\lambda=2/3$. If we limit $\xi<\pi$, the "best" linear antenna with $\lambda=2$ should be the one with the smaller Eigen-value. Thus a "good" linear antenna is anticipated to be the one with a small Eigen-value i.e. of arc angle $\xi<\pi$ and the "worst" is the linear antenna of a straight line where for $\xi=0$, $\varepsilon=\pi^2/4\simeq2.4674$. The relevant MATAB codes for the fundamental Eigen-values and Eigen-functions are shown in Appendix I.

4 Radiation pattern calculation

Given the electric current on a curved linear planar circular arc antenna of constant curvature $\sigma=\xi$ and length L=1, excited by a frequency f=C/(2L) ($C=3\cdot10^8$) its radiation pattern can be calculated according to [1] by the formula
$$P(\theta,\varphi) = \left(\frac{I_E(\theta,\varphi)}{\varepsilon_0\omega} + \frac{I_E(\theta,\varphi)}{\mu_0\omega}\right)\cdot\left({}^{\cos(\theta)}\big/_{\sin(\theta)}\right)^2$$
where φ,θ are respectively the angles in spherical coordinates of the radiation vector $P(\varphi,\theta)$ (i.e.

angle φ in plane (x,y) with the axis x and angle θ with the axis z as shown in fig (5).

As an example we are going to calculate numerically the radiation pattern for a curved planar linear antenna is shown in fig(5) placed on the plane z=1. This antenna is an arc of radius equal to 1.0 cm and of an angle 0.99π, thus of a length 0.99π≃3.11cm. The functions $I_E(\theta,\varphi)$ $I_M(\theta,\varphi)$ can be calculated numerically, for the given planar curved arc antenna using the analysis described in [1]. For the circular antenna of fig (5) its coordinates are given as functions of a variable angle 0<ψ< 0.99π, and *r=1 x and y are given* by the relations *x(ψ)= sin(ψ) , y(ψ) =cos(ψ),* also the following relations are true

$$I_E(\theta,\varphi) = -\int_0^L I(x,y)\cdot\exp(j\alpha x + j\beta y)\cdot(\alpha dx + \beta dy)$$

$$I_{\mathrm{M}}(\theta,\varphi) = \frac{1}{\mathrm{Z_M}}\int_0^L I(x,y)\cdot\exp(j\alpha x + j\beta y)\cdot(\alpha dx - \beta dy)$$

where

$$\alpha = \kappa\sin(\theta)\cos(\phi) \quad \beta = \kappa\sin(\theta)\sin(\phi)$$

$$I(x,y) = I_0\sin(\psi) \quad Z_M = \cos(\theta)/(j\omega\mu_0)$$

$$\omega\mu_0 = 120\pi \quad \omega\varepsilon_0 = 1/120\pi \quad \kappa = 1 \quad L = 0.99\pi$$

Using these equations the numerical MATLAB function (wirerad) calculating the function $P(\varphi,\theta)$ is given in appendix II. In figures (6) and (7) specific radiation pattern curves are shown. Similarly radiation patterns for any other planar thin antenna made by a successive set of arcs of the same curvature can be calculated by the proposed method. We should take into consideration that the connected arcs should have also derivative continuity and its angle sum should be less than π, in order to support the fundamental mode of which its wavelength is double the length of the thin arc antenna [6].

5 Conclusion

By the previous analysis it becomes evident that the curvature effect results in a kind of amplification of the free electron concentration in certain properly designed curved linear antennas made by a set of successive circular arcs of an overall angle less than π radians. Furthermore in order to achieve a very low Eigen-value the overall angle should be less but close to the angle π in radians. Thus the straight linear antenna of zero curvature has a higher Eigen-value for the fundamental mode in comparison to these arc antennas.

We anticipate that the resulting lowering of the energy Eigen-value is possible suggestive of the fact that lower external energy source can excite more easily the fundamental mode. Thus it is possible antennas made of successive circular arcs of the same curvature to replace linear antennas or antennas made of straight line parts in communication mobile and aerial applications. These kinds of very sensitive arc antennas can be very useful in 5G mobile communication networks.

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References:

[1] C. D .Papageorgiou, J. Kanellopoulos J.Physics A: Math. Gen. 15 (1982) 2569-2580

[2] C. D. Papageorgiou. T. E. Raptis, and A. C. Boucouvalas, Proceedings of INASE conference, April 9-11/2017 Athens Greece

[3] J. Stockhofe and P. Schmelcher, Physical Review A89 (033630) (2014).

[4] A. V. Zambetaki *et al.*, *Phys. Rev. E* **92**, 042905 (2015).

[5] arXiv:1504.04177v1 [quant-ph], Quantum single-particle properties in a one-dimensional curved space, J. K. Pedersen, D. V. Fedorov, A. S. Jensen, and N. T. Zinner.

[6] Christos D. Papageorgiou ,"Linear curved antennae of successive circular arcs", Patent upending , 20170100245/30/05/201.

Appendix

```
function y=wire22(e)

% eigenvalue root finder for a curved wire of %length 1, with a constant curvature s %extending in an angle cr=(angle in rad) V=0 %in one end and I=0 at the other end of the wire
```

```
global cr
N=2000;
N1=N/2;
dz=1/N;
s=cr;
z1=10^10;
for n=1:N1
cc=-s^2/4-e;
zb=j*cc*dz/2;
zp=j/dz;
z1=(z1+zb)*zp/(z1+zb+zp)+zb;
end
z2=0;
for n=1:N1;
cc=-s^2/4-e;
zb=j*cc*dz/2;
zp=j/dz;
z2=(z2+zb)*zp/(z2+zb+zp)+zb;
end
y=imag(z1+z2);
```

```
function fz=wirezero22
% calculates the eigenvalues of the curved %symmetric wire of length=1 using the function wire22
global cr fy fz
N=2000;
fz=0;
for n=1:1001;x(n)=(n-1)*2.5/100;
   y(n)=wire22(x(n));
end
L=0;
for n=1:1000; yy(n)=y(n)*y(n+1);
 if yy(n)<0 && y(n)>0;L=L+1;
      fz(L)= fzero(@wire22,[x(n),x(n+1)]);
   end
end
 fy=fz;
```

```
function y=wire32(e)
%eigenfunction of the current for a given %eigenvalue on a wire of angle cr and length 1
global cr

N=2000;
dz=1/N;
s=cr;
iv=[0;1];
f(1)=0;
xx(1)=1;
for n=1:N;
   x=N*dz-(n-1/2)*dz;
   xx(n+1)=x;
   cc=-s^2/4-e;
   A=[1 -j*cc*dz;j*dz 1];
   iv=A*iv;
   f(n+1)=iv(2)*iv(1);
```

```
end

f=imag(f);

f=(f/max(f));

plot(xx,f);grid on


function P=wirerad(q,w)

% The radius of the circular antenna r=1; the arc length is r*cr=cr

% the wavelength of the wave is 2*cr and its wavenumber k=2*pi/(2*cr)=pi/cr

% thus k*r=k=pi/cr

global cr


N=100; D=1/N; kr=pi/cr;

for n=1:N+1;a=(n-1)*cr/N;x(n)=a;y(n)=sin(w)*cos(q)*cos(a)+sin(w)*sin(q)*sin(a);z(n)=sin(w)*sin(q)*cos(a)-sin(w)*cos(q)*sin(a);

end

for n=1:N+1;a=(n-1)*cr/N;A(n)=sin(a)*exp(j*y(n)*kr)*y(n)/cos(w)*D;B(n)=sin(a)*exp(j*y(n)*kr)*z(n)*D;

end

AA=sum(A); BB=sum(B);

K1=abs(AA^2)+abs(BB^2);

P=K1*(cos(w)/sin(w))^2
```

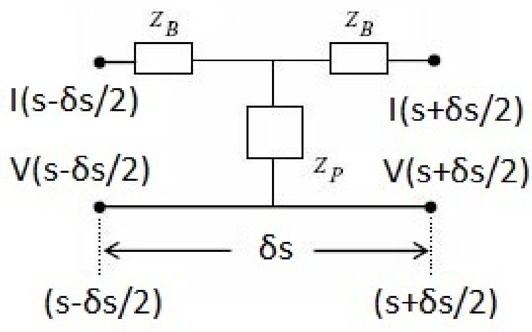


Fig. 1 Schematic diagram of the representative T-circuit.

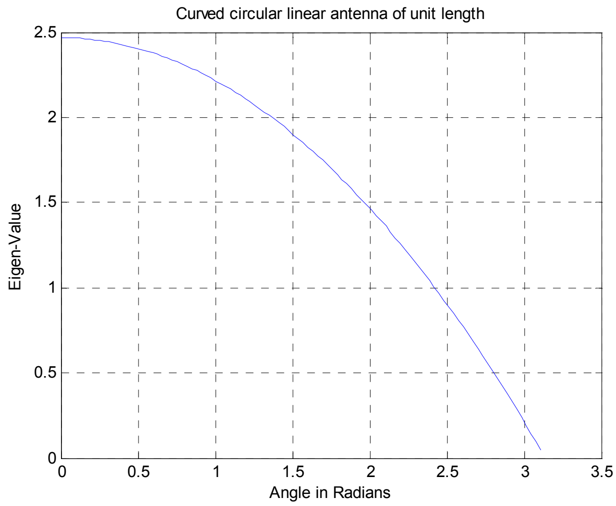


Fig.2 Fundamental Eigen-value as function of the arc angle in radians of a circular arc antenna.

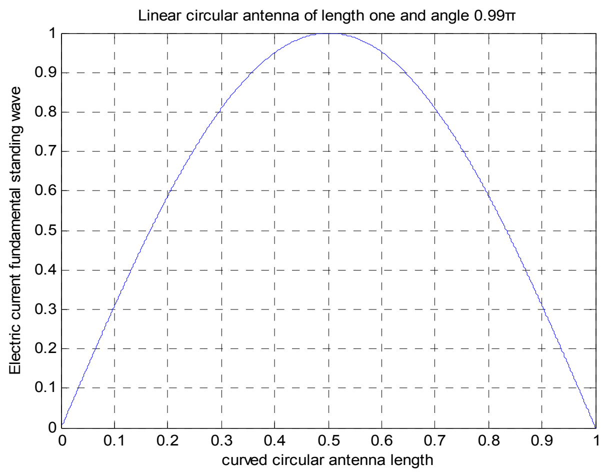
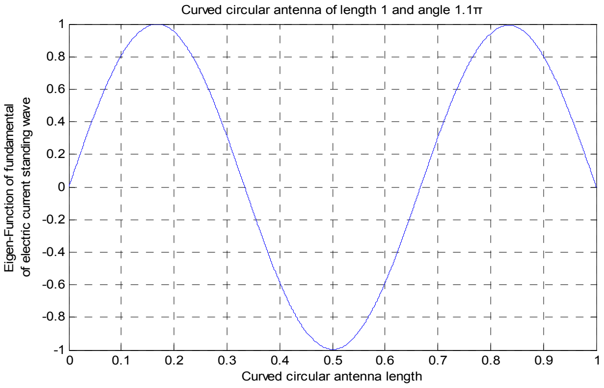


Fig 3. Fundamental standing wave on a circular arc antenna of angle 0.99π



4. Fundamental standing wave on a circular arc antenna of angle 1.1π

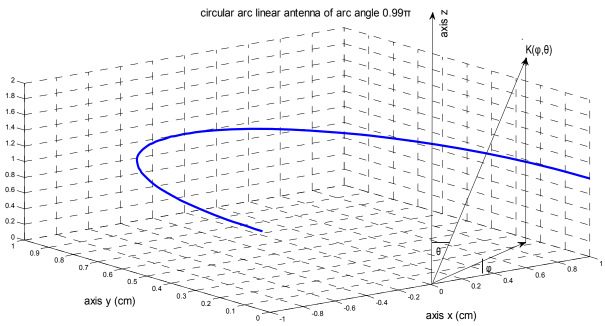


Fig 5. A linear circular arc antenna of arc angle 0.99π

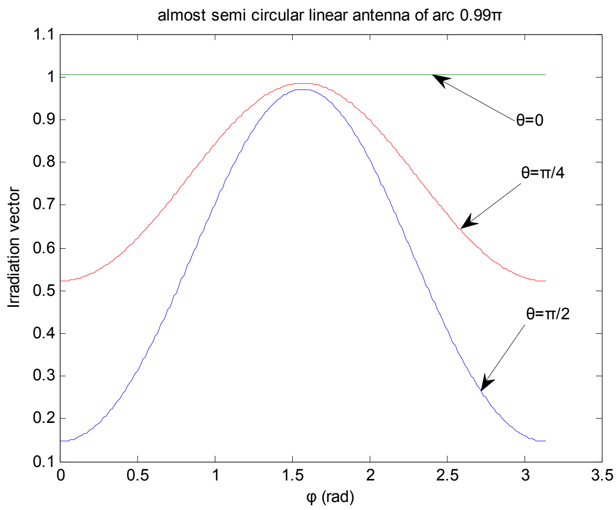


Fig 6. Radiation for variable φ and θ=π/2,π/4,0

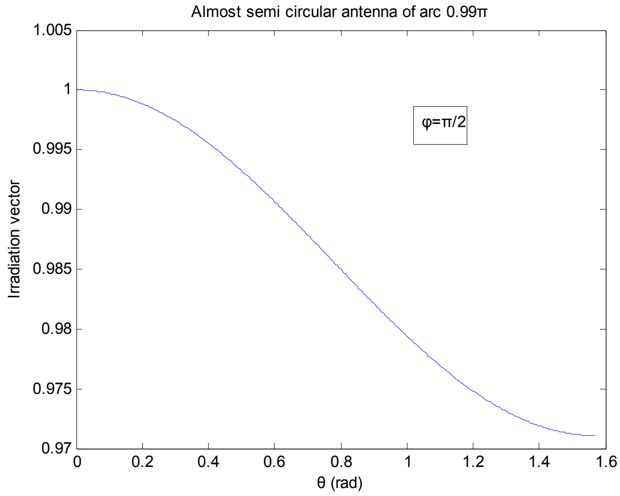


Fig 7. Radiation for variable θ and φ=π/2