# Railway System 'Vehicle-Track': Relation Between the Spectral Density of Excitation vs of Response

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*Abstract:* - The present article examines the relation between the Spectral Density of Excitation to the Spectral Density of response in general and particularly in the case of the system "Railway Vehicle-Railway Track". It begins with the general random excitation proceeds to the stationary and ergodic (random) processes and expresses mathematically the Spectral Density in these cases. Furthermore, it presents the impulsive excitations the Dirac impulse and the relationship between Excitation-Response Spectral Density and specifically for the case of the Track Defects and the Motion of the Railway Vehicle.

*Key-Words:* - Excitation, Response, Spectral, Density, Railways, Track, Vehicle, Random, Functions, stationary, ergodic.

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## 1 Introduction

Train circulation is a random dynamic phenomenon and, according to the different frequencies of the loads it imposes, the corresponding response of track superstructure appears. The dynamic component of the load of the vehicle on the track depends on the mechanical properties (stiffness, damping) of the system "vehicle-track", which acts as an excitation on the vehicle's motion (Figure 1) and vice-versa the vehicle's motion acts as an excitation on the track. The most simplified approach of this motion (vehicle on Track) is simulated by a SDOF system (Figure 2).

The dynamic component of the acting load is primarily caused by the motion of the vehicle's Non-Suspended (Unsprung) Masses, which are excited by track geometry defects, and, to a smaller degree, by the effect of the Suspended (sprung) Masses. In order to evaluate the real defects of the Track and their influence on the acting forces we use Track Recording cars whose reliability was presented recently ([1], [2]).

In order to calculate the magnitude of this dynamic component of the acting Load we use a theoretical analysis based on the Fourier Transform, approaching the phenomenon as Loads owed to forced random oscillations in systems with damping. In the following we will present this procedure. The forms of the excitations are random by nature and not deterministic.

## 2 General Random Excitation

The impulsive loading of structures is caused by forces that act over a short time interval (like e.g. earthquake). Typical examples of this are also the journey of a car along a poor quality road surface, the motion of railway vehicles on railway track, the flight of an airplane in turbulent conditions in the atmosphere, etc. Damping in these cases plays a less significant role than in the harmonic or periodical loads, where the complete response of the system consists of the sum of both a particular and a homogenous part. *The complete response does not matter, in practice, since the homogenous part is significantly affected by the damping and, therefore, the particular solution is of significant relevance in on the engineers' calculations, as long as the excitation is sufficiently far from the resonance area. During impulsive loading, the structures studied by engineers attain the maximum deflection and, consequently, the highest strain for a very short time and hence are minimally affected by damping phenomena. Damping, however, must be taken into consideration in calculations concerning longer periods of time.*

Both the excitation and the response in the system "Railway Vehicle-Railway Track" are random. If $x$ is a random variable and $g(x)$ is a function of the variable $x$, expression $y = g(x)$ is a new random vari-

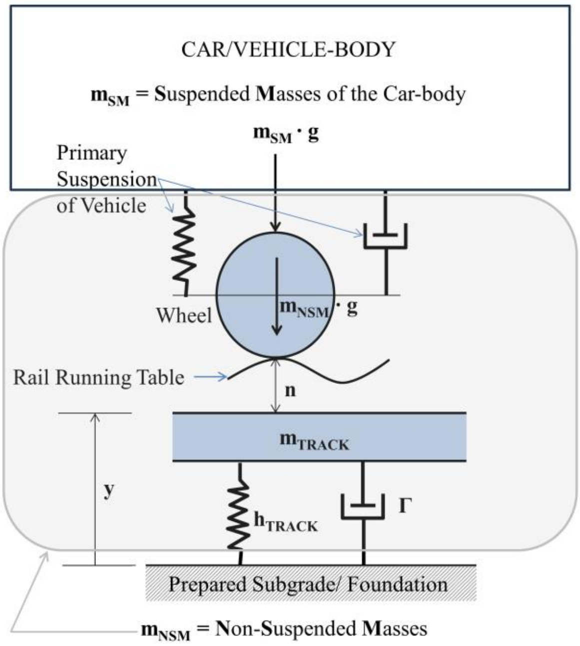


Figure 1. Schematic mapping of a vehicle/car on a Raiway Track: $m_{NSM}$ the Non-Suspended Masses (under the primary suspension) of the vehicle (the not depicted secondary suspension is between the bogie-frame and the car-body); $m_{TRACK}$ the mass of the track that participates in the motion of the Non-Suspended Masses ($m_{NSM}$); $m_{SM}$ the Suspended Masses of the vehicle/car-body (above the primary suspension); Γ damping constant of the track; $h_{TRACK}$ the total dynamic stiffness coefficient of the track; n the fault ordinate of the rail running table, and y the deflection of the track. The dynamic component is owed to the NSM and the SM.

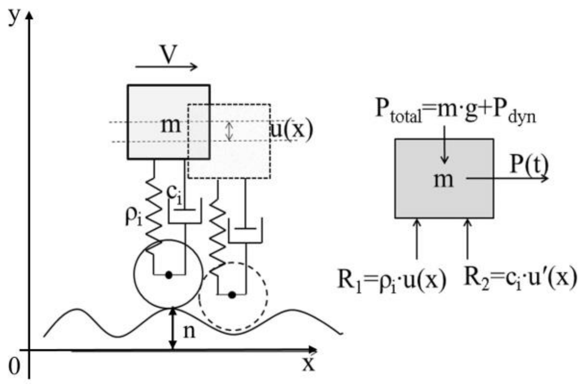


Figure 2. (Left) the motion of a vehicle on a Railway Track simplified as a **S**ingle-**D**egree-**O**f-**F**reedom (SDOF) system; (right) the acting forces on the vehicle: the static weight (m·g) plus the dynamic component $P_{dyn}$.

able defined as follows: for a given ζ, *x(ζ)* is a number, and so is *g[x(ζ)]* that is specified into terms *x(ζ)* and *g(x)*. The latter, is value *y(ζ) = g[x(ζ)]* that corresponds to random variable *y*. Thus, every function of random variable *x* is a composite function *y = g(x) = g[x(ζ)]* into domain set *f of the experimental results oor the measured ones*. Stochastic process *x(t)* is a rule that assignes value *x(t,ζ)* of the function, for every experimental or measured result *ζ*. In other words, a stochastic process is a family of temporal functions depending on the parameter *ζ*, or equivalently, a function of *t* and *ζ*. The domain of *ζ* is the set of all the experimental or measured values and the domain of *t* is the set *R* of real numbers. A stochastic process is called stationary, if its statistical properties remain invariant to a shift of the origin of time. This means that processes *x(t)* and *x(t +Δt)* have the same statistical characteristics for every *Δt* ([[3], pp. 86, 285, 297]; cf. [[6], pp. 14]).

If N series of measurements must be executed to determine if a quantity *x* is, for example, lower than a limit $x_0$ , this is statistically correct but financially disadvantageous. We should then perform tests by measuring and recording 10.000 electric locomotives or airplanes with thousands of engine drivers or pilots per unit to estimate industrial and/or functional tolerances. In the real Railway Track, the defects are random with wavelengths from few centimeters to 100 m and we should measure punctually all the kinds of defects correctly and the (practically infinite) frequencies that they impose on a Vehicle ([[4], pp. 128]; [[5], Ch.2]). As engineers, based on our experience and knowledge, we should be content with measurements, on a few units or some frequencies of the defects [[4], pp. 132], or if they have adequate structural standards and yield adequate statistical data. For this reason we presuppose the existence of measurements over a long period of time.

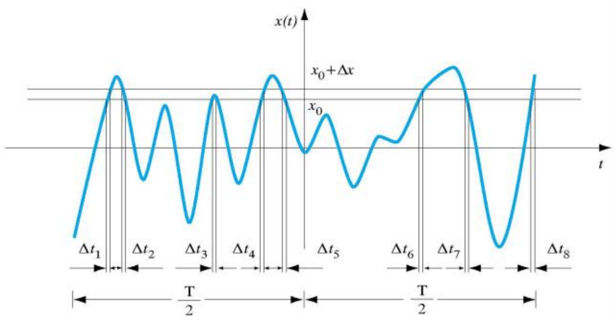


Figure 3. Calculation of the probability density in an ergodic stochastic process (cf. [[4], pp. 133]).

The statistical evaluation of these measurements advances along the time axis and not by the number of items (Figure 3). The statistical processes that allow the use of such methods are called ergodic and these are stationary in nature.

# 3 Stationary Ergodic Processes

The mathematical analysis of the system "Railway Vehicle-Railway Track" consider the system as stationary and ergodic.

We can describe a stationary process *x(t)* as mean-ergodic ([[3], pp. 428]; [[7], pp.473-8]; cf. [8]) when, after we have defined its time average value: $\overline{x}_T(t) = \frac{1}{2T}\int_{-T}^{T} x(t)\cdot dt$, which is also a random variable with mean value:

$$E\left[\overline{x}_T(t)\right] = \frac{1}{2T}\cdot\int_{-T}^{T} E\left[x(t)\right]\cdot dt = \overline{x}(t) \qquad (3.1)$$

Clough and Penzien [[7], pp.473-478] describe a sinusoidal function:

$$x_r(t) = A\cdot\sin\left(\overline{\omega}_0 t + \theta_r\right) \qquad (3.2)$$

where r = 1, 2, …., ∞ , with $x_r(t)$ the rth member of the set, A the fixed amplitude of each waveform, $\overline{\omega}_0$ fixed circular frequency and $\theta_r$ the sampled value of a random phase angle. *This process shows that it can be classified as random, since it contains many frequency components of a waveform* (a corresponding description exists in [8]), and vice-versa any (random) waveform can be approached and analyzed in an infinite series of sinusoidal waveforms.

Then if the variance $\sigma^2_T \to 0$ as $T\to\infty$ then (3.1) gives:

$$\left[\overline{x}_T(t)\right] \to \overline{x}(t) \qquad (3.3)$$

In this case, time average $\overline{x}(\zeta)$ , which is calculated from a single realization of x(t) approaches mean values $\overline{x}(t)$, with a probability that tends to the unity (1). If this is true, we can say that x(t) is mean-ergodic if its time average $\overline{x}_T(t)$ tends to the mean value $\overline{x}(t)$ as T→ ∞.

As illustrated in Figure 3, *p(x₀)·dx* specifies the probability of a measured value (or experimental) lying between $x_0$ and $x_0 + \Delta x$ and the readings being executed in all series of measurements at a given instant. In the case of *ergodic processes*, we must determine how often the measured values lie between $x_0$ and $x_0 + \Delta x$. The ratio of information in one full period of measurements, gives the probability (where T is the largest possible from a mathematical point of view):

$$p(x_0)\cdot\Delta x = \frac{1}{T}\cdot\sum_i \Delta t_i = \frac{\Delta t_1 + \Delta t_2 + \Delta t_3 + ...}{T} \qquad (3.4)$$

If we divide temporal period *T*, into small equal intervals *Δt* and consider the values *x(t)* in the temporal cycle of each *Δt*, we define *N* as the number of intervals ($T = N\cdot\Delta t$) and measure the number of intervals $N_x$, for which x(t) lies between $x_0$ and $x_0 + \Delta x$ where Δx is very small, then:

$$\overline{x} = \lim_{T\to\infty}\frac{1}{T}\cdot\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)\cdot dt,\quad \overline{x}^2 = \lim_{T\to\infty}\frac{1}{T}\cdot\int_{-\frac{T}{2}}^{+\frac{T}{2}} x^2(t)\cdot dt,$$

$$\sigma_x^2 = \lim_{T\to\infty}\frac{1}{T}\cdot\int_{-\frac{T}{2}}^{+\frac{T}{2}}\left[x(t)-\overline{x}\right]^2\cdot dt \qquad (3.5)$$

The above functions, as a rule, converge within a finite time *T*, to a value of *stationary process* and practically $T\to\infty$ is not needed ([[9], Ch.8.1]; [[7], Ch.21-2]; [[4], Ch.13-15]). In addition to the above functions, two more are very helpful in stochastic processes: *τhe auto-correlation function and the spectral density*. These are quantities that can easily be measured experimentally and are useful for the analysis of stochastic excitations in dynamic analysis ([[4], pp.438-441]; [[10], pp. 160-70, 133, 239]). The auto-correlation function for measured values at $t_1$ and $t_1 + \Delta t$:

$$\Phi_{x=}(t_1,\Delta t) = E\left[x(t_1)x(t_1+\Delta t)\right] =$$

$$= \int_{-\infty}^{+\infty} x(t_1)\cdot x(t_1+\Delta t)\cdot p(x)\cdot dx \qquad (3.6)$$

The validity of the following formula is confirmed:

$$\Phi_x(t_1,\Delta t = 0) = \overline{x}^2 \qquad (3.7)$$

For all processes with $\overline{x} = 0$, the following applies:

$$\Phi_x(t_1,\Delta t = 0) = \overline{x}^2 = \sigma_x^2 \quad \text{if} \quad \overline{x} = 0 \qquad (3.8)$$

For stationary processes, $\Phi_x$ does not depend on time $t_1$, but only on the time difference Δt, in the series of measurements. Therefore, time $t_1$, can be ignored and we can have the simplified expression:

$$\Phi_x(0) = \sigma_x^2 \quad \text{if} \quad \overline{x} = 0 \qquad (3.9)$$

and x is a stationary process.

In addition $\Phi_x(\Delta t)$ is a symmetric function for stationary processes:

$$\Phi_x(-\Delta t) = E\left[x(t_1)\cdot x(t_1-\Delta t)\right] =$$

$$= E\left[x(t_1-\Delta t)\cdot x(t_1-\Delta t+\Delta t)\right] =$$

$$= E\left[x(t_2)\cdot x(t_2+\Delta t)\right] = \Phi_x(+\Delta t) \qquad (3.10)$$

where $t_2 = t_1 - \Delta t$.

For stationary processes we can easily prove:

$\Phi_x(0) \geq |\Phi_x(\Delta t)|$, and

$$E\left[\left(x(t_1) \pm x(t_1+\Delta t)\right)^2\right] =$$
$$= \Phi_x(0) \pm 2\Phi_x(\Delta t) + \Phi_x(0) \geq 0 \qquad (3.11)$$

Since, for ergodic functions:

$$\Phi_x(\Delta t) = \lim_{T\to\infty} \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) \cdot x(t+\Delta t) \cdot dt \qquad (3.12)$$

## 4 Spectral Density in Random Processes

*The spectral density function* $S_x$ *is of great importance to the analysis of stochastic processes*. Figure 4 illustrates characteristic cases of spectral density of random functions.

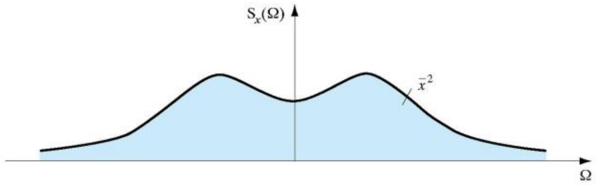


Figure 4. Spectral Density or Power Spectrum $S_x(\Omega)$ and mean square value (average value $\bar{x}^2$) (cf. [[4], pp. 138]).

In the theory of stochastic systems, the spectra are linked to Fourier transforms. For deterministic systems the spectra and the Fourier transformation are used to represent a function as superposition of exponential functions. For random systems (or signals) the concept of spectrum has two interpretations.

a.-The first one includes transforms of averages, and is essentially deterministic.

b.-*The second one* *__includes the representation of the (random) process as a superposition of exponential functions (namely of a sum of infinite sine and cosine functions) with random coefficients__*.

The *power spectrum* or *spectral density* of a stochastic system that is described by a function *x(t)* is the Fourier transform of the system's auto-correlation $\Phi_x(t)$ 12.

If *x(t)* represents the excitation and since a stochastic process, at least theoretically, can last indefinitely, it is not a prerequisite that the following equation will apply: $\int_{-\infty}^{+\infty} |x(t)| \cdot dt < +\infty \qquad (4.1)$
even if the mean value $x = 0$.

There is greater possibility that $\Phi_x(\Delta t)$ will be finite, that is the absolute value of $\Phi_x(\Delta t)$, which is the area below the curve (Figure 5). From the continuity of curve *x(t)*, and for small $\Delta t$, it can be concluded that both *x(t)* and $x(t+\Delta t)$ have the same sign and therefore $\Phi_x(\Delta t)$ must increase with time. There is no predictable behaviour or relationship in a random process between *x(t)* and $x(t+\Delta t)$ for great values of $\Delta t$. $\Phi_x(\Delta t) \to 0$ for great values of $\Delta t$, because contradictive values may arise in this case.

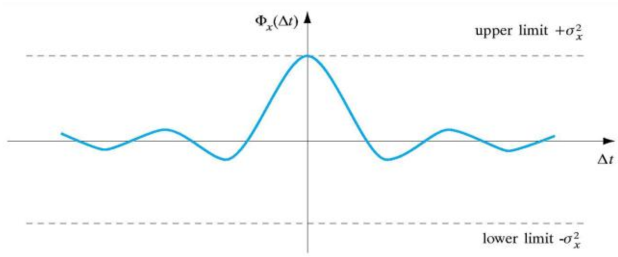


Figure 5. Typical auto-correlation function for stationary process with zero mean value (cf. [[4], pp. 136]).

Let us consider a non-periodic auto-correlation function $\Phi_x(\Delta t)$ that satisfies the equation (4.1). When this equation is valid it allows unlimited use of the Fourier integral (with no restrictions) for the mapping of any function *x(t)*. To calculate this function we can apply the integral for the calculation of the coefficients of the Fourier series [[4], pp. 121, Eqn. 393],

$$f_i(\omega) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} P_i(t) \cdot e^{-i\cdot\omega\cdot t} \cdot dt \qquad (4.2)$$

which is the frequency spectrum of the excitation for non-random processes ([[4], 121-122], [[4], pp. 319]) and we get:

$$\Phi_x(\Delta t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_x(\omega) \cdot e^{i\omega\Delta t} d\omega = E\left[x(t+r) \cdot x(t)\right]$$

where:

$$S_x(\omega) = \int_{-\infty}^{+\infty} \Phi_x(\Delta t) \cdot e^{-i\omega\Delta t} \cdot d(\Delta t) \qquad \textbf{(4.3.a-b)}$$

$S_x(\omega)$ is the spectral density function. Furthermore, $S_x(\omega)$ is the Fourier transform of the function $\Phi_x(\Delta t)$ since $\Phi_x(\Delta t)$ is the inverse Fourier transform of $S_x(\omega)$. It should be noted that, if $\Phi_x(\Delta t)$ is defined as $\Phi_x(\Delta t) = E[x(t_1) \cdot x(t_1+\Delta t)]$, then the factor $1/2\pi$ before the integral does not exist in the Eqn. (4.3.a) but in (4.3.b) (see [[9], pp. 269]).

The WIENER-KHINCHIN theorem [[4], pp. 237] is derived from (3.9) and (4.3.a):

$$\sigma^2(x) = \Phi_x(0) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S_x(\omega) \cdot d\omega = \bar{x}^2 \qquad (4.4)$$

*The shaded area below the spectral density function* (Figure 4) *represents the mean square value of the process*.

Using complex numbers in (4.3.b) and the Euler equation for complex numbers, since the imaginary part is eliminated, because $\Phi_x(\Delta t)$ is symmetrical and $sin(\omega\Delta t)$ is anti-symmetrical with respect to $\Delta t = 0$ and as a result the areas below the anti-symmetrical integral cancel one another out [[9], pp. 271], we get:

$$S_x(\omega) = \int_{-\infty}^{+\infty} \Phi_x(\Delta t)\cdot\cos(\omega\Delta t)\cdot d(\Delta t) -$$

$$-i\cdot\int_{-\infty}^{+\infty} \Phi_x(\Delta t)\cdot\sin(\omega\Delta t)\cdot d(\Delta t) =$$

$$= \frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty} \Phi_x(\Delta t)\cdot\cos(\omega\Delta t)\cdot d(\Delta t) \qquad (4.5)$$

It can be easily proved that:

$$S_x(\omega) = S_x(-\omega) \qquad (4.6)$$

*The spectral density function is real, it does not contain an imaginary part and is symmetrical around position $\omega = 0$, as the autocorrelation function also is*.

# 5 Impulsive Excitation Functions

## 5.1 Impulsive Excitations of General Form

In random phenomena it is not the specific values that a function f(t) takes at a point i, at instant $t_1$ that we are looking for, but, on the contrary, the probability that a function f(t) will be greater than a value $k_0$ at a point i, at instant $t_1$ , where f(t) can be the excitation (acting force) $P_i(t)$ or the response (deplacement) $u_i(t)$ or vice-versa the random waveform of the rail (deflection plus the anomalies of the rail running table) can be the excitation and the acting force the response. A typical result of a stochastic excitation would be, for instance, that $u_i(t)$ (or $P_i(t)$ relevantly) has a 90% probability to be greater than e.g. 0.30.

As it was already mentioned in paragraph 2 above, *during impulsive loading, the structures studied by engineers attain the maximum deflection and, consequently, the highest strain for a very short time and hence are minimally affected by damping phenomena. Damping, however, must be taken into consideration in calculations concerning longer periods of time*. In a Railway Track, with eigenfrequency of 50-75 Hz, the structure does not deform due to the much higher frequency of the running load.

An example of non-periodic excitation is the rectangular impulse of finite duration (Figure 6), which is described by the following equation (5.1):
$P_i = P_0$, for $0 \leq t \leq t_1$ and
$P_i=0$ for $-\infty \leq t < 0$ and $t_1 < t \leq +\infty$ (5.1)

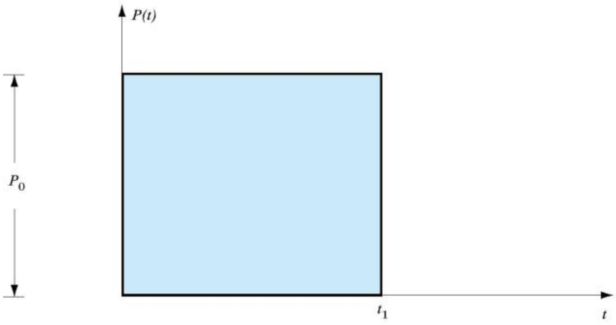


Figure 6. A rectangular impulse of finite duration; an example of non-periodic excitation

The complex frequency spectrum is ([[4], pp. 121, Eqns. 3.9.3, 3.8.3a, 3.9.7]; [[4], pp. 319]):

$$\frac{1}{2\pi}\cdot\int_0^{t_1} P_0\cdot e^{-i\omega t}\cdot dt = \frac{P_0}{2\pi i\omega}\cdot\left(1-e^{i\omega t}\right) = f_i^c - i\cdot f_i^s \quad (5.2)$$

where:

$$f_i^c = \frac{P_0 t_1}{2\pi}\cdot\frac{\sin\omega\, t_1}{\omega\, t_1} \quad \text{and} \quad f_i^s = \frac{P_0 t_1}{2\pi}\cdot\frac{1-\cos\omega\, t_1}{\omega\, t_1} \quad (5.3)$$

$$u_p(t) = \int_{-\infty}^{+\infty} \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\cdot\frac{P_0}{2\pi\cdot\omega i}\cdot\left(1-e^{-i\cdot\omega\cdot t_1}\right)\cdot e^{i\cdot\omega\cdot t}\cdot d\omega = \qquad (5.3.a)$$

$$= \frac{P_0\omega_n^2}{2\pi\cdot i}\int_{-\infty}^{+\infty}\frac{1}{\omega(\omega_n-\omega)\cdot(\omega_n+\omega)}\left(e^{i\cdot\omega\cdot t}-e^{-i\omega(t-t_1)}\right)d\omega$$

By developing the integral factor

$$\frac{1}{\omega(\omega_n-\omega)\cdot(\omega_n+\omega)}\left(e^{i\cdot\omega\cdot t}-e^{-i\omega(t-t_1)}\right)d\omega$$

where we have an integral of the form [[11], pp. 181]:

$$\int_{-\infty}^{+\infty}\frac{e^{iax}}{x}dx = \int_{-\infty}^{+\infty}\left(\frac{\cos ax}{x}+i\frac{\sin ax}{x}\right)dx = 2i\int_{-\infty}^{+\infty}\frac{\sin ax}{x}\cdot dx =$$

$$= \begin{Bmatrix} i\pi & \text{if } a>0 \\ -i\pi & \text{if } a<0 \end{Bmatrix} = \text{sign}(a)\, i\pi \qquad (5.3.1)$$

The solution of the integral $sin(\alpha x)\cdot dx/x$ is calculated in the paragraph (5.1) below.

## 5.2 Integral around a Point of Discontinuity

We have to solve an integral of the form:

$\int \frac{\sin ax}{x}$ with a point of discontinuity at the point $x = 0$; we consider the function: $f(z) = \frac{e^{iz}}{z}$.

Firstly, as an integration line of the function, we consider the contour $C$ of Figure 7, with the path according to the direction of the arrows. From the interior of the contour the point $z = 0$, which is the pole (point where the function is not continuous) of the function $f(z)$ is excluded. There are two half circumferences with radii $\rho$ and $R$. From the Cauchy integral ([[11], pp. 106-27]; [[12], pp.118]) we derive:

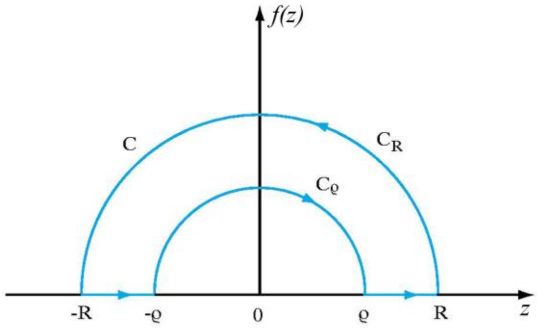


Figure 7. Contour C around the pole ($z=0$) [point of discontinuity] of the function $f(z)$. The integral is calculated for $\rho \to 0$ (cf. [[4], Ch.10, Annex]).

$$\int_C f(z)dz = 0 \Rightarrow$$

$$\Rightarrow \int_{-R}^{-\rho} \frac{e^{iz}}{z}dz + \int_{C_\rho} \frac{e^{iz}}{z}dz + \int_{\rho}^{R} \frac{e^{iz}}{z}dz + \int_{C_R} \frac{e^{iz}}{z}dz = 0 \quad (5.4)$$

If $\rho \to 0$ and $R \to \infty$, then Eqn (5.4) constitutes an integral from $-\infty$ to $+\infty$ with the exception of the point 0. The complex number $z$ has the form:

$$z = r \cdot e^{i\cdot\theta} \Rightarrow \left|\frac{1}{z}\right| = \left|\frac{1}{r} \cdot e^{-i\cdot\theta}\right| = \frac{1}{r} \quad (5.5)$$

Jordan's lemma [[11], pp. 177] states that if a function $f(z)$ is continuous everywhere at the upper side of the complex plane and if $|f(\zeta)| \leq M/\rho^k$ where $\zeta = \rho \cdot e^{i\cdot\theta}$, with $k > 0$ and $M$ constant, then:

$$\lim_{\rho \to \infty} \int_{C_\rho} e^{i\cdot m\cdot\zeta} \cdot f(\zeta) \cdot d\zeta = 0 \quad (5.6)$$

where $C_\rho$ is the semicircle $(0, \rho)$ located at the half plane that was defined above, and $m$ is a positive constant. Hence:

$$\lim_{R \to \infty} \int_{C_R} \frac{e^{i\cdot z}}{z} \cdot dz = 0 \quad (5.7)$$

We have to calculate Eqn (5.6) and for this we develop the function into a Laurent series at the point $z = 0$.

$$\frac{e^{i\cdot z'}}{z'} = \frac{1}{z'} \cdot \left(1 + i \cdot z' + \frac{(i \cdot z')^2}{2!} + \frac{(i \cdot z')^3}{3!} + \ldots\right) =$$

$$= \frac{1}{z'} + P(z') \quad (5.8)$$

Where $P(z')$ is a polynomial, called the normal part of the power series, with $z' = 1/(z-0) = 1/z$ and $P(z') \to 0$. Hence (see Eqn. 5.5):

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{e^{i\cdot z}}{z} \cdot dz = \lim_{\rho \to 0} \int_{C_\rho} \frac{dz}{z} + \lim_{\rho \to 0} \int_{C_\rho} P(z) \cdot dz =$$

$$= \lim_{\rho \to 0} \int_{C_\rho} \frac{dz}{z} = \int_{\pi}^{0} \frac{i \cdot r \cdot e^{i\cdot\theta} \cdot d\theta}{r \cdot e^{i\cdot\theta}} = -\pi \cdot i \quad (5.9)$$

Thus, Eqn (5.4) becomes (for $\rho \to 0$ and $R \to \infty$):

$$\int_{-\infty}^{0} \frac{e^{iz}}{z} \cdot dz - \pi \cdot i + \int_{0}^{+\infty} \frac{e^{iz}}{z} \cdot dz + 0 = 0 \Rightarrow$$

$$\Rightarrow 2 \cdot \int_{0}^{+\infty} \frac{e^{i\cdot z}}{z} \cdot dz = \pi \cdot i \quad (5.10)$$

and, after equalizing the imaginary parts:

$$2 \cdot \int_{0}^{\infty} \frac{\sin z}{z} \cdot dz = \pi \Rightarrow \int_{0}^{\infty} \frac{\sin z}{z} = \frac{\pi}{2} \quad (5.11)$$

Similarly we can prove that:

$$\int \frac{\sin \lambda z}{z} = \frac{\pi}{2} \quad \text{with} \quad \lambda > 0 \quad (5.12)$$

therefore,

$$\int_{-\infty}^{+\infty} \frac{e^{i\cdot z}}{z} \cdot dz = 2 \cdot i \cdot \int_{0}^{\infty} \frac{\sin z}{z} \cdot dz = \pi \cdot i \quad \text{and,}$$

$$\int_{-\infty}^{+\infty} \frac{e^{i\cdot\alpha\cdot z}}{z} \cdot dz = 2 \cdot i \cdot \int_{0}^{\infty} \frac{\sin \alpha z}{z} \cdot dz = \pi \cdot i \quad \text{with} \quad \alpha > 0 \quad (5.13)$$

*Thus we have just calculated one tern of the Eqn. (5.3.a).*

## 5.3 Calculating the Conditions at the Limit

In Eqn. (5.3.a) we have to calculate the second integral:

$$A = \frac{P_0\omega_n^2}{2\pi i} \cdot \int_{-\infty}^{+\infty} \frac{1}{2\omega_n^2} \cdot \left(\frac{2}{\omega} + \frac{1}{\omega_n - \omega} - \frac{1}{\omega_n + \omega}\right) \cdot$$

$$\cdot \left(e^{i\omega t} - e^{-i\omega(t-t_1)}\right) \cdot d\omega \quad (5.14)$$

We name a new variable $\omega' = \omega + \omega_n \Rightarrow \omega = \omega' - \omega_n$. The integral:

$$\int \frac{e^{i\omega t}}{\omega + \omega_n} \cdot d\omega = \int \frac{e^{i(\omega' - \omega_n)t}}{\omega'} \cdot d\omega' = \int \frac{e^{i\omega' t} \cdot e^{-i\omega_n t}}{\omega'} \cdot d\omega' =$$

$$= e^{-i\omega_n t}\int \frac{e^{i\omega' t}}{\omega'}\cdot d\omega' = \pi\cdot i\cdot e^{-i\omega_n t} \qquad (5.15)$$

Respectively, setting $\omega' = \omega + \omega_n$:

$$\int \frac{e^{i\omega t}}{\omega_n-\omega}\cdot d\omega = -\int \frac{e^{i(\omega'+\omega_n)t}}{\omega'}\cdot d\omega' == \int \frac{e^{i\omega' t}\cdot e^{i\omega_n t}}{\omega'}\cdot d\omega' =$$

$$= -e^{i\omega_n t}\int \frac{e^{i\omega' t}}{\omega'}\cdot d\omega' = -\pi\cdot i\cdot e^{i\omega_n t} \qquad (5.16)$$

and also:

$$\int \frac{2e^{i\omega t}}{\omega}\cdot d\omega = 2i\pi \qquad (5.17)$$

Hence:

$t \le 0 \Rightarrow t-t_1 < 0 \Rightarrow sign(t) = sign(t-t_1) = -1$ 1st case.

$$A = \frac{P_0\omega_n^2}{2\pi i}\cdot\frac{1}{2\omega_n^2}\cdot\pi i\cdot[-2+e^{i\omega_n t}+e^{-i\omega_n t}-(-2+$$

$$+e^{i\omega_n(t-t_1)}+e^{-i\omega_n(t-t_1)})] =$$

$$= \frac{P_0}{4}\cdot\left[e^{i\omega_n t}+e^{-i\omega_n t}-e^{i\omega_n(t-t_1)}-e^{-i\omega_n(t-t_1)})\right] =$$

$$= \frac{P_0}{4}\cdot 2\cdot\left[\cos\omega_n t-\cos\omega_n\left(t-t_1\right)\right] =$$

$$= \frac{P_0}{2}\cdot\left[\cos\omega_n t-\cos\omega_n\left(t-t_1\right)\right] \qquad (5.18.a)$$

Respectively, taking the signs of t and $t-t_1$ for the other two cases ($0 \le t \le t_1$ and $t_1 < t$) we arrive at the following two equations **(5.18b)** and **(5.18c)** for $u_p(t)$:

$$0 \le t \le t_1 : u_p(t) = \frac{1}{2}\cdot P_0\cdot\left[2-\cos\omega_n t-\cos\omega_n\left(t-t_1\right)\right]$$

$$t_1 \le t \le +\infty : u_p(t) = \frac{1}{2}\cdot P_0\cdot\left[-\cos\omega_n t+\cos\omega_n\left(t-t_1\right)\right]$$

Consequently the second integral of the Eqn. (5.3a) becomes:

$$\int_{-\infty}^{+\infty}\left(\frac{2}{\omega}+\frac{1}{\omega_n-\omega}-\frac{1}{\omega_n+\omega}\right)\cdot e^{i\cdot\omega\cdot(t-t_1)}\cdot d\omega =$$

$$= sign(t-t_1)\cdot i\cdot\pi\cdot\left(2-e^{i\cdot\omega_n\cdot(t-t_1)}-e^{-i\cdot\omega_n\cdot(t-t_1)}\right) \qquad (5.19)$$

## 5.4 Particular and Complete Response

The particular solution for the response is (**5.20.a**):

$$-\infty < t < 0 : u_p(t) = \frac{1}{2}\cdot P_0\cdot\left[\cos\omega_n t-\cos\omega_n\left(t-t_1\right)\right]$$

$$0 \le t < t_1 : u_p(t) = \frac{1}{2}\cdot P_0\cdot\left[2-\cos\omega_n t-\cos\omega_n\left(t-t_1\right)\right]$$

$$t_1 < t < +\infty : u_p(t) = \frac{1}{2}\cdot P_0\cdot\left[-\cos\omega_n t+\cos\omega_n\left(t-t_1\right)\right]$$

The engineer's sense and inspiration lead to a particular solution of the form:

$$u_p(t) = 0 \quad \text{for} \quad t < 0$$
$$u_p(t) = P_0 \quad \text{for} \quad 0 \le t \le t_1 \qquad (5.20.b)$$
$$u_p(t) = 0 \quad \text{for} \quad t > t_1$$

with points of discontinuity at instances $t=0$ and $t=t_1$.

This particular solution does not contain a homogenous part. Therefore, homogenous terms should be added (*sines*, *cosines*) to cover the whole range of real numbers ($-\infty$, $+\infty$). The complete response of the system can also be obtained with the assumption of initial conditions $u(t) = u'(t) = 0$.

$$u(t) = \frac{1}{2}\cdot P_0\cdot\left[\cos\omega_n t-\cos\omega_n\left(t-t_1\right)\right]+$$

$$+a\cdot\sin\omega_n t+b\cdot\cos\omega_n t \qquad (5.21.a)$$

Hence, at the upper limit $t = 0$:

$$a\cdot\sin\omega_n t+b\cdot\cos\omega_n t =$$
$$= \frac{1}{2}\cdot P_0\cdot\left[\cos\omega_n t-\cos\omega_n\left(t-t_1\right)\right] \qquad (5.21.b)$$

Applying Eqn. (5.21.b) to the Eqn. (5.18.a) for the case $t = 0$, when $0 \le t \le t_1$:

$$u(t) = \frac{1}{2}\cdot P_0\cdot\left[2-\cos\omega_n t-\cos\omega_n\left(t-t_1\right)\right]+$$

$$+a\cdot\sin\omega_n t+b\cdot\cos\omega_n t =$$

$$= \frac{1}{2}\cdot P_0\cdot[2-\cos\omega_n t-\cos\omega_n\left(t-t_1\right)-$$

$$-\cos\omega_n t+\cos\omega_n\left(t-t_1\right)] \Rightarrow$$

$\Rightarrow u(t) = P_0\cdot\left[1-\cos\omega_n t\right]$, when $0 \le t \le t_1$ (5.22).

When $t_1 \le t \le +\infty$, the complete solution is:

$$u(t) = \frac{1}{2}\cdot P_0\cdot\left[-\cos\omega_n t+\cos\omega_n\left(t-t_1\right)\right]+$$

$$+a\cdot\sin\omega_n t+b\cdot\cos\omega_n t \qquad (5.23).$$

Its first derivative:

$$u'(t) = \frac{1}{2}\cdot P_0\cdot\left[\omega_n\cdot\sin\omega_n t-\omega_n\sin\omega_n\left(t-t_1\right)\right]+$$

$$+a\cdot\omega_n\cdot\cos\omega_n t-b\cdot\omega_n\cdot\sin\omega_n t \qquad (5.24)$$

From Eqn. (5.22) the conditions at the limit for $t = t_1$ give:

$$u(t_1) = P_0\cdot\left[1-\cos\omega_n t_1\right],$$
$$u'(t_1) = P_0\cdot\omega_n\cdot\sin\omega_n t_1 \qquad (5.25)$$

For $t = t_1$ Eqns (5.23) and (5.24) have as values from (5.25):

$$(5.23) \Rightarrow P_0\cdot\left[1-\cos\omega_n t\right] = \frac{1}{2}\cdot\left[-\cos\omega_n t+1\right]+$$

$$+a\cdot\sin\omega_n t_1+b\cdot\cos\omega_n t_1=\frac{1}{2}\cdot P_0\cdot\left[1-\cos\omega_n t\right] \quad (5.26a)$$

$$(5.24)\Rightarrow P_0\cdot\omega_n\cdot\sin\omega_n t_1=\frac{1}{2}\cdot P_0\cdot\left[\omega_n\cdot\sin\omega_n t_1-0\right]+$$

$$+a\cdot\omega_n\cdot\cos\omega_n t_1-b\cdot\omega_n\cdot\sin\omega_n t_1\Rightarrow$$

$$\Rightarrow a\cdot\cos\omega_n t_1-b\cdot\sin\omega_n t_1=\frac{1}{2}\cdot P_0\cdot\sin\omega_n t_1 \quad (5.26b)$$

Solving the system of the two Eqns (5.26a) and (5.26b) we find:

$$a=\frac{1}{2}\cdot P_0\cdot\sin\omega_n t_1,\qquad b=\frac{1}{2}\cdot P_0\cdot(\cos\omega_n t_1-1) \quad (5.26c)$$

Hence, the variable part of Eqn. (5.23) for $t = t_1$ becomes:

$$a\cdot\sin\omega_n t_1+b\cdot\cos\omega_n t_1=\frac{1}{2}\cdot P_0\cdot[\sin^2\omega_n t_1+$$

$$+\cos^2\omega_n t_1-\cos\omega_n t_1]=\frac{1}{2}\cdot P_0\cdot\left[1-\cos\omega_n t_1\right] \quad (5.26d)$$

Therefore, the variable part is equal to the steady state (since for $t = t_1$ then $\cos\omega_n(t-t_1) = 1$) (**5.26e**):

$$(5.23)\Rightarrow u(t)=\frac{1}{2}\cdot P_0\cdot\left[\cos\omega_n(t-t_1)-\cos\omega_n t\right]\cdot 2\Rightarrow$$

$$\Rightarrow u(t)=P_0\cdot\left[\cos\omega_n(t-t_1)-\cos\omega_n t\right] \text{ for } t_1\le t\le+\infty$$

(a) We get $u(t) = 0$ for $t \le 0$.

(b) If we introduce these values into the complete solution [Eqn. 5.23] in the time interval $0\le t\le t_1$, we get:

$$u(t)=P_0\cdot(1-\cos\omega_n t)$$

which has the particular solution at $t_1$:

$$u(t_1)=P_0\cdot(1-\cos\omega_n t_1),\quad u'(t_1)=P_0\cdot\omega_n\cdot\sin\omega_n t_1$$

(c ) These values provide the initial conditions for the third time interval $t \ge t_1$, for which we get:

$$u(t)=P_0\cdot\left[\cos\omega_n(t-t_1)-\cos\omega_n t\right].$$

The velocity is the first derivative:

$$u'(t)=P_0\cdot\omega_n\cdot\sin\omega_n t.$$

Its gradient is $P_0\cdot\omega_n^2$ for $t = 0$; therefore, *the velocity is undergoing a significant change as a result of one rectangular impulse of very short duration*.

## 5.5 Strength of Impulse – Duhamel Integral

We define in general form:

$$I_i=\int_0^{t_1}P_i(t)\cdot dt \quad (5.27.a)$$

The *Strength (Power) of the excitation Impulse* and the *Strength of a Rectangular excitation Impulse*:

$$I_i = P_0\cdot t_1 \quad (5.27.b)$$

Integration of the differential equation without taking into account the damping and under the condition that the phenomenon takes place within a very short time interval (for $k = 1$, $m = 1/\omega_n^2$):

$$\frac{1}{\omega_n^2}u''+u=P_0,\quad 0\le t\le t_1,\Rightarrow u''+\omega_n^2\cdot u=\omega_n^2\cdot P_0$$

this leads to the following (for $t = t_1$):

$$\Delta u'=\omega_n^2\cdot\left(P_0\cdot t_1+\int_0^{t_1}u\cdot dt\right)\cong\omega_n^2\cdot I_i \quad (5.28)$$

assuming a system initially at rest. Whereas the power of the impulse, even for a short time duration, can attain a significant value owing to a great amplitude of Force $P_0$, the area below the curve $u(t)$ is practically zero, as a result of the zeroing of the gradient ($t \to 0$) at the beginning of the curve (for this reason $\int u\cdot dt\cong 0$ ).

At the limit $I_i=\lim_{t\to 0}(P_0\cdot t_1)=k$ (5.29)

where $k$ is a finite number.

In this case the applied Force is converted to the *Dirac Impulse*. The Dirac function is defined as P(x) = 0 for x ≠ 0 and P(0) = +∞ since

$$\int_{-\infty}^{+\infty}P(x)\cdot dx=1$$

Physicists (and Engineers) are aware that this is not a real function, but rather a symbolic tool. For Electronic Engineers the *Dirac function* is often considered to be the limit of an Impulse, of amplitude $\Delta t$ and magnitude $1/\Delta t$ when $\Delta t \to 0$. For more, see ([10]; [8]). Different impulses of short duration with a random form can also be approximately defined by the Eqns (5.27.a and b). Since the strength of impulse $I_i$ is known, the exact fluctuation of $P(t)$ is not of interest from a practical point of view.

An important characteristic of the above theoretical analysis is that the excitation is obtained from Eqn. (5.28). We know that, in every case, the maximum displacement occurs at the phase of free oscillation. By applying the initial conditions:

$$u(t_1)=0 \quad\text{and}\quad u'(t_1)=\Delta u'=\omega_n^2\cdot I_i \quad (5.30)$$

The response of the system can be expressed by:

$$u(t)=\omega_n\cdot I_i\cdot\sin\omega_n(t-t_1) \quad\text{for}\quad t>t_1 \quad (5.31)$$

For a *Dirac Impulse* ($t_1 \to 0$) it is further simplified:

$$u(t)=\omega_n\cdot I_i\cdot\sin\omega_n t \quad\text{for}\quad t>0 \quad (5.32)$$

Eqn. (5.32) is analytically precise, whereas Eqn. (5.31) is merely an approximation for impulses that are of finite, but short time-duration $t_1$. We can consider $t_1 < T_i/4$ as a limit.

$$I_i = P_0 \cdot t_1 = \frac{1}{4} \cdot P_0 \cdot T_i = \frac{\pi}{2} \cdot \frac{P_0}{\omega_n} \qquad (5.33)$$

Using Eqn. (5.33) the Eqn. (5.31) gives:

$$u(t) = \frac{\pi}{2} \cdot P_0 \cdot \sin\omega_n (t - t_1) \quad \text{when} \quad t > t_1 \quad (5.34)$$

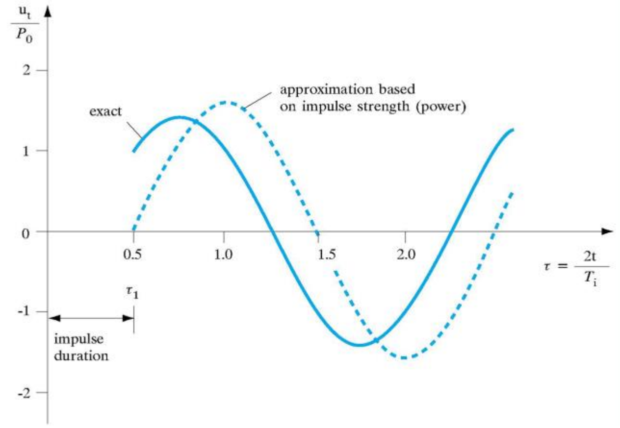


Figure 8. Response to a long duration excitation (cf. [[4], pp.145]).

In Figure 8, we notice, by comparing the precise and the approximate solution –based on the power of the excitation impulse–, a significant deviation for the phase shift, whereas *the maximum amplitude which is of major importance for application to structures*, does not seem to be affected significantly (its deviation does not exceed 11%); see also [[9], pp. 238].

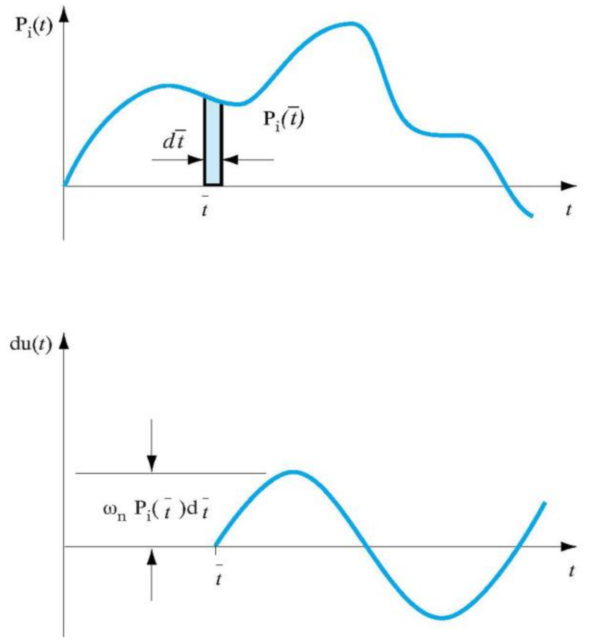

Figure 9. Excitation of generic form, namely, an impulse of finite time-duration (upper illustration) and response (lower illustration); cf. [[4], pp. 146].

Figure 9 illustrates a generic form excitation that can be approached by an infinite sequence of impulses. At time instant $\bar{t}$, an instant infinitesimal impulse is recorded:

$$dI_i = P_i(\bar{t}) \cdot dt \qquad (5.34)$$

that causes the response **(5.35)**:

$$du(t) = \omega_n \cdot dI_i \cdot \sin\omega_n (t - \bar{t}) =$$
$$= \omega_n \cdot P_i(\bar{t}) \cdot \sin\omega_n (t - \bar{t}) \cdot dt \quad \text{where} \quad t > \bar{t}$$

Starting from an *at-rest condition*, by integration of the sequence of all impulses, we get the complete response of the system:

$$u(t) = \omega_n \cdot \int_{\bar{t}=0}^{\bar{t}=t} P_i(\bar{t}) \cdot \sin\omega_n (t - \bar{t}) \cdot d\bar{t} \quad \mathbf{(5.35.a)},$$

which is known as ***the Duhamel integral***, and if we pose:

$$h(t - \bar{t}) = \omega_n \cdot \sin\omega_n (t - \bar{t}) \qquad (5.35.b),$$

it is transformed to:

$$u(t) = \int_{\bar{t}=0}^{\bar{t}=t} P_i(\bar{t}) \cdot h(t - \bar{t}) \cdot d\bar{t} \qquad (5.35.c)$$

If the structure does not start from an at-rest condition at *t = 0*, it is necessary to superimpose an additional free oscillation:

$$u(t) = \omega_n \cdot \int_{\bar{t}=0}^{\bar{t}=t} P_i(\bar{t}) \cdot \sin\omega_n (t - \bar{t}) \cdot d\bar{t} + \frac{u'(0)}{\omega_n} \cdot \sin\omega_n t +$$
$$+ u(0) \cdot \cos\omega_n t \qquad (5.36)$$

Numerical calculation methods, such as the Romberg, Gauss, Simpson, etc, have been developed for the calculation of Duhamel integral.

In any case we can assume that the phenomenon starts at *t = 0*.

## 6 Random excitation in a system with damping, with one degree of freedom

From the *Duhamel integral* Eqns (5.35.c) we find the response in the time domain. The basic second order differential equation of motin which is applied to the system "Railway Vehicle-Railway Track" (see [14], [15]) is:

$$u'' + 2\zeta_i \cdot \omega_n \cdot u' + \omega_n^2 \cdot u = \omega_n^2 \cdot P_i(t) \qquad (6.1)$$

Where $P_i(t)$ is presumed to be an ergodic random force with a mean value equal to zero. The mathematical expression of the term $P_i(t)$ is [[4], pp. 147]:

$$P_i(t) = \frac{P_0}{k} \cdot e^{i\omega t} \qquad (6.2)$$

Using Eqn. (5.35.c) of the *Duhamel integral* we can calculate the response of the system. The Eqn. (5.35.b) represents the response of the system to a *unitary Dirac Impulse*. We should calculate this response; we firstly begin with the solution of Eqn. (6.1) for both cases (a) $P_i(t) = 0$ and (b) $P_i(t) \neq 0$.

**(a).-** $P_i(t) = 0$ and we derive:

$$u(t)_i = e^{-\omega_n \cdot \zeta_i \cdot t} \cdot (a_i \cdot \sin(\omega_{D_i} \cdot t) + b_i \cdot \cos(\omega_{D_i} \cdot t)) \quad (6.3.1).$$

If $\zeta_i < 1$ (*under-critical* damping):

$$u(t)_i = e^{-\omega_n t}\left(a_i + b_i \cdot t\right) \quad (6.3.2).$$

If $\zeta_i = 1$ (*critical* damping):

$$u(t)_i = a_i \cdot e^{-\omega_n \cdot (\zeta_i - \sqrt{\zeta_i^2 - 1}) \cdot t} + b_i \cdot e^{-\omega_n \cdot (\zeta_i + \sqrt{\zeta_i^2 - 1}) \cdot t} \quad (6.3.3).$$

If $\zeta_i > 1$ (*over-critical* damping), where:

$$\omega_{D_i} = \omega_n \cdot \sqrt{1 - \zeta_i^2} \quad (6.3.4).$$

Proportionally to the above, which applies for a free system without damping, we can examine the equation and find a solution for an under-critically damped system, for which the initial conditions are u(0) and u′(0) (see [[4], pp. 100, Eqn. 3.3.5]):

The particular solution becomes (**6.4.1**):

$$b_i = u(0),\ a_i = \frac{u'(0) + \omega_n \zeta_i u(0)}{\omega_n \sqrt{1 - \zeta_i^2}} = \frac{u'(0) + \omega_n \zeta_i u(0)}{\omega_{D_i}}$$

and (**6.4.2**),

$$u(t) = e^{-\omega_n \zeta_i \cdot t} \cdot \left(\frac{u'(0) + \omega_n \zeta_i \cdot u(0)}{\omega_n \sqrt{1 - \zeta_i^2}} \cdot \sin\omega_n \sqrt{1 - \zeta_i^2} \cdot t + \right.$$

$$\left. + u(0) \cdot \cos\omega_n \sqrt{1 - \zeta_i^2} \cdot t\right) =$$

$$= e^{-\omega_n \zeta_i t} \cdot \left(\frac{u'(0) + \omega_n \zeta_i u(0)}{\omega_{D_i}} \cdot \sin\omega_{D_i} t + u(0) \cdot \cos\omega_{D_i} t\right)$$

**(b).-** If we consider a forced oscillation $P_i(t) \neq 0$ with damping, the solution of the differential equation (6.1) now consists of two parts: a homogenous part which we have come across in Eqns. (6.3.1) to (6.3.3), and a particular solution up that depends on the type of excitation (**6.4.3a**):

$$u(t)_i = e^{-\omega_n \cdot \zeta_i \cdot t} \cdot (a_i \cdot \sin(\omega_{D_i} \cdot t) + b_i \cdot \cos(\omega_{D_i} \cdot t)) + u_p(t)_i$$

this equation applies to the most common types of under-critical damping.

If we pose in Eqn. (6.2) $P_0/k = 1$, then:

$$P_i = e^{i\omega t} = \cos\omega t + i \cdot \sin\omega t \quad (6.4.3b)$$

and its response:

$$u(t) = F_i \cdot e^{i\omega t} \quad (6.4.3c)$$

where:

$$F_i = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + i \cdot 2\zeta_i \cdot \left(\frac{\omega}{\omega_n}\right)} \quad (6.4.3d)$$

Recalling from the theory of complex functions:

$$\frac{1}{a + ib} = \frac{1}{r(\cos\varphi + i\sin\varphi)} = \frac{1}{r} \cdot e^{-i\varphi} \quad (6.4.4)$$

where:

$$r = \left(a^2 + b^2\right)^{1/2},\quad \varphi = \tan^{-1}\frac{b}{a} \quad (6.4.5)$$

Value *1/r* gives the absolute value of the complex frequency response characteristic $F_i$ and the response factor of displacement (amplification factor) $R_{d_i}$:

$$R_{d_i} = \left(\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta_i \cdot \left(\frac{\omega}{\omega_n}\right)\right]^2\right)^{-1/2} \quad (6.4.6)$$

Angle $\varphi_i$ specifies the phase shift:

$$\varphi_i = \tan^{-1}\frac{2\zeta_i \cdot \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \quad (6.4.7)$$

The particular solution for an excitation of the form (6.4.3b) is:

$$u_p(t) = R_{d_i} \cdot e^{-i\varphi_i} \cdot e^{i \cdot \omega t} = R_{d_i} \cdot e^{i(\omega t - \varphi_i)} =$$

$$= R_{d_i} \cdot \left(\cos(\omega t - \varphi_i) + i \cdot \sin(\omega t - \varphi_i)\right) \quad (6.4.8)$$

*As we examined before, the homogenous solution of the system with damping, is gradually damped and the particular solution (6.4.8) represents the oscillation of the steady state condition of the system. The damping determines the phase shift of the system's response through Eqn. (6.4.7), which –for a system without damping– is 0 or π. Amplification factor* $R_{d_i}$ *always remains finite, even in the case of resonance* ($\omega = \omega_n$).

After the above analysis, we come back to the solution of Eqn. (6.1) and (5.35.b) that represents the response to a *Dirac impulse*. From the particular solution (**6.4.2**), setting as initial conditions:

$$u(0) = 0,\quad \dot{u}(0) = \omega_n^2 \cdot \int P_i(t) \cdot dt = \omega_n^2 \cdot 1 \quad (6.4.9)$$

that corresponds to the *Dirac impulse*, introducing the necessary phase shift and assuming a weak damping, we have the solution (the only difference to the response is that it decreases exponentially):

$$h(t - \bar{t}) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} \cdot e^{-\omega_n \zeta (t - \bar{t})} \cdot \sin\left(\omega_{D_i}(t - \bar{t})\right) \quad (6.4.10)$$

this formula can be applied up to the critical damping $\zeta_i = 1$. For $\zeta_i = 0$ the solution (6.4.10) turns into (5.35.b). Both Eqns. (6.4.10) and (5.35.b) must be complemented by:

$$h(t - \bar{t}) = 0 \quad \text{for} \quad t < \bar{t} \quad (6.4.11)$$

since no response can exist before the application of *the (unitary) Dirac impulse*.

Nevertheless, it should be noted that random excitation Pi (t) does not start at instant $\bar{t}$, but it could go on –at least in theory– even before $\bar{t}$, for an indefinite period of time. Eqn. (5.35.b) can be written (*Duhamel integral*):

$$u(t) = \int_{\bar{t}=-\infty}^{\bar{t}=t} P_i(\bar{t}) \cdot h(t-\bar{t}) \cdot d\bar{t} = \qquad (6.4.12)$$

$$= \int_{\bar{t}=-\infty}^{\bar{t}=+\infty} P_i(\bar{t}) \cdot h(t-\bar{t}) \cdot d\bar{t} = \int_{-\infty}^{+\infty} h(\theta) \cdot P_i(t-\theta) \cdot d\theta$$

where $\theta = t - \bar{t}$.

The mean value of the response is:

$$\bar{u} = E[u] = E\left[\int_{-\infty}^{+\infty} h(\theta) \cdot P_i(t-\theta) \cdot d\theta\right] =$$

$$= \int_{-\infty}^{+\infty} h(\theta) \cdot E[P_i(t-\theta)] \cdot d\theta =$$

$$= \int_{-\infty}^{+\infty} h(\theta) \cdot \bar{P}_i \cdot d\theta = \bar{P}_i \int_{-\infty}^{+\infty} h(\theta) \cdot d\theta \qquad (6.4.13)$$

Here it should be noted that the calculation of the mean value for an ergodic process *u(t)* is done on the axis of time *t*. But, in this case, since only $P_i(t - \theta)$ depends on time *t*, the mean value can be extracted before the integration. Since excitation function $P_i(t)$ is also ergodic and stationary, *its mean value* $\bar{P}_i$ has to be independent of time. Hence the mean value in (6.4.13) is obtained. A very important rule is derived from the above: if the mean value of the excitation is zero, then the mean value of the response is also zero.

The next step is the calculation of the response autocorrelation function. Here we also face an expectation value:

$$\Phi_u(\Delta t) = E[u(t) \cdot u(t+\Delta t)] = \qquad (6.4.14)$$

$$= E\left[\left(\int_{-\infty}^{+\infty} h(\theta_1) \cdot P(t-\theta_1) \cdot d\theta_1\right) \cdot \left(\int_{-\infty}^{+\infty} h(\theta_2) \cdot P_i(t+\Delta t-\theta_2) \cdot d\theta_2\right)\right] =$$

$$= E\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1) \cdot h(\theta_2) \cdot P_i(t-\theta_1) \cdot P_i(t-\theta_2+\Delta t) d\theta_1 \cdot d\theta_2\right] =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1) \cdot h(\theta_2) E[P_i(t-\theta_1) P_i(t-\theta_2+\Delta t)] d\theta_1 d\theta_2 =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1) \cdot h(\theta_2) \cdot \Phi_p(\Delta t+\theta_1-\theta_2) \cdot d\theta_1 \cdot d\theta_2$$

Here $\Phi_p$ is the excitation autocorrelation function and $\Phi_u$ is the response autocorrelation function. In practice, their detailed calculation is difficult.

From equations (6.4.13) and (6.4.14) it can be derived that response *u(t)* is also stationary for a stationary excitation $P_i(t)$, since neither the mean value $\bar{u}$ nor autocorrelation function $\Phi_u(\Delta t)$ depend on time *t*.

The transformation of spectral density $S_p$ of the excitation into spectral density $S_u$ of the response can be easily calculated. For this purpose, we use the definition of the Spectral Density (Eqn. 4.3.a-b) and the Eqn. (6.4.14) and we calculate the corresponding Spectral Density from the autocorrelation function:

$$S_u(\omega) = \frac{1}{2\pi} \int_{\Delta t=-\infty}^{+\infty} \Phi_u(\Delta t) \cdot e^{-i\omega\Delta t} \cdot d(\Delta t) = \qquad (6.4.15)$$

$$= \frac{1}{2\pi} \int_{\Delta t=-\infty}^{+\infty} \int_{\theta_1=-\infty}^{+\infty} \int_{\theta_2=-\infty}^{+\infty} h(\theta_1) \cdot h(\theta_2) \cdot \Phi_p(\Delta t+\theta_1-\theta_2) \cdot d\theta_2 \cdot d\theta_1 \cdot e^{-i\omega\Delta t} \cdot d(\Delta t) =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1) \cdot h(\theta_2) \cdot \left(\frac{1}{2\pi} \cdot \int_{\Delta t=-\infty}^{+\infty} \Phi_p(\Delta t+\theta_1-\theta_2) \cdot e^{-i\omega\Delta t} \cdot d(\Delta t)\right) \cdot d\theta_1 \cdot d\theta_2 =$$

$$= S_p(\omega) \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1) \cdot h(\theta_2) \cdot e^{-i\omega(\theta_2-\theta_1)} \cdot d\theta_1 \cdot d\theta_2$$

The statistical measurements taken for the response also contain some expressions with integrals. Nevertheless, they can be simplified through the Fourier integral (transform) method, producing impressive results (for an easy calculations of the Fourier integral, see [[13], paragr. 4-1]). Eqn. (6.4.12) is the Duhamel integral and the response in the time domain. If we apply the Fourier transformation as in the non-periodic functions of Eqns. (4.2) to (4.3.a-b) to an excitation impulse with unit value at $\theta = 0$, then we derive the following response:

$$h(\theta) = \int_{-\infty}^{+\infty} F_i(\omega) \cdot f_i(\omega) \cdot e^{i\omega\theta} \cdot d\omega \qquad (6.4.16)$$

which is the response in the *frequency domain*, where the *complex characteristic response frequency*:

$$F_i(\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + i \cdot 2\zeta_i \cdot \frac{\omega}{\omega_n}} \qquad (6.4.17)$$

and the frequency spectrum of the excitation impulse becomes (introducing a new time variable $\theta = t - \bar{t}$) [see [[4], pp. 153, n.21, pp. 119] and [[7], 98-9]):

$$f_i(\omega) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} P_i \cdot e^{-i\omega\theta} \cdot d\theta = \frac{1}{2\pi} \qquad (6.4.18)$$

We get the above result because Pi fluctuates only inside a range of small values of $\theta$, where $e^{-i\omega\theta} \cong 1$. Then the integral gives the impulse and then it is indeed 1. From Eqn. (6.4.16) and (4.4.15) we find:

$$h(\theta) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} F_i(\omega) \cdot e^{-i\omega\theta} \cdot d\theta \qquad (6.4.19)$$

The comparison with equations (4.2) to (4.4) gives the expression $F_i(\omega)/2\pi$ for the Fourier transform of function $h(\theta)$. According to Eqn . (4.4) we have:

$$F_i(\omega) = \int_{-\infty}^{+\infty} h(\theta) \cdot e^{-i\omega\theta} \cdot d\theta \qquad (6.4.20)$$

where the complex characteristic frequency is defined by the simple Eqn (6.4.17). Hence the integrals with infinite limits can be elegantly calculated and, instead of Eqn. (6.4.13) we get for the mean value of the response, when *ω = 0*:

$$\overline{u} = \overline{P_i} \cdot F(0) = \overline{P_i} \qquad (6.4.21)$$

For the Eqn. (6.4.15) we have:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(\theta_1)\cdot h(\theta_2)\cdot e^{-i\omega(\theta_2-\theta_1)}\cdot d\theta_1\cdot d\theta_2 =$$

$$= \int_{-\infty}^{+\infty} h(\theta)\cdot e^{+i\omega\theta}\cdot d\theta \cdot \int_{-\infty}^{+\infty} h(\theta)\cdot e^{-i\omega\theta}\cdot d\theta =$$

$$= F^*(\omega)\cdot F(\omega) = R_d^2 \qquad (6.4.22)$$

here $F_i^*$ is the complex conjugate of $F_i$ and $R_d$ is the real amplification factor of equation (6.4.15), that coincides with the modulus of the complex characteristic frequency.

We have:

$$R_d^2 = \left(\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta_i\cdot\frac{\omega}{\omega_n}\right]^2\right)^{-1} \qquad (6.4.23)$$

Equation (6.4.15) can be now written into the simplified form:

$$S_u(\omega) = R_d^2\cdot S_p(\omega) \qquad (6.4.24)$$

# 7 Transformation Functions in Time and Frequency domains

From the *Duhamel integral* (5.35.a, 5.35.c, 6.4.12) we find the response:

$$u(t) = \int_{+\infty}^{t} P(t)\cdot h(t-\tau)\cdot dt \qquad (7.1)$$

that applies for the time domain, whereas for the frequency domain:

$$u(t) = \int_{-\infty}^{+\infty} F_i(\omega)\cdot f_i(\omega)\cdot e^{-i\omega\cdot t}\cdot d\omega \qquad (7.2)$$

which derives from Eqn. (6.4.16); we should underline that we use the general form $P_i(t)$ instead of a unitary load (as the *Dirac Impulse*). From Eqn. (6.4.18) we derive:

$$f_i(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P_i(t)e^{-i\omega\cdot t}dt \qquad (7.3)$$

and bearing in mind that θ is a time variable, from Eqn. (6.4.17) we have the frequency response function, which is a complex function:

$$F_i(\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + i\cdot 2\zeta_i\cdot\frac{\omega}{\omega_n}} \qquad (7.4)$$

and also the response function to a unitary impulse (from Eqn. 6.4.10):

$$h(t-\overline{t}) = \frac{\omega_n}{\sqrt{1-\zeta^2}}\cdot e^{-\omega_n\zeta(t-\overline{t})}\cdot \sin\left(\omega_{D_i}(t-\overline{t})\right) \quad (7.5)$$

It is characteristic that $F_i(\omega)$ and $h(t-\overline{t})$ are related through the *Fourier transform* and the *inverse Fourier transform*. From Eqn. (6.4.20) we have (**7.6**):

$$F_i(\omega) = \int_{-\infty}^{+\infty} h(\theta)\cdot e^{-i\omega\theta}\cdot d\theta = \int_{-\infty}^{+\infty} h(t-\overline{t})\cdot e^{-i\omega(t-\overline{t})}\cdot dt = H(\nu)$$

and from Eqn. (6.4.19), we have (**7.7**):

$$h(\theta) = \frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty} F_i(\omega)\cdot e^{-i\omega\theta}\cdot d\theta = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_i(\omega)\cdot e^{-i\omega(t-\overline{t})}\cdot dt$$

# 8 Relationship between Excitation-Response Spectral Density: Track Defects and Motion of the Vehicle

If we have the system “Railway Vehicle-Railway Track” then the condition and the position in space of the rail running table is the excitation (Input) and the movement of the vehicle is the response (Output).

From Eqn. (6.4.15) we have (**8.1**):

$$S_u(\omega) = S_p(\omega)\int_{-\infty}^{+\infty} h(\theta_1)\cdot e^{-i\omega\cdot\theta_1}\cdot d\theta_1 \cdot \int_{-\infty}^{+\infty} h(\theta_2)\cdot e^{i\omega\cdot\theta_2}\cdot d\theta_2$$

and using Eqns. (7.6) and (8.1), we find Eqn. (**8.2**) below (see also [[5], Ch.2]):

$$S_u(\omega) = H(-i\omega)\cdot H(i\omega)\cdot S_p(\omega) = \left|H(i\omega)\right|^2\cdot S_p(\omega)$$

where frequency ν has two parts one real and one imaginary.

However, *the spectral density function is real, it does not contain an imaginary part* (see paragraph 4); the Spectral Density of the response (output) Su is related to the Spectral Density of the excitation (input) with a real number $|H(i\omega)|^2$. Consequently if we find or measure the Spectral Density of the input we can calculate the Spectral density of the output.

In the case of the Track defects, should be clarified that different wavelengths address different vehicles’ responses depending on the length of the cord/base of measurement. This is of decisive importance for the wavelengths of 30 – 33 m, which are characteristic for very High Speed Lines ([1], [2]). In the real tracks, the forms of the defects are random with wavelengths from few centimetres to 100 m. The defects constitute the “Input” in the system “Vehicle-Track” since the deflection y and the Action/Reaction R of each support point of the rail/sleeper are the "Output" or "Response" of the system. The accuracy of the measurements of the defects is of utmost importance for the calculation of the deflection y and the Reaction R; this accuracy, due to the bases of the measuring devices/vehicles, is fluctuating. Thus we should pass from the space-time domain to the frequencies' domain through the Fourier transform, in order to use the power spectral density of the defects, especially for defects, with (long) wavelength, larger than the measuring base of the vehicle.

In the case of random defects then we do not use the function f(x) but its Fourier transform:

$$F(\Omega)=\int_{-\infty}^{+\infty} f(x)\cdot e^{-i\Omega x}\cdot dx \qquad (8.3)$$

In practice we don't know the function of real defects y(x) but the measured values f(x) (see Eqns 23 below), from the recording vehicle, and we imply that:

$$S_Z(\Omega)=S_{INPUT}(\Omega)=\frac{S_F(\Omega)}{|K(\Omega)|^2}=\frac{S_{OUTPUT}(\Omega)}{|K(\Omega)|^2} \qquad (8.4)$$

where $S_Z(\Omega)$ is the spectral density of the Fourier transform of the real defects (input in the track recording vehicle), $S_F(\Omega)$ is the spectral density of the Fourier transform of the measured values (output) and $K(\Omega)$ is a complex transfer function (of the Recording Vehicle/Car), called frequency response function, transforming the measured values of defects to the real values. For very High Speed Lines we should analyze the system "railway track – railway vehicle". The calculation of the spectrum of track defects is described in [3] (paragr. 6) and [6] (pp.155-158).

# 9 Conclusions – Spectral Density in Measurements on Railway Track

For the reliability of the measurements via the Track Recording Cars/Vehicles see [1], [2]. In general, in order to approach the matter of the reliability of the measured values of the track-defects by the Track Recording Vehicles/Cars and their Confidence Interval, we should examine the transfer function |H(ω)| of the recording vehicle which presents minimums and zero-points. In the real conditions, the defects are random with wavelengths from few centimeters to 100 m. Since the length of the vehicle's measuring base is much shorter than 100 m, we should pass from the space-time domain to the frequencies' domain through the Fourier transform, in order to use the power spectral density of the defects. Furthermore, in the case of random defects, we cannot and do not use the functions f(x) and z(x) but we can use their Fourier transforms.

In practice through the Spectral Density of the Track defects consist of four separate components 'Spectral Densities' and this is out of the scope of the present paper. Since in the very High Speed Lines (Vmax > 200 km/h) the crucial Track defects are of very long waveform (l > 33 m, cf. [1], [2]), here we are interested in the forms of

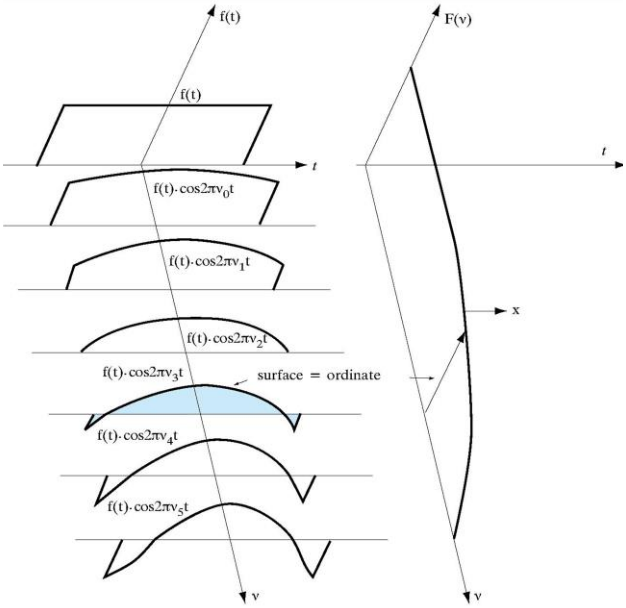


Figure 10. Physical realization of the Fourier transform: the surface *f(t)·cos2πνᵢt* shown sliced in one of two possible ways ([5], pp.160]; [[10], pp.23]; [[16], 20]). (Left) the space domain f(t) and t; (right) the frequency domain: its ordinate is equal to the surface of the component functions.

these defects. Their Spectral Density of the Track Defects is one continuous spectrum diminishing towards the small wavelengths. The French Railways had determined that in one analytic form [[16], pp.333]:

$$S(\Omega)=\frac{a\cdot e^2}{(b+\Omega)^3} \qquad (9.1)$$

where, Ω is equal to 2π/λ, e the average of the total signal recorded, a, b are constants.

In practice this Spectrum can be exploited for wavelengths between 100 m and 3 cm.

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