Towards a K-Theoretic Justification of Fuzzy Logic

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Abstract: - This work develops a categorical pipeline that connects the algebraic semantics of fuzzy logic with the ordered K-theory of AF C^*-algebras (Approximately Finite-Dimensional C^*-algebras). Starting from the classical equivalence between MV-algebras (Many-Valued algebras, i.e., algebras of Łukasiewicz logic) and unital ℓ-groups (lattice-ordered abelian groups with a strong unit) established by Mundici, we show how these ordered structures naturally embed into dimension groups (ordered abelian groups with interpolation and unperforation), the ordered K_0 invariants (Grothendieck ordered K-theory group of projections) that classify AF algebras. By composing the functors Ξ, F, and K_0^{-1}, we construct a functorial correspondence:

$$MV \to u^l G \to DimGrp \to AF$$

that assigns to each MV-algebra a unique AF C^*-algebra whose ordered K_0-group recovers its underlying fuzzy-logical structure. This provides an operator-algebraic semantics for many-valued reasoning, where MV-operations correspond to projection structure and truncated addition in the associated AF-algebra. As an illustrative example, we compute explicitly the AF algebra associated with the three-valued Łukasiewicz algebra $Ł_3$ and show that it corresponds to the matrix algebra $M_2(C)$. The developed framework clarifies the conceptual and categorical role of ordered K-theory in fuzzy logic and suggests new connections between many-valued reasoning, dimension groups, and the structure theory of C^*-algebras.

Key-Words: - K-theory; Fuzzy logic; MV-algebras; AF C*-algebras; Dimension groups; Lattice-ordered groups; Ordered K_0 invariants; Many-valued logic; Functorial equivalence; Operator algebras

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1 Introduction

1.1 Why C^*-algebra is the best match from the main 4 categories in K-theory?

Fuzzy logic algebraic structures (MV-algebras, residuated lattices) → ordered abelian groups. The central algebraic formalism in K-theory for fuzzy logic is the MV-algebra, i.e. Łukasiewicz logic. Mundici's theorem establishes a categorical equivalence between MV-algebras and unital ℓ-groups (lattice-ordered abelian groups with a strong unit). This correspondence already lifts fuzzy truth values from the interval [0, 1] into the world of ordered abelian groups, the natural domain where the functorial invariant K_0 (the Grothendieck ordered K-theory group of projections) lives.

Also, this correspondence rests on the classical foundations [[1], [2], [3], [4], [5], [6], [8], [9]].

The link of K-Theory to Fuzzy-logic is not that new see [7]-this paper generalizes fuzzy sets using **lattices** (L), which is mathematically closer to algebraic structures and might serve as a **bridge toward K-theoretic formalism.**

1.2 AF C^*-algebras ↔ dimension groups (ordered K_0).

AF (Approximately Finite-Dimensional) C^*-algebras possess K_0 groups (Grothendieck ordered K-theory groups) that are scaled, ordered abelian groups — the so-called dimension groups. Elliott's classification theorem states that for AF algebras, the ordered group $(K_0(A), K_0(A)^+, [1_A])$ is a complete invariant. Consequently, the algebraic invariants of AF C^*-algebras coincide precisely with the ordered groups arising naturally from MV-algebras via Mundici's equivalence. This establishes a direct conceptual and categorical bridge between fuzzy logic and operator *K-theory*.

2 Problem Formulation

The central objective of this work is to construct a functorial and mathematically rigorous pipeline that translates fuzzy logical structures—expressed as MV-algebras (Many-Valued algebras) — into operator-algebraic structures governed by ordered K-theory and AF C^*-algebras (Approximately Finite-Dimensional C^*-algebras). Formally, given an MV-algebra M, we seek to identify an AF algebra $A(M)$ such that:

P1. Logical structure preservation.

The algebraic operations of M (MV-sum and negation) correspond functorially to ordered-group operations inside the associated dimension group (ordered abelian group with interpolation).

P2. Categorical compatibility.

The construction must respect morphisms: every MV-homomorphism $\varphi: M \to N$ must induce a unital -*homomorphism* $A(\varphi): A(M) \to A(N)$. Thus, the resulting assignment must be a functor:

$$A: MV \to AF.$$

P3. Ordered K_0 reconstruction.

The AF algebra $A(M)$ must recover the ordered-group structure that encodes the semantics of M:

$$\left(K_0\big(A(M)\big), K_0\big(A(M)\big)^+, [1]\right) \cong (GM, G_M^+, u_M)$$

where $(G_M, u_M) = \Xi(M)$ is the unital ℓ-group

associated to M via Mundici's equivalence.

P4. Computability (finite case).

For finite or finitely presented MV-algebras, the construction must produce explicit, finite-dimensional matrix realisations (e.g. $M_k(\mathbb{C})$ or small inductive limits), suitable for use in computer-scientific applications and numerical experimentation.

These requirements lead directly to the categorical pipeline

$$MV \xrightarrow{\Xi} F \xrightarrow{K_0^{-1}} AF,$$

whose construction and properties form the core of the present work.

3 Background

This section provides the mathematical background required for the conceptual bridge developed later: from MV-algebras (the algebraic semantics of Łukasiewicz fuzzy logic) to ordered abelian groups, and further to AF C^*-algebras via their ordered K_0 invariants. We review the four pillars underlying this correspondence: MV-algebras, unital ℓ-groups and the Mundici equivalence, AF algebras, and ordered K-theory.

3.1 MV-Algebras

MV-algebras were introduced by Chang see also [CDM00, Mun86].) as an algebraic semantics for infinite-valued Łukasiewicz logic. An MV-algebra (Many-Valued algebra) is a structure

$$(M, \oplus, *, 0)$$

satisfying the following axioms:

1. $(M, \oplus, 0)$ is a commutative monoid;

2. the unary operation * (the Łukasiewicz negation) satisfies:

$$(x^*)^* = x, \qquad x \oplus 0^* = 0^*;$$

3. the Łukasiewicz law

$$(x^* \oplus y^*) \oplus y = (y^* \oplus x^*) \oplus x$$

holds for all $x, y \in M$.

The canonical example is the real unit interval [0, 1] with operations:

$$x \oplus y = \min(1, x + y), \quad x^* = 1 - x$$

MV-algebras capture the semantics of fuzzy truth values and allow algebraic treatment of connectives such as conjunction, disjunction,
and negation. Crucially, they admit a faithful representation in terms of lattice-ordered Abelian groups, enabling the use of ordered algebra and K-theory.

3.2 ℓ-Groups and the Mundici Equivalence

A lattice-ordered abelian group (ℓ-group) is an Abelian group G equipped with a lattice order G^+ such that translation preserves the order.
A *unital ℓ-group* Unital ℓ-groups [Goo86] … is a pair (G, u) where $u \in G^+$ is a strong order unit, meaning that for every $g \in G$
there exists $n \in N$ such that:

$$-nu \leq g \leq nu$$

Mundici's celebrated equivalence establishes a categorical duality:

$$\boldsymbol{MV - algebras} \simeq \textbf{unital } \boldsymbol{\ell - groups}$$

The functor

$$\Gamma(G, u) = \{g \in G: 0 \leq g \leq u\}$$

equipped with operations:

$$x \oplus y = \min(u, x + y), \quad x^* = u - x$$

turns each unital ℓ-group into an MV-algebra.
Conversely, every MV-algebra M can be uniquely (up to isomorphism) represented as the unit interval of a unital ℓ-group.
Thus, every fuzzy algebraic structure arising from Łukasiewicz logic has a canonical ordered-group representation, which interfaces naturally with operator algebras and ordered K-theory.

3.3 AF Algebras

An *AF C^*-algebra* (Approximately Finite-Dimensional) is a separable C^*-algebra obtained as an inductive limit of finite-dimensional C^*-algebras:

$$A = \overset{\lim}{\longrightarrow}(A_n, \Phi_n), \ \ A_n = \oplus_{i=1}^{k_n} M_{m_{n,i}}(C)$$

AF algebras form a tractable yet structurally rich class of C^* −algebras. One of the central results of the classification theory of C^* −algebras is:

Elliott's Theorem (AF case). (Elliott's classification [Ell76])
Two AF algebras A and B are * - isomorphic if and only if their ordered scaled K_0 groups:

$$(K_0(A), K_0(A)^+, [1_A]) \text{ and } (K_0(B), K_0(B)^+, [1_B])$$

are order-unit isomorphic.

Thus, AF algebras correspond precisely to *dimension groups*, i.e. ordered abelian groups satisfying:

- unperforation,
- the Riesz interpolation property,
- existence of a distinguished order unit.

As explained later, the dimension groups that arise from MV-algebras via Mundici's equivalence fit naturally into this framework.

3.4 Ordered K0-Theory and the Functor K_0

For a unital C∗-algebra A, the group K0(A) (the Grothendieck ordered K-theory group of projections) is defined using Murray–von Neumann equivalence classes of projections in matrix algebras over A:

$$\mathrm{K}_0(A) = \langle [p] - [q]\text{: p, q projections in } M_n(A) \rangle$$

(see [Bla98, RLL00] for detailed treatments).

The positive cone:

$$\mathrm{K}_0(A)^+ = \{[p]: p\ a\ projection\}$$

gives $\mathrm{K}_0(A)$ the structure of an ordered Abelian group. Identity class,

$$[1_A] \in \mathrm{K}_0(A)^+$$

is a distinguished order unit.

For AF algebras, $\mathrm{K}_0(A)^+$ is a dimension group, and Elliott's classification gives the correspondence"

$$A \leftrightarrow (\mathrm{K}_0(A), \mathrm{K}_0(A),^+, [1_A]).$$

Because dimension groups coincide with unital ℓ-groups (every dimension group is an ℓ-group with interpolation), and MV-algebras correspond to unital ℓ-groups via Mundici's functor Γ, we obtain the categorical chain:

$$\text{MV-algebras} \xrightarrow{\Gamma-1} \text{unital } l\text{-groups} \overset{\cong}{\rightarrow} \text{dimension groups} \xrightarrow{K_0^{-1}} \text{AF } C^*\text{-algebras}$$

This chain provides the theoretical foundation for interpreting fuzzy logical structures via ordered K-theory and AF algebras.

3.5 Further Properties of MV-Algebras

Beyond their basic algebraic presentation, MV-algebras possess several structural features that are essential for their interaction with ordered groups and operator algebras. Each MV-algebra M carries a natural partial order defined by

$$x \le y \text{ iff } x^* \oplus y = 1$$

With this order, M becomes a distributive lattice where:

$$x \wedge y = (x^* \oplus y^*)^*,\ x \vee y = x \oplus (x^* \wedge y).$$

Moreover, every MV-algebra is a subdirect product of MV-chains. Finite MV-chains correspond exactly to the algebras:

$$Ł_n = \{0, \frac{1}{n-1}, \dots, 1\}$$

with truncated addition. This decomposition property often allows reducing structural questions to the totally ordered case. MV-homomorphisms preserve $\oplus, *$, and 0, and therefore preserve the induced lattice order.

The category of MV-algebras is complete and cocomplete, which is useful for constructing adjoint functors and categorical equivalences such as the one due to Mundici.

3.6 Additional Structure of ℓ-Groups and the Mundici Equivalence

Unital ℓ-groups (G, u) form a rich category whose morphisms are positive unital group homomorphisms. The presence of a strong order unit allows one to define the *unit interval*

$$[0, u] = \{g \in G: 0 \le g \le u\},$$

which always forms an MV-algebra under the operations inherited from G.
Mundici's equivalence is realized by two functors:

$$(G, u) \mapsto [0, u],\ \Xi\ \mapsto (G_M, u_M)$$

where G_M is the *universal ℓ-group* generated by M subject to the MV-identities. The unit u_M corresponds to the top element $1 \in M$.
A remarkable feature of this equivalence is that many MV-algebraic constructions (quotients, products, limits) translate cleanly into analogous constructions in the category of unital ℓ-groups. In particular, simple MV-algebras correspond to simple unital ℓ-groups, and finitely generated MV-algebras correspond to finitely generated dimension groups.
This structural compatibility is what enables the bridge to ordered K_0-groups of AF algebras.

3.7 Structural and Categorical Aspects of AF Algebras

From a categorical viewpoint, AF algebras form a coreflective subcategory of all C^*-algebras.
Background on C^*-algebras may be found in [Mur90, KR97]. Every AF algebra may be described by a Bratteli diagram, an infinite directed graph encoding the finite-dimensional building blocks and the connecting maps of an inductive system.
A Bratteli diagram determines, and is determined by, a scaled ordered dimension group $(K_0(A), K_0(A),^+, [1_A])$. Thus, both the algebra A and its K-theory can be read off directly from the diagram.
This combinatorial nature is one of the reasons AF algebras provide such a natural landing place for logical and algebraic invariants coming from MV-algebras.
Many classical C∗ structures arise as AF algebras:

- finite-dimensional algebras,
- UHF algebras (Uniformly Hyperfinite algebras),
- certain crossed products and inductive limits associated with Cantor minimal systems.

The tractability of AF algebras, coupled with their complete classification by ordered K0, makes them ideal for providing a C^*-algebraic semantics of MV-algebras and their associated fuzzy logical theories.

3.8 Ordered K-Theory and Categorical Properties of K0

Ordered K-theory enriches the usual Grothendieck group K_0 with a positive cone and a distinguished order unit. In the AF case, this additional structure is not an optional refinement but an essential feature: the ordered scaled group completely determines the algebra.
Morphisms of ordered groups induced by *-homomorphisms preserve the positive cones and the order units, so the functor

$$K_0 : AF \longrightarrow DimGrp$$

is fully faithful. In fact, Elliott's theorem states that K_0 is an equivalence of categories between AF algebras and dimension groups.
Important consequences include:

- every dimension group arises as the K0 of an AF algebra;
- limits and colimits in the AF category correspond to the respective limits and colimits of dimension groups;
- interpolation and unperforation in the ordered group correspond to deep structural features of projections and inductive limits inside the algebra.

Because MV-algebras correspond to unital ℓ-groups, and unital ℓ-groups coincide with dimension groups with a distinguished order unit, the functorial passage:

$$MV - algebras \longrightarrow unital\ \ell - groups \longrightarrow dimension\ grooups \longrightarrow AF\ C^* - algebras$$

places fuzzy logic naturally within the landscape of ordered *K*-theory.

4 Problem Solution

Our solution is based on the introduction of a new Pipeline:

$$MV \longrightarrow \ell - Group \longrightarrow K_0 \longrightarrow AF$$

In this section we develop the functorial bridge that connects MV-algebras, unital ℓ-groups, dimension groups, and AF C^*-algebras. This establishes a canonical pipeline

$$MV \xrightarrow{\Xi} u\ell G \xrightarrow{F} DimGrp \xrightarrow{K_0^{-1}} AF$$

which takes a fuzzy logical structure and assigns to it an approximately finite-dimensional operator algebra. Each step of the pipeline
is functorial, ensuring compatibility between homomorphisms, logical connectives, order structure, and operator-algebraic data.

4.1 Overview of the Functorial Bridge

The starting point is an MV-algebra M, representing the algebraic semantics of Łukasiewicz fuzzy logic.

Mundici's equivalence associates to M a unique (up to isomorphism) unital ℓ-group (G, u) whose unit interval [0, u] is naturally isomorphic to M.
The ordered group (G, u) is then recognized as a dimension group, i.e. an unperforated ordered Abelian group with the Riesz interpolation property and a distinguished order unit.
Dimension groups are precisely the ordered K_0-groups of AF C^*-algebras. Elliott's classification theorem for AF algebras establishes a categorical equivalence between AF algebras and dimension groups, thereby completing the pipeline.

4.2 Stage 1: The Mundici Functor $\Xi : MV \longrightarrow u\ell G$

Given an MV-algebra $(M, \oplus, *, 0)$, Mundici constructs a unital ℓ-group $\Xi(M) = (G_M, u_M)$ satisfying:

$$M \simeq \Gamma(G_M, u_M) = \{g \in G_M : 0 \leq g \leq u_M\}$$

with MV-operations given by:

$$x \oplus y = \min(u_M, x + y), x^* = u_M - x$$

Every MV-homomorphism $\varphi : M \longrightarrow N$ lifts uniquely to a positive unital ℓ-group homomorphism

$$\Xi(\varphi) : (G_M, u_M) \longrightarrow (G_N, u_N)$$

Mundici's dual functor

$$\Gamma : u\,\ell G \rightarrow MV$$

satisfies

$$\Gamma \circ \Xi \cong Id_{MV},\ \Xi \circ \Gamma \cong Id_{u\,\ell G}$$

so Ξ and Γ form an equivalence of categories. Thus, the category of MV-algebras is faithfully represented inside the category of unital ℓ-groups.

4.3 Stage 2: From Unital ℓ-Groups to Dimension Groups

Unital ℓ-groups automatically enjoy two key properties:

- *unperforation*: if $ng \geq 0$ for some $n \in \mathbb{N}$, then $g \geq 0$;
- *interpolation* (Riesz decomposition): whenever $a_1, a_2 \leq b_1, b_2$, there exists c with $a_i \leq c \leq b_j$.

These are precisely the axioms of a dimension group.
Hence the forgetful functor:

$$\text{F: u}\ell G \longrightarrow DimGrp$$

is fully faithful and essentially surjective: every unital ℓ-group is canonically a dimension group, and conversely every dimension group carries a unique ℓ-group structure compatible with its positive cone.
This step of the pipeline is therefore not merely an embedding but an identification of categories:

$$\text{u}\ell G \simeq DimGrp.$$

4.4 Stage 3: Dimension Groups as Ordered K_0-Groups

Ordered abelian groups arising as K0-groups of AF algebras are exactly the dimension groups. For an AF algebra A, the ordered group

$$(K_0(A), K_0(A)^+, [1_A])$$

encodes all of the projection structure of A. Conversely, every dimension group (G, G+, u) arises as the ordered K0-group of a unique (up to *-isomorphism) AF algebra A with $[1_A] = u$.

This correspondence is typically realized via Bratteli diagrams: to a dimension group one associates a Bratteli diagram who's inductive limit is the desired AF algebra. Positive group homomorphisms preserving order units correspond exactly to unital *-homomorphisms between the associated AF algebras.

4.5 Stage 4: Recovering AF Algebras via K_0^{-1}

Elliott's classification theorem yields an equivalence of categories

$$K_0: AF \longrightarrow DimGrp, \quad K_0^{-1}: DimGrp \longrightarrow AF$$

The inverse functor K_0^{-1} assigns to a dimension group its unique AF algebra. Given a morphism of dimension groups

$$\theta: (G, G^+, u) \to (H, H^+, v),$$

the functor K_0^{-1} produces a unital *-homomorphism between the corresponding AF algebras.
Thus, the reconstruction is entirely functorial.

4.6 The Composed Functor and Its Interpretation

The entire pipeline is summarized by the composite functor

$$\boldsymbol{\mathcal{A}} = K_0^{-1} \circ F \circ \Xi: \text{MV} \longrightarrow AF$$

To each MV-algebra M, the functor $\boldsymbol{\mathcal{A}}$ assigns a unique AF algebra A(M). To each MV-homomorphism $\varphi: \text{M} \to \text{N}$, the functor assigns a unital *-homomorphism $\boldsymbol{\mathcal{A}}(\varphi): \boldsymbol{\mathcal{A}}(\text{M}) \to \boldsymbol{\mathcal{A}}(\text{N})$.

This construction provides an operator-algebraic semantics for fuzzy logic: logical operations in M correspond to positive group operations in Ξ(M) and, ultimately, to projection and inductive limit structures within the AF algebra $\boldsymbol{\mathcal{A}}$(M).

4.7 Examples

4.7.1 Finite MV-Chains.

Let Łn be the finite MV-chain with n truth values. (Here Łn denotes the Łukasiewicz n-valued MV-chain.) Then:

$$\Xi(\text{Łn}) = (\square, \mathbf{n-1})$$

and the corresponding dimension group is $(\square, \square_{\geq 0}, \mathbf{n-1})$. Under K_0^{-1} this yields the finite-dimensional C^*-algebra C^n.

4.7.2 The Standard MV-Algebra [0, 1].

The MV-algebra [0, 1] corresponds to the unital ℓ-group $(\mathbb{R}, \mathbf{1})$. The dimension group $(\mathbb{R}, \mathbb{R}_{\geq 0}, 1)$ is realized as the ordered K_0-group of an AF algebra

associated with a suitable UHF algebra (Uniformly Hyperfinite algebra), for example the
2^{∞} UHF algebra.
These examples illustrate how MV-structures translate into concrete operator-algebraic objects through the functorial pipeline.

5 Practical translation layer: from fuzzy rules to ordered K_0

The preceding development exhibits a clean categorical pipeline:

$$MV \longrightarrow \mathbf{u}\ell \longrightarrow DimGrp \longrightarrow AF$$

To make this pathway actionable for computer scientists (CS) working with fuzzy logic (FL) we now describe a practical translation layer. The focus here is on *finite* or *finitely presented* MV-algebras (including finite MV- chains and quantized truth systems) because these are the objects that most directly appear in CS applications (rule bases, quantized controllers, and discrete fuzzy classifiers).

5.1 Intuition for CS readers

Translate the operator-theoretic language to CS-friendly metaphors:

- **Truth values** (MV-elements) correspond to *resource units* or ranks in an algebraic model. In AF algebras these become projection ranks — concrete integers that count finite-dimensional subspaces.
- **MV-addition with truncation** models clipped accumulation (e.g. saturating confidence). In the ordered group this is addition truncated at an order unit (a capacity). In the AF algebra it is rank addition clipped by the full projection (the unit).
- **Negation** is a complement relative to the order unit (useful for thresholding and" what remains" after allocation).

5.2 A simple algorithmic pipeline (finite MV-chains)

For many CS tasks one works with quantized truth values Łn $= \left\{0, \frac{1}{n-1}, \ldots, 1\right\}$. The following steps convert such an MV-chain into a small AF algebra and explicit projections.

Input: finite MV-chain Łn (or a small finite MV-algebra M).

Output: a finite-dimensional AF algebra (matrix algebra), its projection representatives, and the K_0 classes.

1. **Mundici step**. For Łn take $\Xi(Ł_n) = (□, n-1)$. Interpret each MV-value $k/(n-1)$ as integer rank k.
2. **Dimension group**. Positive cone is $□_{\geq 0}$ with unit $n-1$
3. **AF reconstruction**. The ordered K_0-triple $(□, □_{\geq 0}, n-1)$ is realised by the matrix algebra $M_{n-1}(\mathbb{C})$.

   Choose projection representatives of ranks $0, 1, \ldots n-1$.
4. **Interpretation**. MV-operations map to addition/complement of ranks.

5.3 Pseudocode (constructive toy implementation)

The following pseudocode produces a concrete matrix model for a finite MV-chain.

```
# MATLAB-style pseudocode

function [P-ranks] = mvchain_to_matrices(n)
    % n = number of truth values in  Łn
    m = n - 1; "% matrix size"
    % define rank-k projection as diagonal matrix
    % with k ones
    P = cell (m+1,1);
    ranks = 0:m;
    for k = 0:m:
       diag_vec =[ones(1,k), zeros(1,m-k)];
      P{k+1} = diag(diag_vec);
          % k-rank projection in M_m(C)
      end
end
```

This trivial construction already exhibits the correspondence: the MV-sum $a \oplus b$ corresponds to min(m, rank(pa) + rank(pb)).

5.4 Computational remarks and complexity

- For *finite* MV-algebras the construction is linear-time in the number of atoms (building diagonal/projective representatives is trivial).

- For finitely presented MV-algebras (generators + relations) the main cost is solving an integer linear system to obtain the lattice-ordered group presentation; this is polynomial-time using standard integer linear algebra and Smith normal form (suitable libraries exist).
- Building Bratteli diagrams for small inductive systems is combinatorial and feasible for sizes typically used in CS prototypes (tens–low hundreds of nodes).

6 Applications, Experiments and Worked examples for Computer Science

This section proposes simple, reproducible experiments and application scenarios that demonstrate why translating FL (Fuzzy Logic) into ordered K-theory and AF algebras is beneficial for CS (Computer Science) research.
Three short experiments are suggested:

(1) interpretability via rank semantics,
(2) robustness to quantization noise,
(3) compact model representation and incremental updates.

6.1 Experiment 1 — Interpretability: projection-rank semantics

Goal show that fuzzy truth computations (saturation, negation, combination) admit a transparent integer/rank interpretation that can aid explainability of fuzzy classifiers.

6.2 Protocol

1. Implement a simple fuzzy rule evaluator using a finite MV-chain (e.g. Ł5 with 5 truth-values).
2. Convert the chain to matrix projections using the mvchain_to_matrices routine.
3. For a set of sample inputs, compute fuzzy outputs and show the corresponding projection ranks; visualize inputs $x \mapsto rank(k)$.

Expected outcome mapping to ranks gives an immediate integer explanation (e.g. “rule fired at strength 3 of 4”), making the fuzzy decision traceable.

6.3 Experiment 2 — Robustness analysis under quantization noise

Goal test whether the rank interpretation yields insight into tolerance of fuzzy computations to perturbations (noise) in truth-values.

Protocol

1. Generate synthetic inputs and compute fuzzy outputs in a continuous MV model.
2. Quantize outputs to a chosen Łn and map to ranks.
3. Apply random perturbations to inputs and measure how often ranks flip.
4. Compare rank-flip rates across different n (coarser vs finer quantization).

Expected outcome coarse quantization gives stable ranks but loses nuance; rank-stability can be used as a metric for robustness.

6.4 Experiment 3 — Compact representation and incremental updates

Goal show AF / Bratteli style inductive representations are natural for incrementally growing fuzzy rule-bases (add/remove rules without re-training whole model).

Protocol

1. Represent a small rule-base as a direct sum of matrix blocks (finite-dimensional pieces).
2. Simulate adding/removing rules as adding/removing summands in the inductive system.
3. Show how K0 (rank distributions) updates trivially; measure update cost vs a monolithic representation.

6.5 Why CS scientists should care (short list)

- **Explainability**: ranks provide integer, human-interpretable tokens.
- **Model modularity**: AF inductive picture is naturally modular.
- **Robustness metrics**: ordered-group distances and rank stability become formal measures

• **Hardware friendliness**: finite-dimensional matrix models map easily to quantized/digital hardware.

7 Worked Example: The 3-valued Łukasiewicz Algebra

In this section we illustrate the pipeline $MV \longrightarrow \mathbf{u}\ell \longrightarrow DimGrp \longrightarrow AF$ on the simplest nontrivial finite example, the three-valued Łukasiewicz algebra Ł3. The computations are explicit and show how MV-operations are realized in the ordered K_0-picture and finally by a concrete finite-dimensional C^*-algebra.

7.1 Definition of Ł3

The MV-chain Ł3 consists of three truth-values Ł$3 = \{0, \frac{1}{2}, 1\}$, with MV-operations defined by truncated addition and negation:

$$x \oplus y = \min(1, x + y), x^* = 1 - x.$$

Concretely, $\frac{1}{2} \oplus \frac{1}{2} = 1,\ \ \frac{1}{2} \oplus 1 = 1,\ \left(\frac{1}{2}\right)^* = \frac{1}{2}$.

7.2 Stage 1: Ł3 ↦ Ξ(Ł3)

Mundici's construction identifies Ł3 with the unit interval of the unital ℓ-group (□, 2); that is,

Ξ(Ł3) = (G, u) = (□, 2), and Γ(□, 2)={0,1,2} where we identify the integer $k \in \{0,1,2\}$ with the MV-element $k/_2 \in \{0, 1/_2, 1\}$.

The MV-operations correspond to group addition truncated at the unit u = 2:

$$\frac{k}{2} \oplus \frac{l}{2} \leftrightarrow \boldsymbol{min(2, k + l)}$$

(then reinterpreted as a fraction over 2).

Thus, the dictionary is: $0 \leftrightarrow 0,\ \ \frac{1}{2} \leftrightarrow 1,\ 1 \leftrightarrow 2$ (in the □-model).

7.3 Stage 2: (□, 2) as a dimension group

The ordered group $(□, □_{\geq \mathbf{0}})$ with distinguished order unit u = 2 is a dimension group: it is unperforated and has the interpolation property trivially (being
totally ordered). Hence we may view $(□, □_{\geq \mathbf{0}}, \mathbf{2})$ as an object of **DimGrp**.

7.4 Stage 3: Realizing the ordered K_0-group

For a unital finite-dimensional matrix algebra $M_m(\mathbb{C})$ we have: $K_0(M(\mathbb{C})) \cong$ □, $K_0\big(M(\mathbb{C})\big)^+ \cong □_{\geq \mathbf{0}}, \big[\mathbf{1}_{M_m}\big] = \mathbf{m} \in □$

Comparing with $(□, □_{\geq \mathbf{0}}, 2)$, we see that the unique (up to *-isomorphism) unital C-algebra who's ordered K_0-invariant equals (Z, Z≥0, 2) is $M_2(\mathbb{C})$.

Therefore, the final stage of the pipeline assigns to Ł3 the AF algebra $\boldsymbol{\mathcal{A}}$(Ł3) $\cong M_2(\mathbb{C})$.

7.5 Interpreting MV-operations via projections

Inside $M_2(\mathbb{C})$, consider projections of rank 0, 1, 2. Their K_0-classes are 0,1,2 ∈ □ ≅ $K_0(M_2)0$,

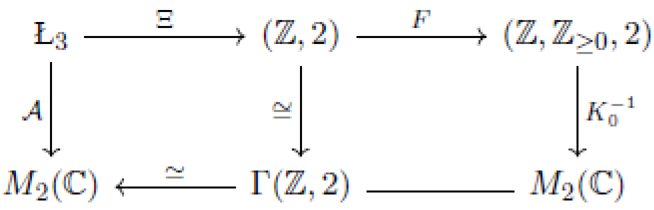
The MV-sum $\frac{1}{2} \oplus \frac{1}{2} = 1$ corresponds to the relation: $[p1] + [p1] = [p2]$, *where* p_1 is a rank-1 projection and $p_2 = 1_{M_2}$ is the rank-2 projection (the unit). Truncation at the unit is exactly the clipping of sum in the ordered group at the distinguished order unit.

Hence the MV-logic computations on $\{0, \frac{1}{2}, 1\}$ are realized concretely by ranks of projections (and their classes in K_0) inside the AF algebra $M_2(\mathbb{C})$.

7.6 Remarks

- The example is prototypical: for the finite MV-chain Łn $= \left\{0, \frac{1}{\mathrm{n-1}}, \ldots, 1\right\}$ one obtains Ξ(Łn) = (□, $\mathbf{n-1}$), and the associated AF algebra produced by K_0^{-1} is $M_{n-1}(\mathbb{C})$.
- Finite MV-chains therefore correspond to finite-dimensional matrix algebras under the pipeline; more complicated (infinite) MV-algebras produce infinite-dimensional AF algebras (or more intricate inductive limit algebras).

We now illustrate the entire functorial pipeline with a compact commutative diagram.



8 Conclusion

The functorial pipeline developed in this work,

$$MV \xrightarrow{\Xi} u\ell G \xrightarrow{F} DimGrp \xrightarrow{K_0^{-1}} AF$$

reveals a deep structural connection between the algebraic semantics of fuzzy logic and the classification theory of AF C^*-algebras. Rather than treating MV-algebras as isolated logical structures, the pipeline situates them within a landscape governed by ordered groups, interpolation properties, and K-theoretic invariants. Two conceptual insights emerge clearly:

1. **Logical semantics as ordered geometry.**

Mundici's equivalence shows that an MV-algebra is already the unit interval of an ordered abelian group with a strong order unit. The logical operations $\oplus$ and $*$ are therefore shadows of algebraic and order-theoretic structures on the ambient group. This allows logical transformations to be studied via interpolation, convexity, and homomorphisms of ordered groups.

2. **Fuzzy truth values as projection data.**

Dimension groups are precisely the ordered $K0$-groups of AF algebras. Since projections in AF algebras encode finite-rank phenomena, MV-elements may be interpreted (functorially) as classes of projections or positive elements inside the associated AF algebra. Logical combination becomes rank-sum, truncated by the order unit, and negation corresponds to taking complements inside K_0.

The worked example of the three-valued Łukasiewicz algebra illustrates this viewpoint vividly: the truth values $\{0,1,2\}$ become the projection ranks $\left\{0, \frac{1}{2}, 1\right\}$ in $M_2(\mathbb{C})$, with MV-addition realized as truncated rank-sum. For more complicated MV-algebras, the AF algebra $\boldsymbol{\mathcal{A}}(\mathbf{M})$ may be infinite-dimensional, and its Bratteli diagram provides a geometric visualization of the logical structure of M.

Finally, because each step of the pipeline is functorial, the entire construction is stable under morphisms: MV-homomorphisms become unital *-homomorphisms, so logical transformations correspond to operator algebra morphisms. This offers a new semantics of fuzzy logic inside operator algebras and suggests the possibility of extending fuzzy reasoning into noncommutative settings.

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