On Invariant Potential Fields in Partially Occupied Charged Media

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Abstract: In three-dimensional potential theory, classical analytic methods result in force fields exhibiting discontinuities, particularly at the boundary regions of volumes and surfaces occupied by charges and currents. Utilizing a specific topological structure introduced by E. Zeeman [1], I aim to augment the conventional methods of standard analysis, which are typically limited to differential and integral approaches. A specific bitopological structure in Euclidean space—namely the Zeeman topology Zpl with respect to the family of all piecewise linear arcs—is employed. The constructive proof of the non-regularity of this topology, along with the auto-homeomorphism group representing its invariance, provides the foundation for the physical applications presented here. We consider test particles on the boundary of a specific region occupied by charged particles. I propose an invariant solution that prioritizes force interactions restricted to straight lines over traditional coordinate invariance.

Keywords: invariant properties, special charges in potential theory, Zeeman topologies and applications

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1 Introduction

The foundational mathematical and geometric structure in this paper is based on a topological treatment by E. Zeeman [1] in 4D Minkowski space.
E. Zeeman pointed out that the topological structure is incredibly rich and may lead to many generalizations.
We consider auto-transformations of R^3 having invariant sub structures that are discontinuous in standard analysis.
In this paper we try to find physical applications in Euclidean space R^3.
We are using the topology Zpl = $Z(R^3, \Sigma pl)$ that is strictly finer than the Euclidean topology R^3, defined by the family of sub spaces Σpl, consisting of piecewise straight lines (having finitely many corners).
We also remember the theorem of E.Noether:
„Physical actions are local and are invariant with respect to the Pointcaregroup". The inverse of this theorem seems to be more interesting. We will start with the auto-homeomorphism group H(Zpl) and look for possible physical applications in R^3.
In the theory of potentials and fields there are many well developed theorems using standard mathematical analysis.[4].

There are also many known problems concerning the compatibility of differently charged regions and analytic solutions in occupied and non occupied regions. [4] Morey-Nirenberg]. We try to provide an alternative method to treat the problems in analysis.
The use of Zeno-sequences has to be explained to explore physical properties.

We will explain the difference between our "Zpl-method" and the standard analytic method using a simple example:
The field of a homogeneous charged disc in R^3. [4, pp.81-83].

2 The Standard Method *(Martensen)*

The classical theory of current and charge distributions dates back to the end of 19th century [Poincaré, Lyapunov] and has been explored using analytical and differential geometric methods. Martensen [4] worked out many problems in an excellent book, especially using integration and differential geometry.

The fundamental real valued function is 1/d(P1,P2), where d is the metric distance function and P1,P2 are two points in Euclidean space R^3. It was introduced to derive the potential and the force between two charged particles. Extending this function into charged regions there arise singularities which were mostly excluded by small balls.

An instructing example is the potential and the force field induced by a homogenous charged disc in R^3.

Martensen computes an overall existing continuous potential with these formulas:

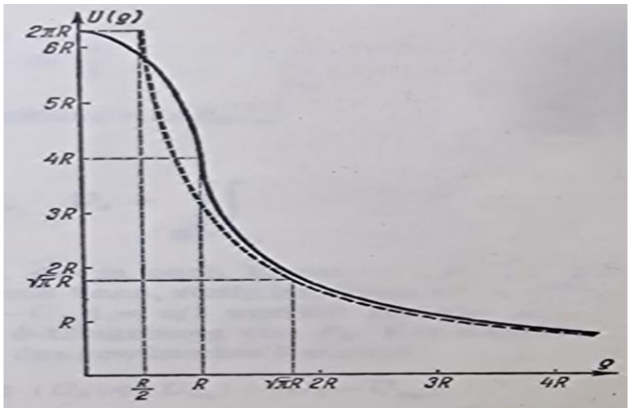

Fig. 29
Potential einer homogen belegten Kreisscheibe (gestrichelt: Potential einer im Mittelpunkt konzentrierten Ladung gleicher Größe)

$$U(\varrho) = \lim_{\alpha \to 0} \int_0^{2\pi} \int_\alpha^{\varrho^*(\varphi)} \frac{\varrho'\, d\varrho'\, d\varphi}{\varrho'} = \int_0^{2\pi} \varrho^*(\varphi)\, d\varphi = 4 \int_0^{\frac{\pi}{2}} \sqrt{R^2 - \varrho^2 \sin^2 \varphi}\, d\varphi,$$

$$0 \leqq \varrho < R.$$

$$U(R) = \lim_{\alpha \to 0} \int_{\frac{\pi}{2}+\beta}^{\frac{3\pi}{2}-\beta} \int_\alpha^{\varrho^{**}(\varphi)} \frac{\varrho'\, d\varrho'\, d\varphi}{\varrho'} = \lim_{\alpha \to 0} \int_{\frac{\pi}{2}+\beta}^{\frac{3\pi}{2}-\beta} (\varrho^{**}(\varphi) - \alpha)\, d\varphi$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \varrho^{**}(\varphi)\, d\varphi = 4R.$$

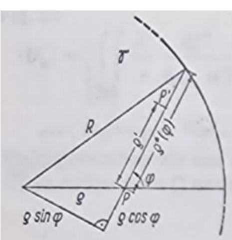

Fig. 27
Zur Berechnung des Potentials im Innern einer Kreisscheibe

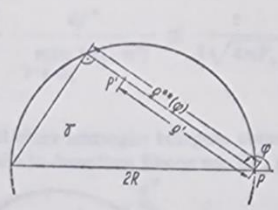

Fig. 28
Zur Berechnung des Potentials auf dem Rande einer Kreisscheibe

Note: On the border of the disc the gradient is singular.

We observe that the usual assumption of charged elements of volumes leads to this discontinuity in the gradient field of force on a testing charge.

A mathematical tool modeling the physical connections may be useful for computer analyses. It can be explained on the simple example above.

3 The Zpl-Method

The overall potential, which is affected by different charge and current distributions is accompanied by discontinuities in analytical methods using potential functions. We consider the local structure of these discontinuities to potentially enable to describe the force fields in a more realistic way. The discontinuity of the gradient shown above will be avoided by using our Zpl-method.

We consider a convex Euclidean open set U of Euclidean space R^3. We reduce this set U to a Zpl-closed subset A which is dense in the Euclidean closure of A. To get the set A we use an explicit construction used in topology by Popvassilev [2] proving the non regularity of Zpl. Therefore we remove on every line γ that crosses U, all points leaving back at most two points on γ inside of U. It can be proved that A is dense generated on this way.

To clarify our method, we must distinguish the purely mathematical procedure, including technical proof details, from the physical aspect we intend to introduce. The complete mathematical proof has already been established by Popvasilev in [2] using mathematical induction. This formal aspect is now to be supplemented by the following physical considerations. If we consider the subspace A of R^3 as a space where charges are fixed, it becomes evident through the construction that these charges maintain a finite distance along straight-line connections. The lines themselves lie within the Euclidean space R^3. Examining the Coulomb force relation between two charges, one observes that the distance is measured along the straight line connecting them. The significance of straight lines is to be particularly emphasized in this context. Furthermore, it can be easily shown that the Zeeman topology Zpl, generated by piecewise linear arcs, coincides with the topology Zl generated by the family of all straight lines. For the remainder of this work, we shall exclusively use the notation Zpl to refer to this topology.

The invariance of this construct is characterized by the auto-homeomorphism group H(Zpl). This group can be represented by the set of auto-bijections Bij(Σpl) that leave the family of piecewise linear sets Σpl invariant. It should be noted that the group of auto-bijections which leave the family of all straight lines invariant consists of linear and continuous transformations. However, regarding the physical significance of the auto-homeomorphism group, the preservation of the non-regularity of Zpl remains fundamental. This implies that a ray passing through a Euclidean boundary point of the charged set A must maintain an arbitrarily small distance to A. Furthermore, the inherent problems of coordinate representations become apparent when considering this group of auto-bijections, as it preserves the construction while neglecting specific coordinate systems. The existence of this invariance group demonstrates that coordinate transformations are not necessary for the Zpl method, particularly as discontinuous transformations may exist. Crucially, however, the physical content remains invariant.

We remark that every point R in the closure of A is the accumulation point of a Zeno-sequence consisting of members in A. A Zeno-Sequence is defined by a convergent sequence with respect to the Euclidean topology which is not convergent with respect to Zpl topology.

For further applications, the study of different Zeno-sequences may be useful- especially the fact that a Zeno-sequence converging to a point R needs infinitely many lines through R to cover all members of the sequence.

Take a compact sphere S^2 around R and study the accumulation points of the intersection with S^2.

4 Solution

We propose the following algorithm to avoid singularities as a physical application of this pure topological and geometrical construction in R^3 using the example above:

The potential of a Zpl-homogenous charged disc has to be constructed by the following algorithm:

A. Remove all points from the disc leaving back the Zpl-closed subset A which is dense in the Euclidean closure of A.

B. Put charges homogeneously in all points of A

C. Take a test charged point R of R^3 and draw all straight lines γ through R. The position of R can be inside or outside of the disc[Fig 1 and Fig 2].

D. Lines γ meeting the disc will be connected with charges in at most two points along a section between the crossing points with the border of the disk. The border points will be called B(γ).

E. Take the midpoints of these points in their segments [Fig 1and 2] and integrate 1/d(R,M(γ)) for all γ to get the Zpl-potential *Zpl-U(R)* in R.

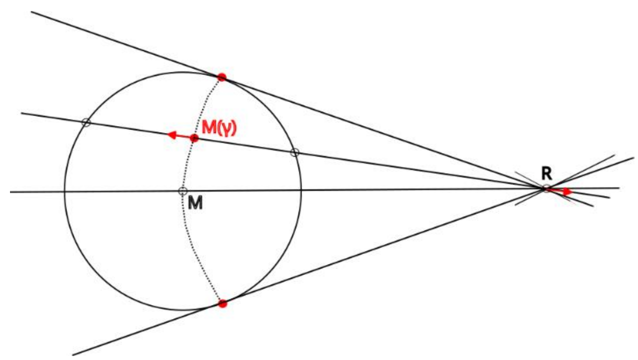

Fig 1: Test charge far outside the disc

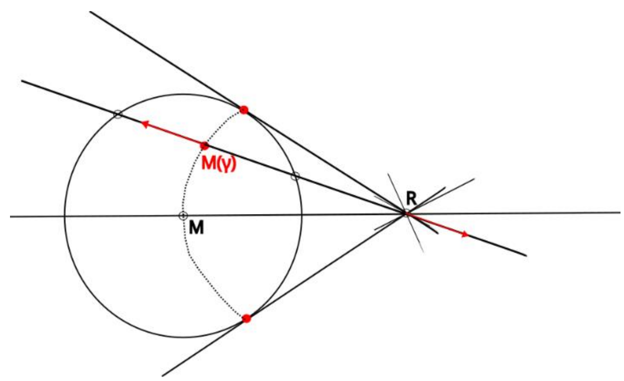

Fig 2: Test charge closer to the disc

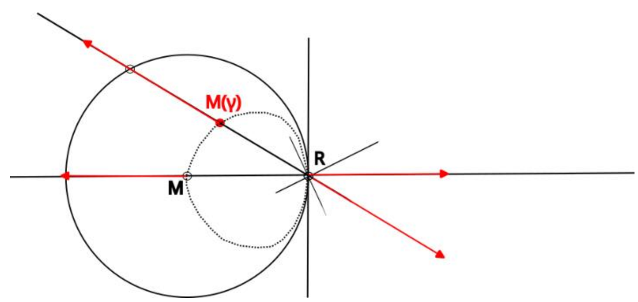

Fig 3: Test charge on the disc outline

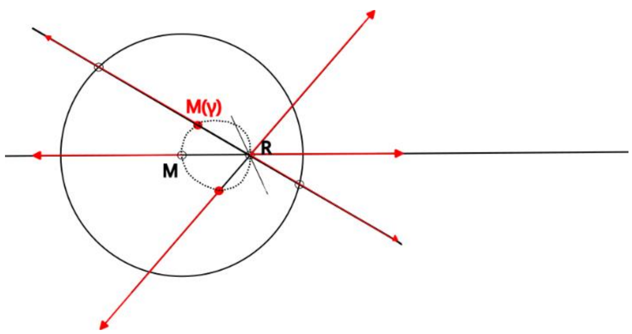

Fig 4: Test charge inside the disc

Remarks:
Ad A.: The “geometric” straight line between two charges gets an important “physical” notion on this way.
Ad B.: Generalizations have to be worked out. The charged set A consists of all Zeno-sequences without their Euclidean accumulation points. These fill out the rest of points(geometrically) in the disc.
Ad C.: From an arbitrary Zeno-sequence in A the accumulation point can be separated by a Zpl-neighborhood. This means that on every line through this point we can find a finite interval not meeting the Zeno-sequence. But the whole dense set A cannot be separated by Zpl-open sets from this point. Therefore exists a new physical connection between the test charge in R and the Zpl-charged set A.
Ad D.: Taking the medium-points on the segment between the boundary-points B(γ) seems to be justified by the density-property of A in the disc. The exact position of the charged particle of A on γ can not be determined.
Ad E.: The Zpl-potential can be constructed due to real physical situation going through the border of the disc. The gradient representing the force has to be finite. To avoid the singularity of the differentiable force field at the boundary of the disc, one should consider that along each ray through R, the finite force interaction can be calculated by $(1/d^2(R, M(\gamma))$ using the Coulomb formula. The selection of midpoints on the segments is statistically justifiable, bypassing a further discussion of the uncertainty of their precise location within those segments.
Ad Fig. 1-4: Charges C and R are assumed to be positive.

5 Conclusion

Finally we remark that the invariance group H(Zpl) of auto-homeomorphisms that can be represented by the set of auto- bijections Bij(Σpl) leaving the family of piecewise linear arcs Σpl invariant, contains non-continuous transformations. The representation in coordinate-systems does not seem to be important.

From a physical perspective, the Zpl-closed set A is characterized by the requirement that all charged points are separated by finite distances.

OPEN PROBLEMS

A. Treatment of static Vector Potentials and magnetic fields

B. Computer Simulation of different sources

References

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