# A Study on Neutrosophic Intuitionistic Fuzzy Subsemiring of a Semiring

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*Abstract:* This study introduces neutrosophic intuitionistic fuzzy subsemirings in semiring theory and establishes essential algebraic properties. Particular attention is given to their behavior with respect to homomorphism and anti-homomorphism.
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## 1 Introduction

Various notions in universal algebra expand upon the structure of an associative ring (□; +, .). Among these, nearrings(NR's) and different forms of semirings have proven to be highly valuable. An algebra (□;+,.) is defined as a semiring (SR) when (□; +) and (□; .) form semigroups and necessarily adhere to the distributive laws$\varkappa.(y + z) = \varkappa.y + \varkappa.z$ and $(y + z).\varkappa = y.\varkappa + z.\varkappa$ without exception for $\varkappa$, $y$, and $z$ in □. A SR□ is deemed additively commutative if $\varkappa + y = y + \varkappa$ is true without exception for $\varkappa$, $y$ in □. Furthermore, a SR□might include a distinguished identity element 1, where $1.\varkappa = \varkappa = \varkappa.1$, and a zero element 0, where $0 + \varkappa = \varkappa = \varkappa + 0$ and $\varkappa.0 = 0 = 0.\varkappa$ hold without exception for $\varkappa$ in □. After L.A. Zadeh introduced fuzzy sets(FS's) in his work [19], scholars began exploring extensions of this idea. [16,17]Smarandache, in publications from 2002 and 2005, put forth Neutrosophic fuzzy set (NFS) as an innovative philosophical perspective. Later, S. Abou Zaid [1] introduced the notions of fuzzy subnearrings(SNR) and ideals.. The concept of intuitionistic fuzzy subset(IFS) was introduced by K.T. Atanassov [ 7,8], as a generalization of the notion of FS. In this paper, we introduce the some theorems in neutrosophicintuitionistic fuzzy subsemiring (NIFSSR) of a SR under homomorphism and anti-homomorphism.

## 2 Preliminaries

**2.1 Definition[14]:**A NFS Đ over a universal set X is described in terms of a truth membership function$\check{T}_{Đ}(\varkappa)$, an indeterminacy function $\hat{I}_{Đ}(\varkappa)$, and a falsity membership function $Ƒ_{Đ}(\varkappa)$, represented as $Đ = \{ < \varkappa, \check{T}_{Đ}(\varkappa), \hat{I}_{Đ}(\varkappa), Ƒ_{Đ}(\varkappa) > ; \varkappa \in X \}$, where $\check{T}_{Đ}, \hat{I}_{Đ}, Ƒ_{Đ} : X \to [0, 1]$ and the condition $0 \leq \check{T}_{Đ}(\varkappa) + \hat{I}_{Đ}(\varkappa) + Ƒ_{Đ}(\varkappa) \leq 3$ holds.

**2.1.1 Example:** Let ( N , + , .) be a SR . Then NFSĐ = $\{< \varkappa, \check{T}_{Đ}(\varkappa), \hat{I}_{Đ}(\varkappa), Ƒ_{Đ}(\varkappa) > ; \varkappa \in X \}$ of N, where

$$\check{T}_{Đ}(\varkappa) = \begin{cases} 0.4 & \text{if } \varkappa \text{ is even} \\ 0.6 & \text{if } \varkappa \text{ is odd} \end{cases}$$

$$\hat{I}_{Đ}(\varkappa) = \begin{cases} 0.3 & \text{if } \varkappa \text{ is even} \\ 0.3 & \text{if } \varkappa \text{ is odd} \end{cases}$$

and

$$Ƒ_{Đ}(\varkappa) = \begin{cases} 0.3 & \text{if } \varkappa \text{is even} \\ 0.1 & \text{if } \varkappa \text{ is odd} \end{cases}$$

Clearly Đ is a Neutrosophic anti-fuzzy SSR (NAFSSR).

**2.2 Definition:** Let R be a SR. An NIFSĐ of R is said to be an **neutrosophic intuitionistic fuzzy SSR (NIFSSR)** of R if it satisfies the following conditions:

(i) (a) $\mu_{Đ}(\varkappa + y) \geq$ min$\{ \mu_{Đ}(\varkappa), \mu_{Đ}(y) \}$,

(b) $\nu_{Đ}(\varkappa + y) \leq$ max $\{ \nu_{Đ}(\varkappa), \nu_{Đ}(y) \}$,

(c ) $\iota_{Đ}(x + y) \le \max\{ \iota_{Đ}(x), \iota_{Đ}(y) \}$,
(ii) (a) $\mu_{Đ}(xy) \ge \min\{ \mu_{Đ}(x), \mu_{Đ}(y) \}$,
(b) $\nu_{Đ}(xy) \le \max\{ \nu_{Đ}(x), \nu_{Đ}(y) \}$,
(c ) $\iota_{Đ}(xy) \le \max\{ \iota_{Đ}(x), \iota_{Đ}(y) \}$, for all $x$ and $y$ in R.

**2.2.1 Example:** Let ( $Z_3$ , + , .) be a SR . Then NIFSĐ = { (( x, $\mu_D(x)$, $\iota_D(x)$,$\nu_D(x)$ ) / x∈ $Z_3$ } of $Z_3$ , where

$$\mu_D(x) = \begin{cases} 0.6 & \text{if } x=0 \\ 0.3 & \text{if } x=1,2 \end{cases}$$

$$\iota_D(x) = \begin{cases} 0.2 & \text{if } x=0 \\ 0.2 & \text{if } x=1,2 \end{cases}$$

and

$$\nu_D(x) = \begin{cases} 0.2 & \text{if } x=0 \\ 0.5 & \text{if } x=1,2 \end{cases}$$

Clearly Đ is a NIFSSR.

**2.3 Definition:** Let ( R, +, · ) and ( $R^{I}$, +, · ) be any two SR's. Let f : R → $R^{I}$ be any function and Đ be an NIFSSR in R, V be an NIFSSR in f(R) = $R^{I}$, defined by $\mu_V(y) = \sup_{x\in f^{-1}(y)} \mu_{Đ}(x)$ , $\nu_V(y) = \inf_{x\in f^{-1}(y)} \nu_{Đ}(x)$ and $\iota_V(y) = \inf_{x\in f^{-1}(y)} \iota_{Đ}(x)$ , for all x in R and y in $R^{I}$. Then Đ is called a preimage of V under f and is denoted by $f^{-1}(V)$.

# 3 Properties of neutrosophic intuitionistic fuzzy subsemiring of a semiring

**3.1 Theorem:** Intersection of any two NIFSSR of a SR R is a NIFSSR of R.

**Proof:**

Let Đ1 and B1 be any two NIFSSR's of a SR R and $x$ and $y$ in R. Let Đ1 = { ( $x$, $\mu_{Đ1}(x)$, $\nu_{Đ1}(x)$,$\iota_{Đ1}(x)$ ) / $x$∈R } and B1 = { ( $x$, $\mu_{B1}(x)$, $\nu_{B1}(x)$,$\iota_{B1}(x)$ ) / $x$∈R } and also let C = Đ1∩B1 = { ( $x$, $\mu_C(x)$, $\nu_C(x)$,$\iota_C(x)$ ) / $x$∈R }, where $\min\{ \mu_{Đ1}(x), \mu_{B1}(x) \} = \mu_C(x)$ , $\max\{ \nu_{Đ1}(x), \nu_{B1}(x) \} = \nu_C(x)$ and $\max\{ \iota_{Đ1}(x), \iota_{B11}(x) \} = \iota_C(x)$.Now,(i) (a) $\mu_C( x + y ) = \min\{ \mu_{Đ1}(x + y), \mu_{B11}(x + y) \} \ge \min\{ \min\{ \mu_{Đ1}(x), \mu_{Đ1}(y) \}, \min\{ \mu_{B11}(x), \mu_{B11}(y) \} \} = \min\{ \min\{ \mu_{Đ1}(x), \mu_{B11}(x) \}, \min\{ \mu_{Đ1}(y), \mu_{B11}(y) \} \} = \min\{ \mu_C(x),\mu_C(y) \}$.Consequently, $\mu_C(x +y) \ge \min\{ \mu_C(x), \mu_C(y) \}$, without exception for $x$ and $y$ in R.(b) $\nu_C( x + y ) = \max\{ \nu_{Đ1}(x + y), \nu_{B1}(x + y) \} \le \max\{ \max\{ \nu_{Đ1}(x), \nu_{Đ1}(y) \}, \max\{ \nu_{B1}(x), \nu_{B1}(y) \} \} = \max\{ \max\{ \nu_{Đ1}(x), \nu_{B1}(x) \}, \max\{ \nu_{Đ1}(y), \nu_{B1}(y) \} \} = \max\{ \nu_C(x),\nu_C(y) \}$.Consequently, $\nu_C(x + y) \le \max\{ \nu_C(x), \nu_C(y) \}$, without exception for $x$ and $y$ in R.(c) $\iota_C( x + y ) = \max\{ \iota_{Đ1}(x + y), \iota_{B1}(x + y) \} \le \max\{ \max\{ \iota_{Đ1}(x), \iota_{Đ1}(y) \}, \max\{ \iota_{B1}(x), \iota_{B1}(y) \} \} = \max\{ \max\{ \iota_{Đ1}(x), \iota_{B1}(x) \}, \max\{ \iota_{Đ1}(y), \iota_{B1}(y) \} \} = \max\{ \iota_C(x),\iota_C(y) \}$.Consequently, $\iota_C(x + y) \le \max\{ \iota_C(x), \iota_C(y) \}$, without exception for $x$ and $y$ in R And, (ii) (a) $\mu_C(xy) = \min\{ \mu_{Đ1}(xy), \mu_{B1}(xy) \} \ge \min\{ \min\{ \mu_{Đ1}(x), \mu_{Đ1}(y) \}, \min\{ \mu_{B1}(x), \mu_{B1}(y) \} \} = \min\{ \min\{ \mu_{Đ1}(x), \mu_{B1}(x) \}, \min\{ \mu_{Đ1}(y), \mu_{B1}(y) \} \} = \min\{ \mu_C(x), \mu_C(y) \}$.Consequently, $\mu_C(xy) \ge \min\{ \mu_C(x),\mu_C(y) \}$, without exception for $x$ and $y$in R. (b) $\nu_C(xy) = \max\{ \nu_{Đ1}(xy), \nu_{B1}(xy) \} \le \max\{ \max\{ \nu_{Đ1}(x), \nu_{Đ1}(y) \}, \max\{ \nu_{B1}(x), \nu_{B1}(y) \} \} = \max\{ \max\{ \nu_{Đ1}(x), \nu_{B1}(x) \}, \max\{ \nu_{Đ1}(y), \nu_{B1}(y) \} \} = \max\{ \nu_C(x), \nu_C(y) \}$.Consequently, $\nu_C(xy) \le \max\{ \nu_C(x),\nu_C(y) \}$, without exception for $x$ and $y$in R. (c) $\iota_C(xy) = \max\{ \iota_{Đ1}(xy), \iota_{B1}(xy) \} \le \max\{ \max\{ \iota_{Đ1}(x), \iota_{Đ1}(y) \}, \max\{ \iota_{B1}(x), \iota_{B1}(y) \} \} = \max\{ \max\{ \iota_{Đ1}(x), \iota_{B1}(x) \}, \max\{ \iota_{Đ1}(y), \iota_{B1}(y) \} \} = \max\{ \iota_C(x), \iota_C(y) \}$.Consequently,$\iota_C(xy) \le \max\{ \iota_C(x),\iota_C(y) \}$, without exception for $x$ and $y$in R.Consequently C is an NIFSSR of a SR R.Hence the intersection of any two NIFSSR's of a SR R is an NIFSSR of R.

**3.2 Theorem:** The intersection of a family of NIFSSR's of SR R is an NIFSSRof R.

**Proof:**

Let { $V_i$: i∈I} be a family of NIFSSR's of a SR R and let Đ1 = $\bigcap_{i\in I} V_i$ . Let $x$ and $y$ in R.Then , (i) (a) $\mu_{Đ1}(x + y) = \inf_{i\in I} \mu_{V_i}(x + y) \ge \inf_{i\in I} \min\{ \mu_{V_i}(x), \mu_{V_i}(y) \} = \min\{ \inf_{i\in I} \mu_{V_i}(x), \inf_{i\in I} \mu_{V_i}(y) \} = \min\{ \mu_{Đ1}(x), \mu_{Đ1}(y) \}$.Consequently, $\mu_{Đ1}(x+y) \ge \min\{ \mu_{Đ1}(x),\mu_{Đ1}(y) \}$, without exception for $x$ and $y$in R.(b) $\nu_{Đ1}(x + y) = \sup_{i\in I} \nu_{V_i}(x + y) \le \sup_{i\in I} \max\{ \nu_{V_i}(x), \nu_{V_i}(y) \} = \max\{ \sup_{i\in I} \nu_{V_i}(x), \sup_{i\in I} \nu_{V_i}(y) \} = \max\{ \nu_{Đ1}(x), \nu_{Đ1}(y) \}$.Consequently, $\nu_{Đ1}(x+y) \le \max\{ \nu_{Đ1}(x),\nu_{Đ1}(y) \}$, without exception for$x$ and $y$in R.(c) $\iota_{Đ1}(x + y) = \sup_{i\in I} \iota_{V_i}(x + y) \le \sup_{i\in I} \max\{ \iota_{V_i}(x), \iota_{V_i}(y) \} = \max$

$\{ \sup_{i\in I} \acute{\iota}_{Vi}(\varkappa), \sup_{i\in I} \acute{\iota}_{Vi}(y) \} = \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa), \acute{\iota}_{\text{Đ}1}(y) \}$. Consequently, $\acute{\iota}_{\text{Đ}1}(\varkappa+y) \le \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa), \acute{\iota}_{\text{Đ}1}(y) \}$, without exception for $\varkappa$ and $y$ in R. And, (ii) (a) $\mu_{\text{Đ}1}(\varkappa y) = \inf_{i\in I} \mu_{Vi}(\varkappa y) \ge \inf_{i\in I} \min\{ \mu_{Vi}(\varkappa), \mu_{Vi}(y) \} = \min\{ \inf_{i\in I} \mu_{Vi}(\varkappa), \inf_{i\in I} \mu_{Vi}(y) \} = \min\{ \mu_{\text{Đ}1}(\varkappa), \mu_{\text{Đ}1}(y) \}$. Consequently, $\mu_{\text{Đ}1}(\varkappa y) \ge \min\{ \mu_{\text{Đ}1}(\varkappa), \mu_{\text{Đ}1}(y) \}$, without exception for $\varkappa$ and $y$ in R. (b) $\nu_{\text{Đ}1}(\varkappa y) = \sup_{i\in I} \nu_{Vi}(\varkappa y) \le \sup_{i\in I} \max\{ \nu_{Vi}(\varkappa), \nu_{Vi}(y) \} = \max\{ \sup_{i\in I} \nu_{Vi}(\varkappa), \sup_{i\in I} \nu_{Vi}(y) \} = \max\{ \nu_{\text{Đ}1}(\varkappa), \nu_{\text{Đ}1}(y) \}$. Consequently, $\nu_{\text{Đ}1}(\varkappa y) \le \max\{ \nu_{\text{Đ}1}(\varkappa), \nu_{\text{Đ}1}(y) \}$, without exception for $\varkappa$ and $y$ in R. (c) $\acute{\iota}_{\text{Đ}1}(\varkappa y) = \sup_{i\in I} \acute{\iota}_{Vi}(\varkappa y) \le \sup_{i\in I} \max\{ \acute{\iota}_{Vi}(\varkappa), \acute{\iota}_{Vi}(y) \} = \max\{ \sup_{i\in I} \acute{\iota}_{Vi}(\varkappa), \sup_{i\in I} \acute{\iota}_{Vi}(y) \} = \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa), \acute{\iota}_{\text{Đ}1}(y) \}$. Consequently, $\acute{\iota}_{\text{Đ}1}(\varkappa y) \le \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa), \acute{\iota}_{\text{Đ}1}(y) \}$, without exception for $\varkappa$ and $y$ in R. That is, Đ1 is an NIFSSR of a SR R. Hence, the intersection of a family of NIFSSR's of R is an NIFSSRof R.

**3.3 Theorem:** If Đ1 and B1 are any two NIFSSR's of the SR's $R_1$ and $R_2$respectively, then Đ1×B1 is an NIFSSRof $R_1 \times R_2$.

**Proof:**

Let Đ1 and B1 be two NIFSSR's of the SR's $R_1$ and $R_2$respectively. Let $\varkappa_1$ and $\varkappa_2$ be in $R_1$, $y_1$ and $y_2$ be in $R_2$. Then $(\varkappa_1, y_1)$ and $(\varkappa_2, y_2)$ are in $R_1 \times R_2$. Now, (i) (a) $\mu_{\text{Đ}1\times B1}[(\varkappa_1, y_1) + (\varkappa_2, y_2)] = \mu_{\text{Đ}1\times B1}(\varkappa_1 + \varkappa_2, y_1 + y_2) = \min\{ \mu_{\text{Đ}1}(\varkappa_1 + \varkappa_2), \mu_{B1}(y_1 + y_2) \} \ge \min\{ \min\{ \mu_{\text{Đ}1}(\varkappa_1), \mu_{\text{Đ}1}(\varkappa_2) \}, \min\{ \mu_{B1}(y_1), \mu_{B1}(y_2) \} \} = \min\{ \min\{ \mu_{\text{Đ}1}(\varkappa_1), \mu_{B1}(y_1) \}, \min\{ \mu_{\text{Đ}1}(\varkappa_2), \mu_{B1}(y_2) \} \} = \min\{ \mu_{\text{Đ}1\times B1}(\varkappa_1, y_1), \mu_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. Consequently, $\mu_{\text{Đ}1\times B1}[(\varkappa_1, y_1) + (\varkappa_2, y_2)] \ge \min\{ \mu_{\text{Đ}1\times B1}(\varkappa_1, y_1), \mu_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. (b) $\nu_{\text{Đ}1\times B1}[(\varkappa_1, y_1) + (\varkappa_2, y_2)] = \nu_{\text{Đ}1\times B1}(\varkappa_1 + \varkappa_2, y_1 + y_2) = \max\{ \nu_{\text{Đ}1}(\varkappa_1 + \varkappa_2), \nu_{B1}(y_1 + y_2) \} \le \max\{ \max\{ \nu_{\text{Đ}1}(\varkappa_1), \nu_{\text{Đ}1}(\varkappa_2) \}, \max\{ \nu_{B1}(y_1), \nu_{B1}(y_2) \} \} = \max\{ \max\{ \nu_{\text{Đ}1}(\varkappa_1), \nu_{B1}(y_1) \}, \max\{ \nu_{\text{Đ}1}(\varkappa_2), \nu_{B1}(y_2) \} \} = \max\{ \nu_{\text{Đ}1\times B1}(\varkappa_1, y_1), \nu_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. Consequently, $\nu_{\text{Đ}1\times B1}[(\varkappa_1, y_1) + (\varkappa_2, y_2)] \le \max\{ \nu_{\text{Đ}1\times B1}(\varkappa_1, y_1), \nu_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. (c) $\acute{\iota}_{\text{Đ}1\times B1}[(\varkappa_1, y_1) + (\varkappa_2, y_2)] = \acute{\iota}_{\text{Đ}1\times B1}(\varkappa_1 + \varkappa_2, y_1 + y_2) = \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_1 + \varkappa_2), \acute{\iota}_{B1}(y_1 + y_2) \} \le \max\{ \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_1), \acute{\iota}_{\text{Đ}1}(\varkappa_2) \}, \max\{ \acute{\iota}_{B1}(y_1), \acute{\iota}_{B1}(y_2) \} \} = \max\{ \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_1), \acute{\iota}_{B1}(y_1) \}, \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_2), \acute{\iota}_{B1}(y_2) \} \} = \max\{ \acute{\iota}_{\text{Đ}1\times B1}(\varkappa_1, y_1), \acute{\iota}_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. Consequently, $\acute{\iota}_{\text{Đ}1\times B1}[(\varkappa_1, y_1) + (\varkappa_2, y_2)] \le \max\{ \acute{\iota}_{\text{Đ}1\times B1}(\varkappa_1, y_1), \acute{\iota}_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. Also, (ii) (a) $\mu_{\text{Đ}1\times B1}[(\varkappa_1, y_1)(\varkappa_2, y_2)] = \mu_{\text{Đ}1\times B1}(\varkappa_1\varkappa_2, y_1 y_2) = \min\{ \mu_{\text{Đ}1}(\varkappa_1\varkappa_2), \mu_{B1}(y_1 y_2) \} \ge \min\{ \min\{ \mu_{\text{Đ}1}(\varkappa_1), \mu_{\text{Đ}1}(\varkappa_2) \}, \min\{ \mu_{B1}(y_1), \mu_{B1}(y_2) \} \} = \min\{ \min\{ \mu_{\text{Đ}1}(\varkappa_1), \mu_{B1}(y_1) \}, \min\{ \mu_{\text{Đ}1}(\varkappa_2), \mu_{B1}(y_2) \} \} = \min\{ \mu_{\text{Đ}1\times B1}(\varkappa_1, y_1), \mu_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. Consequently, $\mu_{\text{Đ}1\times B1}[(\varkappa_1, y_1)(\varkappa_2, y_2)] \ge \min\{ \mu_{\text{Đ}1\times B1}(\varkappa_1, y_1), \mu_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. (b) $\nu_{\text{Đ}1\times B1}[(\varkappa_1, y_1)(\varkappa_2, y_2)] = \nu_{\text{Đ}1\times B1}(\varkappa_1\varkappa_2, y_1 y_2) = \max\{ \nu_{\text{Đ}1}(\varkappa_1\varkappa_2), \nu_{B1}(y_1 y_2) \} \le \max\{ \max\{ \nu_{\text{Đ}1}(\varkappa_1), \nu_{\text{Đ}1}(\varkappa_2) \}, \max\{ \nu_{B1}(y_1), \nu_{B1}(y_2) \} \} = \max\{ \max\{ \nu_{\text{Đ}1}(\varkappa_1), \nu_{B1}(y_1) \}, \max\{ \nu_{\text{Đ}1}(\varkappa_2), \nu_{B1}(y_2) \} \} = \max\{ \nu_{\text{Đ}1\times B1}(\varkappa_1, y_1), \nu_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. Consequently, $\nu_{\text{Đ}1\times B1}[(\varkappa_1, y_1)(\varkappa_2, y_2)] \le \max\{ \nu_{\text{Đ}1\times B1}(\varkappa_1 y_1), \nu_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. (c) $\acute{\iota}_{\text{Đ}1\times B1}[(\varkappa_1, y_1)(\varkappa_2, y_2)] = \acute{\iota}_{\text{Đ}1\times B1}(\varkappa_1\varkappa_2, y_1 y_2) = \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_1\varkappa_2), \acute{\iota}_{B1}(y_1 y_2) \} \le \max\{ \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_1), \acute{\iota}_{\text{Đ}1}(\varkappa_2) \}, \max\{ \acute{\iota}_{B1}(y_1), \acute{\iota}_{B1}(y_2) \} \} = \max\{ \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_1), \acute{\iota}_{B1}(y_1) \}, \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_2), \acute{\iota}_{B1}(y_2) \} \} = \max\{ \acute{\iota}_{\text{Đ}1\times B1}(\varkappa_1, y_1), \acute{\iota}_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. Consequently, $\acute{\iota}_{\text{Đ}1\times B1}[(\varkappa_1, y_1)(\varkappa_2, y_2)] \le \max\{ \nu_{\text{Đ}1\times B1}(\varkappa_1, y_1), \nu_{\text{Đ}1\times B1}(\varkappa_2, y_2) \}$. Hence Đ1×B1 is an NIFSSR of SR of $R_1 \times R_2$.

**3.4 Theorem:** Let Đ1 be an NIFS of a SR R and V be the strongest NIFR of R. Then Đ1 is an NIFSSRof R if and only if V is an NIFSSR of R×R.

**Proof:** Suppose that Đ1 is an NIFSSR of a SR R. Then for any $\varkappa = (\varkappa_1, \varkappa_2)$ and $y = (y_1, y_2)$ are in R×R. We have, (i) (a) $\mu_V(\varkappa + y) = \mu_V[(\varkappa_1, \varkappa_2) + (y_1, y_2)] = \mu_V(\varkappa_1 + y_1, \varkappa_2 + y_2) = \min\{ \mu_{\text{Đ}1}(\varkappa_1 + y_1), \mu_{\text{Đ}1}(\varkappa_2 + y_2) \} \ge \min\{ \min\{ \mu_{\text{Đ}1}(\varkappa_1), \mu_{\text{Đ}1}(y_1) \}, \min\{ \mu_{\text{Đ}1}(\varkappa_2), \mu_{\text{Đ}1}(y_2) \} \} = \min\{ \min\{ \mu_{\text{Đ}1}(\varkappa_1), \mu_{\text{Đ}1}(\varkappa_2) \}, \min\{ \mu_{\text{Đ}1}(y_1), \mu_{\text{Đ}1}(y_2) \} \} = \min\{ \mu_V(\varkappa_1, \varkappa_2), \mu_V(y_1, y_2) \} = \min\{ \mu_V(\varkappa), \mu_V(y) \}$. Consequently, $\mu_V(\varkappa + y) \ge \min\{ \mu_V(\varkappa), \mu_V(y) \}$, without exception for $\varkappa$ and $y$ in R×R. (b) $\nu_V(\varkappa + y) = \nu_V[(\varkappa_1, \varkappa_2) + (y_1, y_2)] = \nu_V(\varkappa_1 + y_1, \varkappa_2 + y_2) = \max\{ \nu_{\text{Đ}1}(\varkappa_1 + y_1), \nu_{\text{Đ}1}(\varkappa_2 + y_2) \} \le \max\{ \max\{ \nu_{\text{Đ}1}(\varkappa_1), \nu_{\text{Đ}1}(y_1) \}, \max\{ \nu_{\text{Đ}1}(\varkappa_2), \nu_{\text{Đ}1}(y_2) \} \} = \max\{ \max\{ \nu_{\text{Đ}1}(\varkappa_1), \nu_{\text{Đ}1}(\varkappa_2) \}, \max\{ \nu_{\text{Đ}1}(y_1), \nu_{\text{Đ}1}(y_2) \} \} = \max\{ \nu_V(\varkappa_1, \varkappa_2), \nu_V(y_1, y_2) \} = \max\{ \nu_V(\varkappa), \nu_V(y) \}$. Consequently, $\nu_V(\varkappa + y) \le \max\{ \nu_V(\varkappa), \nu_V(y) \}$, without exception for $\varkappa$ and $y$ in R×R. (c) $\acute{\iota}_V(\varkappa + y) = \acute{\iota}_V[(\varkappa_1, \varkappa_2) + (y_1, y_2)] = \acute{\iota}_V(\varkappa_1 + y_1, \varkappa_2 + y_2) = \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_1 + y_1), \acute{\iota}_{\text{Đ}1}(\varkappa_2 + y_2) \} \le \max\{ \max\{ \acute{\iota}_{\text{Đ}1}(\varkappa_1), \acute{\iota}_{\text{Đ}1}(y_1) \},$

$\max\{ ί_{Đ1}(\varkappa_2), ί_{Đ1}(ɏ_2) \} \} = \max\{ \max\{ ί_{Đ1}(\varkappa_1), ί_{Đ1}(\varkappa_2) \}, \max\{ ί_{Đ1}(ɏ_1), ί_{Đ1}(ɏ_2) \} \} = \max\{ ί_V(\varkappa_1, \varkappa_2), ί_V(ɏ_1, ɏ_2) \} = \max\{ί_V(\varkappa), ί_V(ɏ)\}$.Consequently, $ί_V(\varkappa + ɏ) \le \max\{ ί_V(\varkappa), ί_V(ɏ) \}$, without exception for $\varkappa$ and $ɏ$ in R×R.And,(ii) (a) $\mu_V(\varkappa ɏ) = \mu_V[(\varkappa_1, \varkappa_2)(ɏ_1, ɏ_2)] = \mu_V(\varkappa_1 ɏ_1, \varkappa_2 ɏ_2) = \min\{ \mu_{Đ1}(\varkappa_1 ɏ_1), \mu_{Đ1}(\varkappa_2 ɏ_2) \} \ge \min\{ \min\{ \mu_{Đ1}(\varkappa_1), \mu_{Đ1}(ɏ_1) \}, \min\{ \mu_{Đ1}(\varkappa_2), \mu_{Đ1}(ɏ_2) \} \} = \min\{ \min\{ \mu_{Đ1}(\varkappa_1), \mu_{Đ1}(\varkappa_2) \}, \min\{ \mu_{Đ1}(ɏ_1), \mu_{Đ1}(ɏ_2) \} \} = \min\{ \mu_V(\varkappa_1, \varkappa_2), \mu_V(ɏ_1, ɏ_2) \} = \min\{ \mu_V(\varkappa), \mu_V(ɏ) \}$.Consequently, $\mu_V(\varkappa ɏ) \ge \min\{ \mu_V(\varkappa), \mu_V(ɏ) \}$, without exception for $\varkappa$ and $ɏ$ in R×R.(b) $\nu_V(\varkappa ɏ) = \nu_V[(\varkappa_1, \varkappa_2)(ɏ_1, ɏ_2)] = \nu_V(\varkappa_1 ɏ_1, \varkappa_2 ɏ_2) = \max\{ \nu_{Đ1}(\varkappa_1 ɏ_1), \nu_{Đ1}(\varkappa_2 ɏ_2) \} \le \max\{ \max\{ \nu_{Đ1}(\varkappa_1), \nu_{Đ1}(ɏ_1) \}, \max\{ \nu_{Đ1}(\varkappa_2), \nu_{Đ1}(ɏ_2) \} \} = \max\{ \max\{ \nu_{Đ1}(\varkappa_1), \nu_{Đ1}(\varkappa_2) \}, \max\{ \nu_{Đ1}(ɏ_1), \nu_{Đ1}(ɏ_2) \} \} = \max\{ \nu_V(\varkappa_1, \varkappa_2), \nu_V(ɏ_1, ɏ_2) \} = \max\{ \nu_V(\varkappa), \nu_V(ɏ) \}$.Consequently, $\nu_V(\varkappa ɏ) \le \max\{ \nu_V(\varkappa), \nu_V(ɏ) \}$, without exception for $\varkappa$ and $ɏ$ in R×R.(c) $ί_V(\varkappa ɏ) = ί_V[(\varkappa_1, \varkappa_2)(ɏ_1, ɏ_2)] = ί_V(\varkappa_1 ɏ_1, \varkappa_2 ɏ_2) = \max\{ ί_{Đ1}(\varkappa_1 ɏ_1), ί_{Đ1}(\varkappa_2 ɏ_2) \} \le \max\{ \max\{ ί_{Đ1}(\varkappa_1), ί_{Đ1}(ɏ_1) \}, \max\{ ί_{Đ1}(\varkappa_2), ί_{Đ1}(ɏ_2) \} \} = \max\{ \max\{ ί_{Đ1}(\varkappa_1), ί_{Đ1}(\varkappa_2) \}, \max\{ ί_{Đ1}(ɏ_1), ί_{Đ1}(ɏ_2) \} \} = \max\{ ί_V(\varkappa_1, \varkappa_2), ί_V(ɏ_1, ɏ_2) \} = \max\{ ί_V(\varkappa), ί_V(ɏ) \}$.Consequently, $ί_V(\varkappa ɏ) \le \max\{ ί_V(\varkappa), ί_V(ɏ) \}$, without exception for $\varkappa$ and $ɏ$ in R×R.Thisprovesthat V isanNIFSSRof R×R.Conversely asume that V isanNIFSSRof R×R, thenforany$\varkappa = (\varkappa_1, \varkappa_2)$ and $ɏ = (ɏ_1, ɏ_2)$ are in R×R, wehave(i) (a) $\min\{ \mu_{Đ1}(\varkappa_1 + ɏ_1), \mu_{Đ1}(\varkappa_2 + ɏ_2) \} = \mu_V(\varkappa_1 + ɏ_1, \varkappa_2 + ɏ_2) = \mu_V[(\varkappa_1, \varkappa_2) + (ɏ_1, ɏ_2)] = \mu_V(\varkappa + ɏ) \ge \min\{ \mu_V(\varkappa), \mu_V(ɏ) \} = \min\{ \mu_V(\varkappa_1, \varkappa_2), \mu_V(ɏ_1, ɏ_2) \} = \min\{ \min\{ \mu_{Đ1}(\varkappa_1), \mu_{Đ1}(\varkappa_2) \}, \min\{ \mu_{Đ1}(ɏ_1), \mu_{Đ1}(ɏ_2) \} \}$.If$\mu_{Đ1}(\varkappa_1 + ɏ_1) \le \mu_{Đ1}(\varkappa_2 + ɏ_2)$, $\mu_{Đ1}(\varkappa_1) \le \mu_{Đ1}(\varkappa_2)$, $\mu_{Đ1}(ɏ_1) \le \mu_{Đ1}(ɏ_2)$, weget, $\mu_{Đ1}(\varkappa_1 + ɏ_1) \ge \min\{ \mu_{Đ1}(\varkappa_1), \mu_{Đ1}(ɏ_1) \}$, without exception for $\varkappa_1$ and $ɏ_1$ in R.(b) $\max\{ \nu_{Đ1}(\varkappa_1 + ɏ_1), \nu_{Đ1}(\varkappa_2 + ɏ_2) \} = \nu_V(\varkappa_1 + ɏ_1, \varkappa_2 + ɏ_2) = \nu_V[(\varkappa_1, \varkappa_2) + (ɏ_1, ɏ_2)] = \nu_V(\varkappa + ɏ) \le \max\{ \nu_V(\varkappa), \nu_V(ɏ) \} = \max\{ \nu_V(\varkappa_1, \varkappa_2), \nu_V(ɏ_1, ɏ_2) \} = \max\{ \max\{ \nu_{Đ1}(\varkappa_1), \nu_{Đ1}(\varkappa_2) \}, \max\{ \nu_{Đ1}(ɏ_1), \nu_{Đ1}(ɏ_2) \} \}$.If$\nu_{Đ1}(\varkappa_1 + ɏ_1) \ge \nu_{Đ1}(\varkappa_2 + ɏ_2)$, $\nu_{Đ1}(\varkappa_1) \ge \nu_{Đ1}(\varkappa_2)$, $\nu_{Đ1}(ɏ_1) \ge \nu_{Đ1}(ɏ_2)$, weget, $\nu_{Đ1}(\varkappa_1 + ɏ_1) \le \max\{ \nu_{Đ1}(\varkappa_1), \nu_{Đ1}(ɏ_1) \}$, without exception for $\varkappa_1$ and $ɏ_1$ in R.(c) $\max\{ ί_{Đ1}(\varkappa_1 + ɏ_1), ί_{Đ1}(\varkappa_2 + ɏ_2) \} = ί_V(\varkappa_1 + ɏ_1, \varkappa_2 + ɏ_2) = ί_V[(\varkappa_1, \varkappa_2) + (ɏ_1, ɏ_2)] = ί_V(\varkappa + ɏ) \le \max\{ ί_V(\varkappa), ί_V(ɏ) \} = \max\{ ί_V(\varkappa_1, \varkappa_2), ί_V(ɏ_1, ɏ_2) \} = \max\{ \max\{ ί_{Đ1}(\varkappa_1), ί_{Đ1}(\varkappa_2) \}, \max\{ ί_{Đ1}(ɏ_1), ί_{Đ1}(ɏ_2) \} \}$.If$ί_{Đ1}(\varkappa_1 + ɏ_1) \ge ί_{Đ1}(\varkappa_2 + ɏ_2)$, $ί_{Đ1}(\varkappa_1) \ge ί_{Đ1}(\varkappa_2)$, $ί_{Đ1}(ɏ_1) \ge ί_{Đ1}(ɏ_2)$, we get, $ί_{Đ1}(\varkappa_1 + ɏ_1) \le \max\{ ί_{Đ1}(\varkappa_1), ί_{Đ1}(ɏ_1) \}$, without exception for $\varkappa_1$ and $ɏ_1$ in R.And,(ii) (a) $\min\{ \mu_{Đ1}(\varkappa_1 ɏ_1), \mu_{Đ1}(\varkappa_2 ɏ_2) \} = \mu_V(\varkappa_1 ɏ_1, \varkappa_2 ɏ_2) = \mu_V[(\varkappa_1, \varkappa_2)(ɏ_1, ɏ_2)] = \mu_V(\varkappa ɏ) \ge \min\{ \mu_V(\varkappa), \mu_V(ɏ) \} = \min\{ \mu_V(\varkappa_1, \varkappa_2), \mu_V(ɏ_1, ɏ_2) \} = \min\{ \min\{ \mu_{Đ1}(\varkappa_1), \mu_{Đ1}(\varkappa_2) \}, \min\{ \mu_{Đ1}(ɏ_1), \mu_{Đ1}(ɏ_2) \} \}$.If$\mu_{Đ1}(\varkappa_1 ɏ_1) \le \mu_{Đ1}(\varkappa_2 ɏ_2)$, $\mu_{Đ1}(\varkappa_1) \le \mu_{Đ1}(\varkappa_2)$, $\mu_{Đ1}(ɏ_1) \le \mu_{Đ1}(ɏ_2)$,we get$\mu_{Đ1}(\varkappa_1 ɏ_1) \ge \min\{ \mu_{Đ1}(\varkappa_1), \mu_{Đ1}(ɏ_1) \}$, without exception for $\varkappa_1$ and $ɏ_1$ in R.(b) $\max\{ \nu_{Đ1}(\varkappa_1 ɏ_1), \nu_{Đ1}(\varkappa_2 ɏ_2) \} = \nu_V(\varkappa_1 ɏ_1, \varkappa_2 ɏ_2) = \nu_V[(\varkappa_1, \varkappa_2)(ɏ_1, ɏ_2)] = \nu_V(\varkappa ɏ) \le \max\{ \nu_V(\varkappa), \nu_V(ɏ) \} = \max\{ \nu_V(\varkappa_1, \varkappa_2), \nu_V(ɏ_1, ɏ_2) \} = \max\{ \max\{ \nu_{Đ1}(\varkappa_1), \nu_{Đ1}(\varkappa_2) \}, \max\{ \nu_{Đ1}(ɏ_1), \nu_{Đ1}(ɏ_2) \} \}$.If$\nu_{Đ1}(\varkappa_1 ɏ_1) \ge \nu_{Đ1}(\varkappa_2 ɏ_2)$, $\nu_{Đ1}(\varkappa_1) \ge \nu_{Đ1}(\varkappa_2)$, $\nu_{Đ1}(ɏ_1) \ge \nu_{Đ1}(ɏ_2)$, we get$\nu_{Đ1}(\varkappa_1 ɏ_1) \le \max\{ \nu_{Đ1}(\varkappa_1), \nu_{Đ1}(ɏ_1) \}$, without exception for $\varkappa_1$ and $ɏ_1$ in R.(c) $\max\{ ί_{Đ1}(\varkappa_1 ɏ_1), ί_{Đ1}(\varkappa_2 ɏ_2) \} = ί_V(\varkappa_1 ɏ_1, \varkappa_2 ɏ_2) = ί_V[(\varkappa_1, \varkappa_2)(ɏ_1, ɏ_2)] = ί_V(\varkappa ɏ) \le \max\{ ί_V(\varkappa), ί_V(ɏ) \} = \max\{ ί_V(\varkappa_1, \varkappa_2), ί_V(ɏ_1, ɏ_2) \} = \max\{ \max\{ ί_{Đ1}(\varkappa_1), ί_{Đ1}(\varkappa_2) \}, \max\{ ί_{Đ1}(ɏ_1), ί_{Đ1}(ɏ_2) \} \}$.If$ί_{Đ1}(\varkappa_1 ɏ_1) \ge ί_{Đ1}(\varkappa_2 ɏ_2)$, $ί_{Đ1}(\varkappa_1) \ge ί_{Đ1}(\varkappa_2)$, $ί_{Đ1}(ɏ_1) \ge ί_{Đ1}(ɏ_2)$, we get$ί_{Đ1}(\varkappa_1 ɏ_1) \le \max\{ ί_{Đ1}(\varkappa_1), ί_{Đ1}(ɏ_1) \}$, without exception for $\varkappa_1$ and $ɏ_1$ in R.ConsequentlyĐ1isanNIFSSRof R.

**3.5 Theorem:**If Đ1 is an NIFSSR of a SR (R, +, · ), then H $=\{ \varkappa / \varkappa \in R: \mu_{Đ1}(\varkappa) = 1, \nu_{Đ1}(\varkappa) = 0, ί_{Đ1}(\varkappa) = 0\}$ is either empty or is a SSR of R.

**Proof:** If no element of the underlying set satisfies the stated condition, H is empty.If$\varkappa$ and $ɏ$ in H, then(i) (a) $\mu_{Đ1}(\varkappa + ɏ) \ge \min\{ \mu_{Đ1}(\varkappa), \mu_{Đ1}(ɏ) \} = \min\{ 1, 1 \} = 1$.Consequently, $\mu_{Đ1}(\varkappa + ɏ) = 1$.(b) $\nu_{Đ1}(\varkappa + ɏ) \le \max\{ \nu_{Đ1}(\varkappa), \nu_{Đ1}(ɏ) \} = \max\{ 0,0\} = 0$.Consequently, $\nu_{Đ1}(\varkappa + ɏ) = 0$.(c)$ί_{Đ1}(\varkappa + ɏ) \le \max\{ ί_{Đ1}(\varkappa), ί_{Đ1}(ɏ) \} = \max\{ 0,0\} = 0$.Consequently, $ί_{Đ1}(\varkappa + ɏ) = 0$.And (ii) (a) $\mu_{Đ1}(\varkappa ɏ) \ge \min\{ \mu_{Đ1}(\varkappa), \mu_{Đ1}(ɏ) \} = \min\{ 1,1 \} = 1$.Consequently, $\mu_{Đ1}(\varkappa ɏ) = 1$.(b) $\nu_{Đ1}(\varkappa ɏ) \le \max\{ \nu_{Đ1}(\varkappa), \nu_{Đ1}(ɏ) \} = \max\{ 0,0\} = 0$.Consequently, $\nu_{Đ1}(\varkappa ɏ) = 0$.(c) $ί_{Đ1}(\varkappa ɏ) \le \max\{ ί_{Đ1}(\varkappa), ί_{Đ1}(ɏ) \} = \max\{ 0,0\} = 0$.Consequently, $ί_{Đ1}(\varkappa ɏ) = 0$.We get $\varkappa + ɏ, \varkappa ɏ$ inH.Consequently, H is a SSR of R.
Hence H is either empty or is a SSR of R.

**3.6 Theorem:**If Đ1 be an NIFSSR of a SR ( R, +, · ), then

(i) if $\mu_{Đ1}(\varkappa + ɏ) = 0$, then either $\mu_{Đ1}(\varkappa) = 0$ or $\mu_{Đ1}(ɏ) = 0$, without exception for$\varkappa$ and $ɏ$ in R.

(ii) if $\mu_{Đ1}(\varkappa + ɏ) = 1$, then either $\mu_{Đ1}(\varkappa) = 1$ or $\mu_{Đ1}(ɏ) = 1$, without exception for$\varkappa$ and $ɏ$ in R.

**Proof:**Let ϰ and ɏinR.(i) B1ɏ thedefinition $\mu_{Đ1}(ϰ+ɏ) \geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$,this entails that $0 \geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$.Consequently, either $\mu_{Đ1}(ϰ) = 0$ or $\mu_{Đ1}(ɏ) = 0$.(ii)B1ɏ the definition$\mu_{Đ1}(ϰ+ɏ) \leq ɱaϰ\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$,which implies that $1 \leq ɱaϰ\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$.Consequently, either $\mu_{Đ1}(ϰ) = 1$ or $\mu_{Đ1}(ɏ) = 1$.

**3.7 Theorem:**If Đ1 is an NIFSSR of a SR( R, +, · ), then $H=\{\langle ϰ,\mu_{Đ1}(ϰ)\rangle : 0<\mu_{Đ1}(ϰ)\leq 1, \nu_{Đ1}(ϰ)=0$ and $\acute{\imath}_{Đ1}(ϰ)=0\}$ is either empty or a FSSRof R.

**Proof:**If no element of the underlying set satisfies the stated condition,H is empty.If ϰ and ɏsatisfies this condition,then$\nu_{Đ1}(ϰ+ɏ)\leq ɱaϰ\{\nu_{Đ1}(ϰ),\nu_{Đ1}(ɏ)\}=ɱaϰ\{0,0\}=0$.Consequently, $\nu_{Đ1}(ϰ+ɏ)=0$, without exception forϰ and ɏ in R. And, $\nu_{Đ1}(ϰɏ)\leq ɱaϰ\{\nu_{Đ1}(ϰ),\nu_{Đ1}(ɏ)\}=ɱaϰ\{0,0\}=0$. Consequently, $\nu_{Đ1}(ϰɏ) = 0$, without exception for ϰ and ɏ in R.We have, $\acute{\imath}_{Đ1}(ϰ+ɏ)\leq ɱaϰ\{\acute{\imath}_{Đ1}(ϰ),\acute{\imath}_{Đ1}(ɏ)\}=ɱaϰ\{0,0\}=0$.Consequently, $\acute{\imath}_{Đ1}(ϰ+ɏ) = 0$, without exception forϰ and ɏ in R.And,$\acute{\imath}_{Đ1}(ϰɏ)\leq ɱaϰ\{\acute{\imath}_{Đ1}(ϰ),\acute{\imath}_{Đ1}(ɏ)\}=ɱaϰ\{0,0\}= 0$. Consequently, $\acute{\imath}_{Đ1}(ϰɏ) = 0$, without exception forϰ and ɏ in R.And, $\mu_{Đ1}(ϰ+ɏ)\geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$. Consequently, $\mu_{Đ1}(ϰ+ɏ) \geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$, without exception forϰ and ɏ in R.And, $\mu_{Đ1}(ϰɏ)\geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$ Consequently, $\mu_{Đ1}(ϰɏ) \geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$,without exception for ϰ and ɏ in R.Hence H is a FSSR of R.Consequently, H is either empty or a FSSR of R.

**3.8 Theorem:**If Đ1 is an NIFSSR of a SR( R, +, · )then $H = \{\langle ϰ, \mu_{Đ1}(ϰ)\rangle : 0<\mu_{Đ1}(ϰ)\leq 1\}$ is either empty or aFSSRof R.

**Proof:**If no element of the underlying set satisfies the stated condition,H is empty.If ϰ and ɏ satisfies this condition, then $\mu_{Đ1}(ϰ+ɏ)\geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$.Consequently, $\mu_{Đ1}(ϰ+ɏ)\geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$, without exception for ϰ and ɏ in R.And$\mu_{Đ1}(ϰɏ)\geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$.Consequently, $\mu_{Đ1}(ϰɏ) \geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$, without exception for ϰ and ɏ in R.Consequently, H is either empty or a FSSR of R.

**3.9 Theorem:**If Đ1 is an NIFSSR of a SR(R, +, ·), then $H=\{\langle ϰ,\nu_{Đ1}(ϰ)\rangle : 0<\nu_{Đ1}(ϰ)\leq 1\}$ is either empty or anAFSSRof R.

**Proof:**If no element of the underlying set satisfies the stated conditionH is empty.If ϰ and ɏ satisfies this condition, then$\nu_{Đ1}(ϰ+ɏ)\leq ɱaϰ\{\nu_{Đ1}(ϰ),\nu_{Đ1}(ɏ)\}$.Consequently, $\nu_{Đ1}(ϰ+ɏ)\leq ɱaϰ\{\nu_{Đ1}(ϰ),\nu_{Đ1}(ɏ)\}$, without exception for ϰ and ɏ in R. And $\nu_{Đ1}(ϰɏ)\leq ɱaϰ\{\nu_{Đ1}(ϰ),\nu_{Đ1}(ɏ)\}$Consequently, $\nu_{Đ1}(ϰɏ)\leq ɱaϰ\{\nu_{Đ1}(ϰ),\nu_{Đ1}(ɏ)\}$, without exception for ϰ and ɏ in R.Hence H is either empty or an AFSSR of R.

**3.10 Theorem:**If Đ1 is an NIFSSR of a SR (R, +, ·), then $H=\{\langle ϰ,\acute{\imath}_{Đ1}(ϰ)\rangle : 0<\acute{\imath}_{Đ1}(ϰ) \leq 1\}$ is either empty or anAFSSRof R.

**Proof:**If no element of the underlying set satisfies the stated condition,H is empty.If ϰ and ɏ satisfies this condition, then$\acute{\imath}_{Đ1}(ϰ+ɏ)\leq ɱaϰ\{\acute{\imath}_{Đ1}(ϰ),\acute{\imath}_{Đ1}(ɏ)\}$.Consequently, $\acute{\imath}_{Đ1}(ϰ+ɏ)\leq ɱaϰ\{\acute{\imath}_{Đ1}(ϰ),\acute{\imath}_{Đ1}(ɏ)\}$, without exception for ϰ and ɏ in R.And$\acute{\imath}_{Đ1}(ϰɏ)\leq ɱaϰ\{\acute{\imath}_{Đ1}(ϰ),\acute{\imath}_{Đ1}(ɏ)\}$.Consequently, $\acute{\imath}_{Đ1}(ϰɏ) \leq ɱaϰ\{\acute{\imath}_{Đ1}(ϰ),\acute{\imath}_{Đ1}(ɏ)\}$, without exception for ϰ and ɏ in R.Hence H is either empty or an AFSSR of R.

**3.11 Theorem:**If Đ1 is an NIFSSRof a SR( R, +, · ), then □Đ1 is an NIFSSRof R.

**Proof:** Let Đ1 be an NIFSSRof a SRR.Consider $Đ1=\{\langle ϰ,\mu_{Đ1}(ϰ),\nu_{Đ1}(ϰ),\acute{\imath}_{Đ1}(ϰ)\rangle\}$, without exception for ϰ in R, we take □Đ1 $=B1=\{\langle ϰ,\mu_{B1}(ϰ),\nu_{B1}(ϰ),\acute{\imath}_{B1}(ϰ)\rangle\}$, where $\mu_{B1}(ϰ) = \mu_{Đ1}(ϰ)$, $\nu_{B1}(ϰ) = 1-\mu_{Đ1}(ϰ)$, $\acute{\imath}_{B1}(ϰ) = 1-\mu_{Đ1}(ϰ)$. Clearly, $\mu_{B1}(ϰ+ɏ)\geq ɱ\text{in}\{\mu_{B1}(ϰ),\mu_{B1}(ɏ)\}$, without exception for ϰ and ɏ in R and $\mu_{B1}(ϰɏ) \geq ɱ\text{in}\{\mu_{B1}(ϰ),\mu_{B1}(ɏ)\}$, without exception for ϰ and ɏ in R. Since Đ1 is an NIFSSRof R, we have $\mu_{Đ1}(ϰ+ɏ) \geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$, without exception for ϰ and ɏ in R,this entails that $1-\nu_{B1}(ϰ+ɏ) \geq ɱ\text{in}\{(1-\nu_{B1}(ϰ)),(1-\nu_{B1}(ɏ))\}$,this entails that $\nu_{B1}(ϰ+ɏ) \leq 1-ɱ\text{in}\{(1-\nu_{B1}(ϰ)),(1-\nu_{B1}(ɏ))\}=ɱaϰ\{\nu_{B1}(ϰ),\nu_{B1}(ɏ)\}$.Consequently, $\nu_{B1}(ϰ+ɏ) \leq ɱaϰ\{\nu_{B1}(ϰ),\nu_{B1}(ɏ)\}$,without exception for ϰ and ɏ in R.Also, we have $\mu_{Đ1}(ϰ+ɏ) \geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$this entails that $1-\acute{\imath}_{B1}(ϰ+ɏ) \geq ɱ\text{in}\{(1-\acute{\imath}_{B1}(ϰ)),(1-\acute{\imath}_{B1}(ɏ))\}$,this entails that $\acute{\imath}_{B1}(ϰ+ɏ) \leq 1-ɱ\text{in}\{(1-\acute{\imath}_{B1}(ϰ)),(1-\acute{\imath}_{B1}(ɏ))\}=ɱaϰ\{\acute{\imath}_{B1}(ϰ),\acute{\imath}_{B1}(ɏ)\}$.Consequently, $\acute{\imath}_{B1}(ϰ+ɏ) \leq ɱaϰ\{\acute{\imath}_{B1}(ϰ),\acute{\imath}_{B1}(ɏ)\}$,without exception for ϰ and ɏ in R.And $\mu_{Đ1}(ϰɏ) \geq ɱ\text{in}\{\mu_{Đ1}(ϰ),\mu_{Đ1}(ɏ)\}$, without exception for ϰ and ɏ in R,this entails that $1-\nu_{B1}(ϰɏ) \geq ɱ\text{in}\{(1-\nu_{B1}(ϰ)),(1-\nu_{B1}(ɏ))\}$this entails that $\nu_{B1}(ϰɏ)\leq 1-$

ɱin{(1– $\nu_{B1}(ϰ)$), (1– $\nu_{B1}(ɏ)$)}=ɱaϰ {$\nu_{B1}(ϰ)$,$\nu_{B1}(ɏ)$ }.Consequently, $\nu_{B1}(ϰɏ) \le$ ɱaϰ { $\nu_{B1}(ϰ)$,$\nu_{B1}(ɏ)$ },without exception for ϰ and ɏ in R.Also, we have $\mu_{Ð1}(ϰɏ) \ge$ ɱin { $\mu_{Ð1}(ϰ)$,$\mu_{Ð1}(ɏ)$ }, without exception for ϰ and ɏ in R,this entails that 1– $ί_{B1}(ϰɏ) \ge$ ɱin { ( 1– $ί_{B1}(ϰ)$ ), ( 1– $ί_{B1}(ɏ)$ ) }this entails that $ί_{B1}(ϰɏ) \le$ 1– ɱin{ ( 1– $ί_{B1}(ϰ)$ ), ( 1– $ί_{B1}(ɏ)$ ) }=ɱaϰ {$ί_{B1}(ϰ)$,$ί_{B1}(ɏ)$}.Consequently, $ί_{B1}(ϰɏ) \le$ ɱaϰ { $ί_{B1}(ϰ)$,$ί_{B1}(ɏ)$ },without exception for ϰ and ɏ in R.Hence B1 = □Ð1 is an NIFSSRof a SRR.

**3.12 Theorem:**If Ð1 is an NIFSSRof a SR( R, +, · ), then ◊Ð1 is an NIFSSRof R.

**Proof:**Let Ð1 be an NIFSSRof a SRR.

That is Ð1={⟨ϰ,$\mu_{Ð1}(ϰ)$,$\nu_{Ð1}(ϰ)$,$ί_{Ð1}(ϰ)$⟩}, without exception for ϰ in R.Let ◊Ð1 = B1 = { ⟨ϰ, $\mu_{B1}(ϰ)$, $\nu_{B1}(ϰ)$,$ί_{B1}(ϰ)$⟩ }, where $\mu_{B1}(ϰ) = 1–\nu_{D}(ϰ)$, $\nu_{B1}(ϰ) = \nu_{D}(ϰ)$, $ί_{B1}(ϰ) = 1–\nu_{D}(ϰ)$.Clearly, $\nu_{B1}(ϰ+ɏ) \le$ ɱaϰ{ $\nu_{B1}(ϰ)$,$\nu_{B1}(ɏ)$ },without exception for ϰ and ɏ in R and $\nu_{B1}(ϰɏ) \le$ɱaϰ{ $\nu_{B1}(ϰ)$,$\nu_{B1}(ɏ)$ }, without exception for ϰ and ɏ in R.

Since Ð1 is an NIFSSRof R, we have$\nu_{Ð1}(ϰ+ɏ) \le$ ɱaϰ{ $\nu_{Ð1}(ϰ)$,$\nu_{Ð1}(ɏ)$ },without exception for ϰ and ɏ in R,this entails that $1–\mu_{B1}(ϰ+ɏ) \le$ ɱaϰ { ( 1– $\mu_{B1}(ϰ)$ ) , ( 1– $\mu_{B1}(ɏ)$ ) }this entails that $\mu_{B1}(ϰ+ɏ) \ge$ 1– ɱaϰ { ( 1– $\mu_{B1}(ϰ)$ ), ( 1– $\mu_{B1}(ɏ)$ ) }=ɱin { $\mu_{B1}(ϰ)$,$\mu_{B1}(ɏ)$ }.Consequently, $\mu_{B1}(ϰ+ɏ) \ge$ ɱin{ $\mu_{B1}(ϰ)$,$\mu_{B1}(ɏ)$ }, without exception for ϰ and ɏ in R.

Also, we have $\nu_{Ð1}(ϰ+ɏ) \le$ ɱaϰ{ $\nu_{Ð1}(ϰ)$, $\nu_{Ð1}(ɏ)$ }, without exception for ϰ and ɏ in R,this entails that $1–ί_{B1}(ϰ+ɏ) \le$ ɱaϰ { ( 1– $ί_{B1}(ϰ)$ ) , ( 1– $ί_{B1}(ɏ)$ ) }this entails that $ί_{B1}(ϰ+ɏ) \ge$ 1– ɱaϰ { ( 1– $ί_{B1}(ϰ)$ ), ( 1– $ί_{B1}(ɏ)$ ) }= ɱin { $\mu_{B1}(ϰ)$, $\mu_{B1}(ɏ)$ }.Consequently, $\mu_{B1}(ϰ+ɏ) \ge$ ɱin{ $\mu_{B1}(ϰ)$, $\mu_{B1}(ɏ)$ }, without exception for ϰ and ɏ in R.And $\nu_{Ð1}(ϰɏ) \le$ ɱaϰ { $\nu_{Ð1}(ϰ)$, $\nu_{Ð1}(ɏ)$ }, without exception for ϰ and ɏ in R,this entails that $1–\mu_{B1}(ϰɏ) \le$ ɱaϰ { ( $1–\mu_{B1}(ϰ)$ ) , ( $1–\mu_{B1}(ɏ)$ ) },this entails that $\mu_{B1}(ϰɏ) \ge$ 1– ɱaϰ { ($1–\mu_{B1}(ϰ)$ ), ( 1– $\mu_{B1}(ɏ)$ ) }= ɱin { $\mu_{B1}(ϰ)$, $\mu_{B1}(ɏ)$ }.Consequently, $\mu_{B1}(ϰɏ) \ge$ ɱin { $\mu_{B1}(ϰ)$, $\mu_{B1}(ɏ)$ }, without exception for ϰ and ɏ in R.Also, $\nu_{Ð1}(ϰɏ) \le$ ɱaϰ { $\nu_{Ð1}(ϰ)$, $\nu_{Ð1}(ɏ)$ }, without exception for ϰ and ɏ in R,this entails that $1–ί_{B1}(ϰɏ) \le$ ɱaϰ { ( $1–ί_{B1}(ϰ)$ ) , ( $1–ί_{B1}(ɏ)$ ) },this entails that $ί_{B1}(ϰɏ) \ge$ 1– ɱaϰ { ($1–ί_{B1}(ϰ)$ ), ( $1–ί_{B1}(ɏ)$ ) }= ɱin { $\mu_{B1}(ϰ)$, $\mu_{B1}(ɏ)$ }.Consequently, $\mu_{B1}(ϰɏ) \ge$ ɱin { $\mu_{B1}(ϰ)$, $\mu_{B1}(ɏ)$ }, without exception for ϰ and ɏ in R.Hence B1 = ◊Ð1 is an NIFSSRof a SR R.

## 4 Conclusion

In this paper, we introduced and studied the concept of Neutrosophic Intuitionistic Fuzzy Subsemiring (NIFSSR) of a semiring. This notion combines the ideas of semirings with neutrosophic and intuitionistic fuzzy theories, which helps in handling uncertainty, indeterminacy, and inconsistency in algebraic structures. Basic definitions and properties of NIFSSR were discussed, and several results were established to illustrate their algebraic behavior. The study shows that NIFSSRs generalize classical fuzzy and intuitionistic fuzzy subsemirings. Hence, this framework provides a broader and more flexible mathematical model, which may be useful for further theoretical research and potential applications in decision-making and information systems under uncertainty.

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