Functional curves of high quality – innovation in geometric modeling

VALERIAN MUFTEJEV[1], FAIRUZA ZIGANSHINA[2], VADIM GUMEROV[3]

[1]C3D Labs
Ufa, RUSSIA

[2]Department of Complex Engineering and Computer Graphics
Ufa State Petroleum Technological University
Ufa, RUSSIA

[3]Department of Complex Engineering and Computer Graphics
Ufa State Petroleum Technological University
Ufa, RUSSIA

Abstract: Curves and surfaces that form the geometry of technical products often directly determine the functional characteristics of the designed products. It is logical to call such curves and surfaces functional. Often, the aesthetics of a product is one of the important consumer properties of the product. Therefore, aesthetic curves can also be classified as functional.

The optimal curve is not always defined by an analytical curve, such as the profile of a tooth (involute of a circle), the trajectory of a load transportation as a line of the fastest descent (brachistochrone), or the profile of a dome (catenary). Free-form curves in the form of spline curves are more commonly used.

Methods for constructing functional curves must satisfy the following requirements:

- Isogeometric construction of a curve on the initial polyline with fixed end and intermediate parameters.
- Construction of a fair curve.
- Low value of potential energy of the curve.

Regardless of the specific product, functional curves must be fair. Functional curves must meet the following fairness criteria:

- High order of smoothness (not lower than the 4th order).
- Minimum number of curve vertices (or minimum number of curvature extremes).
- Low value of curvature variation and rate of curvature change.
- Low value of potential energy of the curve.

Spline curves that meet these criteria are called F-curves or curves of class F.

The authors have developed the FairCurveModeler software and methodological complex (SMC) for modeling F-curves. Based on the functionality of the FairCurveModeler SMC, universal and specialized applications for CAD systems (KOMPAS 3D, nanoCAD / ZWCAD / AutoCAD), mathematical systems (MathCAD / Mathematica / Wolfram Cloud), an Excel application, and a Web application have been developed.

The FairCurveModeler SMC has been adapted and implemented into the C3D geometric core as the C3D FairCurveModeler section.

The philosophy of the FairCurveModeler SMC is based on the theory of calculating parameters of the Soviet school of applied geometry. The initial data for constructing or editing curves are presented in the form of geometric determinants (GD).

The following innovations have been implemented based on the parametric approach:

- A new paradigm for constructing spline curves based on the theory of parameter calculus has been proposed. A spline basis is formed as a sequence of 5-parametric conical curves of double contact, with 4 common parameters of adjacent conical curves. Then, on the spline basis, points of a virtual curve are generated in the lenses of contacting conical curves. It is shown that the generated points belong to the curve of class C5.
- The method for isogeometric approximation of a virtual curve by means of a rational cubic Bezier spline curve has been developed.
- The method for isogeometric approximation of a virtual curve by means of a B-spline curve has been developed.

The FairCurveModeler SMC is characterized by the following system properties:

1) The methods for constructing F-curves ensure isogeometricity (shape preserving) of the constructing curves on the original polylines. The shape of the modeled curve is similar to the shape of the original polyline.

The designer is provided with a wide range of tools:
- Base polyline. The spline curve passes through the vertices of the base polyline. In the general case, the spline nodes do not coincide with the vertices of the polyline.
- A set of tangent lines (in particular, in the form of a tangent polyline). The curve passes tangent to the lines (tangent to the links of the tangent polyline).
- Hermite GD. The base polyline is equipped with tangent vectors and curvature vectors at its vertices.
- GB-polygons of Bezier spline curves.
- S-polygons of B-spline curves.

2) The methods provide flexibility in construction and editing. This is the ability to locally control the shape of a global spline with fixed parameters at intermediate points of the polyline.

3) The unique feature of the methods is the ability to geometrically accurately model circles and, in general, conical curves.

4) The methods are invariant with respect to affine transformations.

The article substantiates the importance of the property of minimizing the potential energy of curves in F-curves. The works of Mehlum and Livien are analyzed in detail. An experiment with a physical spline is conducted. The advantages of the methods for constructing F-curves in FairCurveModeler over spline curves of class A and over a physical spline and its approximation methods are proven.

Innovative methods for constructing surfaces are proposed: a frame-kinematic method for constructing a spline surface, a method for constructing a topologically complex surface.

Keywords: — isogeometric approximation, fair curves, spline curves, FairCurveModeler, F-curves, B-spline surface, complex surfaces.

Received: July 23, 2025. Revised: August 27, 2025. Accepted: September 29, 2025. Published: November 5, 2025.

1. Introduction

Curved lines and surfaces that form the geometry of technical products often directly determine the functional characteristics of the designed products. It is logical to call such curves and surfaces functional.

Examples of functional curves and surfaces:
- External contours of ships, cars, airplanes.
- Profiles of cams in cam mechanisms.
- Working surfaces of tillage machines and units.
- Profiles and surfaces of airplane and UAV wings, propellers, turbines, and compressor blades.
- Centerlines of road routes.

Often, the aesthetic properties of curves and surfaces determine the consumer properties of products (for example, the aesthetics of body surfaces in the automotive industry, the aesthetics of architectural forms, and forms of industrial design products). Therefore, aesthetic curves can be considered as functional curves.

Among the types of functional curves, one can distinguish a subclass of engineering analytical curves, which provide some design characteristic of an object optimally. Such curves include, for example, the involute of a circle used to construct the profile of the teeth of a gear wheel, [1], the inverted catenary line of the profile of the dome of the cathedral in London, [2], [3], as well as the brachistochrone - the curve of the steepest descent for transporting objects, [4].

In the general case, functional curves are described by spline curves of free form.

2. Requirements for the quality of functional curves

In applied geometry, a distinction is made between fairness and smoothness. Smoothness is the order of differentiability of a spline curve. A smooth curve is a curve of the first order of smoothness. Fairness includes many criteria, including the order of smoothness.

Increased requirements are imposed on the smoothness parameters of functional curves, which are universal and do not depend on the specifics of the designed objects. The following is a list of these requirements with explanations.

1. High, at least 4th order of smoothness

Smoothness is a characteristic of a function or geometric figure (curve, surface, etc.) that indicates whether the function is differentiable over its entire domain of definition or allows us to reduce the points of the figure to a neighborhood described by differentiable functions.

In different designs, splines with different orders of smoothness are used. For example,

- clothoid splines are used for modeling of road routes, and smoothness is provided at least of the 2nd order;

- to profile a high-speed camshaft cam, smoothness of at least the third order is required, so profile design begins with the construction of a smooth graphic of the derivative part, [5]. This approach eliminates “jerks” [6].

- when modeling spatial curves, the curve should have 3rd order smoothness to ensure continuity of the torsion function, and 4th order smoothness to ensure smoothness of the torsion;

2. No or minimum number of curvature extrema

The smoothness of the curve also depends on the shape of the graph of curvature variation along the line of motion.

Since the oscillation of the curvature function according to the basic equation of dynamics, [7], [8] will cause a pulsation of centrifugal forces acting on the material point, the section of the line of motion should have a minimum number of curvature extrema or curve vertices.

The presence of unnecessary extremes of curvature, for example, in the shape of technical products and design objects, can cause the following negative phenomena:

- Unwarranted runout of the cam mechanism tappet, the consequence of which is premature wear of the mechanism.

- Soil sticking on the plow section with a concentration of curvature near the soil trajectory, which leads to an increase in plow resistance and, consequently, to an increase in the energy intensity of the plowing process.

- When the extreme values of the curvature profile are excessive, excessive pulsation of the flowing medium occurs, which leads to the appearance of resistance and can cause flow breakdown.

- The need for unnecessary braking and acceleration, which increases the energy costs of moving along the vehicle track.

- The effect of curved mirrors at curved body surfaces, [9].

3. Small values of curvature variation and its rate of change

In some application domains, the requirement to minimize the curvature variation is introduced, and hence the curvature concentration must be constrained to a maximum value.

For example, such a constraint on the minimum value of the radius of curvature (maximum curvature) is naturally introduced in road design: the minimum radius of a curve is limited by the calculation of the allowable vehicle speed, [10].

An important parameter of curve quality is the rate of curvature change. When designing a road alignment, this parameter governs the rate of increase of centrifugal force acting on the vehicle on curves and is easily controlled through the use of clothoid segments with a linear change in curvature function, [10], [11].

Note that these requirements are contradictory. When the variation (the difference between the maximum and minimum curvature value) decreases, the rate of curvature change may increase, and vice versa.

4. Low value of the potential energy curve

The smoothness of a curve is considered to be directly related to its potential energy.

The necessity of choosing a functional curve with a minimum value of potential energy is explained by the assumption that when an object with a functional surface moves at high speed, the surrounding environment behaves like an elastic body. Obviously, less work is expended on the deformation of such an elastic medium along the flow lines with lower potential energy.

When a material point moves along a concave curvilinear trajectory, taking friction into account, the work expended on movement decreases with a decrease in the potential energy of the trajectory itself, [12].

5. Aesthetics of the curve

The authors of this study hold the view that the priority is the evaluation according to the criteria of fairness. Expert evaluation from the standpoint of the laws of technical aesthetics (conciseness, integrity, expressiveness, proportional consistency, compositional balance, structural organization, imagery, rationality, dynamism, scale, plasticity, harmony) is valid only after the evaluation for fairness.

6. Accurate modeling of roundness

In [13], several types of splines for aesthetic design are compared: minimum potential energy curve MEC, clothoidal, circular, polynomial degree 3, and log aesthetic curves. The prize is the honor of the best spline for design. The contest begins with the ability to accurately model a circle.

The author's next conclusion is absolutely correct: a spline constructed on points of a circle must exactly coincide with the circle. For some technical objects, this requirement is essential. For example, a smooth road route must have curves with a section coinciding with the arc of a circle. In tracing methods, such sections are modeled by composite curves - clotoid - circle - clotoid.

Another example is the profile of a flat cam of an internal combustion engine camshaft. The profile should have two sections of exact arcs of circles connected by smooth transition curves.

2.1 FairCurveModeler

Functional curves whose parameters meet the above requirements are called class F curves or F-curves, [14].

It is important to note here that it is these strict requirements for smoothness parameters that distinguish curves of this class from class A curves. The latter are the shape-forming curves of class A surfaces - high-quality surfaces of external body surfaces according to the criteria of aesthetics. A “good” curve for these surfaces will have a curvature graph with a small number of areas of monotonic curvature change, [15]. This requirement can be compared to the requirement to minimize the number of curvature extrema in functional curves.

Thus, class A curves are curves for surface shaping in industrial design, while functional curves are engineering curves. High-quality curves are also commonly referred to as faired curves (faired curves, fairing curves). Here, it is important not to confuse the latter with smooth curves, which are low-quality curves of first-order smoothness, [16].

However, even if a CAD system supports the modeling of class A curves and surfaces, it does not provide a proper quality of functional curves according to fairness criteria.

As a result of our research in the field of geometric modeling, class F methods (F-methods) were developed for modeling class F curves.

Based on the F class methods, the FairCurveModeler software and methodology complex (SMC), [14], FairCurveModeler has by now been implemented in the form of two complexes:

- Software and methodological complex (SMC) FairCurveModeler, [14]. The SMC FairCurveModeler is implemented in the C++ language. The functionality is available through the COM component FairCurveModeler.exe. COM-component FairCurveModeler.exe is included in universal and specialized applications and can be considered as a geometric core of FairCurveModeler.exe for applications. Universal and specialized applications can be ordered and downloaded on the developer's website http:/Spliner.ru.
- C3D FairCurveModeler is a section of C3D geometric core. It is an adaptation of the SMC FairCurveModeler according to C3D software standards. The functionality is available to developers through the C3D ToolKit, [19 C3D ToolKit].

Only the FairCurveModeler.exe command invocation interface is developed for release applications. All applications access the COM component of FairCurveModeler.exe.

Applications are divided into universal and specialized ones.

Universal applications implement the functionality of the FairCurveModeler.exe geometrical kernel.

Specialized applications are developed on the basis of FairCurveModeler.exe geometrical kernel functionality to solve specific design tasks.

The following universal applications have been developed [14]:

- Cloud WEB-release WebFairCurveModeler based on COM-component.

- FairCurveModeler application based on a COM component on CAD-system platforms (KOMPAS 3D, nanoCAD, ZWCAD, AutoCAD),

- FairCurveModeler application based on a COM component on the Excel platform.

- FairCurveModeler applications in the computer math systems MathCAD, Mathematica, WolframCloud.

Specialized applications are developed on CAD-system platforms, nanoCAD / ZWCAD / AutoCAD, based on FairCurveModeler.exe core functionality + options for modeling specific objects. These include:
- Applied CAD Plow.
- An application for modeling aerofoils based on Abbott's improved method.
- Application for road tracing.
- Application for profiling cam profiles.
- Application for profiling of steam turbine blades.

Further, the paper describes innovative solutions used in the development of SMC, reveals, and substantiates significant advantages over existing methods of geometric modeling in CAD.

3. Innovative solutions in the development of F-methods

Let's consider the innovative characteristics of F-methods.

A set of geometrically oriented methods forms the basis of the FairCurveModeler software package. This article presents the methodological justification of FairCurveModeler's methods in the field of engineering geometry.

The use of computer systems has radically expanded the scope of engineering geometry. We can speak of the emergence of computer-aided engineering geometry. Computer-aided engineering geometry draws on such areas of engineering as descriptive geometry and engineering graphics, computational geometry, and computer graphics. The development of the theory and methods of computer-aided engineering geometry has resulted in the development of geometric modeling subsystems for CAD systems and their geometric kernels.

A geometrically oriented approach to the algorithmization of solutions to geometric modeling problems, in contrast to the algebraic approach, is the main one in the methods of computer engineering geometry.

With the algebraic approach, any problem is reduced to some standard algebraic systems, which are solved using the methods of computational mathematics.

The geometrically oriented approach utilizes a geometric interpretation of formulas and equations, and formulates the problem in terms of geometric objects and their geometric relationships. A solution to the problem may also be found using geometric algorithms.

A classic example of such a successful geometrically oriented approach to problem formulation and solution was demonstrated by Bezier and Casteljau [17], [18].

The geometric interpretation of the coefficients in the formula of the Russian mathematician Bernstein allowed them to create a new paradigm for curve modeling. Bernstein's parametric equations [19 Bernstein 1], [20 Bernstein 2] are now called Bézier curves. Calculating points on a Bézier curve using the Casteljau algorithm also has a geometrically clear interpretation [17], [18].

The generalization of the Bezier method to B-spline curves is also an example of a geometrically oriented approach, [21]. Carl de Boer [22] and, independently of him, Cox [23] established a connection between the geometric form of representation of spline coefficients and the form of the parametric Schoenberg spline, [24].

The theoretical basis of engineering geometry is the theory of parametrization or the theory of parameter calculus.

The theory of parametric calculus, developed by the scientific school of Academician N.F. Chetverukhin and Professor I.I. Kotov, [25], [26], [27], [28], is an outstanding scientific and practical result of the development of engineering geometry in the USSR. The theory of parametric calculus was used by Professor Samuel Geisberg of Leningrad University to create the successful Pro/ENGINEER CAD system [29]. Parametric modeling is currently the basis of any CAD system.

The theory of parametric calculus underlies the scientific school of Professor V.A. Osipov. The methods of two and three relations [30], [31], created for flexible editing of curves, are essentially geometrically oriented interpretations of rational quadratic curves with controlled weights.

The scientific school of Academician V.S. Polozov and Professor S.I. Rotkov raised the theory of parametric calculus to the highest level as a mathematical apparatus for analyzing the parametric relationships of two-dimensional and three-dimensional objects, [32]. Based on this apparatus, heuristic algorithms (artificial intelligence algorithms) were developed for synthesizing a three-dimensional object from flat projections and constructing an optimized drawing based on a three-dimensional model of the object.

Of extreme importance for practical design is A.E. Klevensky's [33] idea of deferred calculation of a geometric object's parameters until sufficient parametric relationships between geometric objects have been established. At the Graphics Department of the Ufa Aviation Institute, a team of developers led by Associate Professor V.I. Makutov successfully implemented this idea in the "Alpha" program. The program was used at a number of leading machine-building enterprises in the USSR.

Let us consider the modeling of spline curves in light of parametrization theory.

In parametrization theory, a spline curve model is called a determinant (D). A determinant consists of a set of geometric parameters, called a geometric determinant (GD), and a procedure for constructing a spline curve using the GD or a procedure for generating curve points. The number of GD parameters that uniquely define a curve is called the parametric number of the determinant.

Geometric determinants in the FairCurveModeler system:

- Base polyline. The curve passes through the vertices of the polyline.
- Tangent lines/tangent polyline. The curve passes tangent to the lines or links of the polyline.
- Hermite determinant. The support polyline is provided with tangent vectors [and curvature vectors] at the vertices of the polyline. Accordingly, it is considered a first- [and second-] order determinant.

In addition to the GD, certain requirements for the shape and quality of the curve can be added: isogeometricity of the shape of the polyline and the modeled curve, the order of smoothness of the modeled curve, etc.

How do we apply parameterization theory to model spline curves?

In the theory of spline curves, by the method of construction, curves are divided into local and global. Let us explain the terms local and global splines by the example of cubic splines.

A segment of a cubic parametric polynomial (Ferguson curve, cubic Bezier curve) is defined by 4 geometric parameters of the form of 2 points and 2 tangents or 4 vertices of a B-polygon, [9].

A local geometric cubic spline is constructed on a polyline by specifying the direction of the tangents at each point of the polyline. Changing an individual segment of the spline does not affect the shape of other segments. It is a curve of the 1st order of smoothness.

To construct a cubic spline with continuity of curvature at the junction points of analytic curves, a system of equations is solved. As a result, we obtain a global spline of the 2nd order of smoothness. Globality means that changing the coordinates of one point causes a change in the entire spline curve.

In traditional spline theory, it is considered that the analog of the energy functional is minimized when constructing a spline, [34], [35], [36], [37], [38], [39], [40].

3.1 Spline Basis

The innovation in our scheme of spline construction: any configuration of connections of geometrical parameters (not only at the ends of segments) between adjacent segments can be used. We have developed an innovative method of construction on the polyline $\{P_i\}$, the set of segments $\{R_i\}$ of double osculated conic curves (Fig. 1).

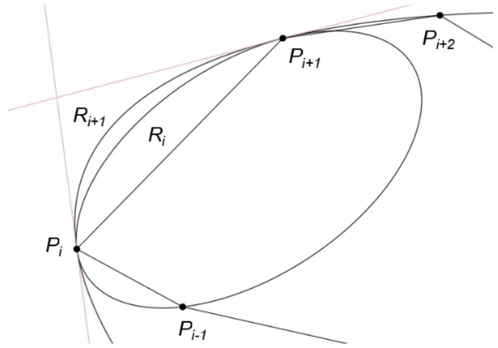

Fig. 1. Configuration of adjacent segments of the conic curves.

The adjacent 5-parametric conic curves R_i, R_{i+1} have 4 common parameters. Further increase of the number is possible only at full coincidence. This is a global spline.

3.2 Virtual curve

And so, we have constructed a spline as a set of double-osculating conic curves. This is a semi-finished product that we call the **spline basis**. The algorithm for constructing the spline basis is described in [41].

Next, on the spline basis, we construct a virtual curve. New points of the virtual curve are generated in the centers of "lenses" formed by adjacent conic curves (Fig. 2).

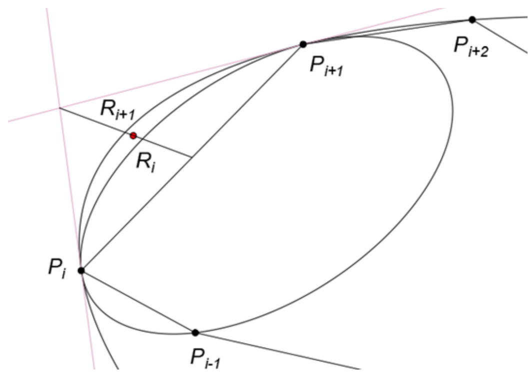

Fig. 2. Generation of a point in a "lens" formed by adjacent curves.

The lens (see Fig. 1) formed by adjacent analytic curves can be viewed as the magnitude of the deviation/difference of adjacent analytic segments.

In algorithms, the difference of engineering discriminants is used as the deviation value. In the Soviet school of applied geometry, the engineering discriminant d is the ratio of the length of the segment from the curve to the base of a contiguous triangle along the median to the length of the median, $0 < d < 1$, [42].

The second innovation: generation of new curve points in the middle of lenses formed by adjacent double-contact conic curves (see Fig. 1).

Recursive generation of curve points in applied geometry is not a new idea. For example, Chaikin's idea of fast point generation, [43]. Also, the idea of estimating the deviations of two adjacent segments by the lens is not new, [44].

An innovative method for generating a point in the middle of a lens combines these two approaches.

The generated points in the limit form a virtual curve (V-curve) of class C^5, [41].

Let us note some important properties of the spline basis and virtual curve.

The method of construction of the spline basis is invariant with respect to projective transformations.

The method of constructing a virtual curve is invariant with respect to affine transformations.

These properties are necessary for adequate estimation of the shape of a spatial curve from images and the possibility of adequate editing of the shape on projections of the curve and its GD.

An important property of this global spline is that if the original polyline belongs to a conic curve, the spline degenerates into a particular conic curve.

This means that conic curves can be modeled geometrically accurately using this method.

This is a big plus for the method. Moreover, connoisseurs, [13], believe that for CAD spline modeling methods, it is a necessary property.

It can be assumed that the construction of this global spline also minimizes some energy functional.

Another important property is the possibility of constructing a basis on tangent lines. This property follows from the duality property of the definition of 5-parameter conic curves. A conic curve can be defined by 5 points or 5 tangents, [45].

3.3 A geometrically oriented approach to solving ill-posed problems

The method for constructing a spline on set of double-osculating conic curves implements a geometrically oriented solution to the ill-posed problem of constructing a nonlinear spline.

A robust algorithm for determining the set of double-osculating conic curves on a convex base polyline based on a priori information about the desired solution [41] can be interpreted as a regularizing algorithm based on Tikhonov's scheme [46].

The formal formulation of the problem of constructing a set of double-osculating conical curves leads to a system of nonlinear equations.

In the author's approach [41], the problem is formulated in terms of parametric theory as follows.

The desired geometric parameters are the sets of tangents at the base points. The initial set of tangents is determined in the feasible solution space. In particular, the directions of the tangents must correspond to the shape of the polyline. In the algorithms, the direction of the chord is used when specifying the initial tangent at the current point.

The result of constructing conical curves of double osculating can be considered a set of tangents $\{T_i\}$ at base points $\{P_i\}$ such that the conical curves constructed from local geometric determinants $G_i = \{P_{i-1}, T_{i-1}, P_i, P_{i+1}, T_{i+1}\}$ are conical curves of double-osculating. An iterative procedure for fitting the directions of tangents at points of the base polyline is proposed. In the current iteration m, to reduce the residual of tangents at points $\{P_i\}$, the conical $\{R_i\}$ and tangents $\{T_i\}$ are redefined on local geometric determinants $G_i = \{P_{i-1}, T_{i-1}, P_i, P_{i+1}, T_{i+1}\}$. The procedure can be written like this

$\{T_i\}_m = \alpha\{R_i\} = \alpha\beta\{P_i,T_{i,}\}_{m-1} = \gamma\{T_i\}_{m-1},$, where

α – operation of determining tangents on conical curves $\{R_i\}_{m-1}$ at points $\{P_i\}_{m-1}$;

β - is the operation of determining conical curves $\{R_i\}$ at points $\{P_i\}$ on local GDs $G_i = \{P_{i-1}, T_{i-1}, P_i, P_{i+1}, T_{i+1}\}$.

γ - is a superposition of operations α, β.

This approach allows you to replace complex schemes for solving a system of nonlinear equations with a compact formula for recurrent calculations

$\{T_i\}_{m+1} = \gamma\{T_i\}_m$.

It should be noted that the article [41] was republished in the US. Later, 15 years later, a US Army military mathematician attempted to replicate the algorithm for constructing this nonlinear spline [47].

Mufteev's envelope method and R.W. Soanes's method are extremely similar. R.W. Soanes's method also uses a scheme for constructing two families of mutually osculating conical curves. The operations for generating additional points in the curve point generation scheme are identical. However, Soanes uses computational mathematics algorithms to solve the problem. A disadvantage of Soanes's method is the need for extensive compaction of the V-curve to obtain an acceptable quality spline curve in the form of conical curve segments.

Credit must be given to the military mathematician's mathematical intuition. In his reasoning, he suggests that the resulting curve has a higher order of smoothness than that stated in the title of the article, "Thrice Differentiable Affine Conic Spline Interpolation." Indeed, as the author demonstrated in [41], the envelope curve belongs to class C^5 curves.

4. Isogeometric approximation of virtual curve by means of NURBS curves

In the Soviet school of applied geometry, when constructing a curve on a polyline, the term isogeometricity means the similarity of the shape of the modeled curve to the shape of the original polyline, [48],[49].

In foreign literature, the term "Isogeometric" is used to denote the method of "Isogeometric Analysis" [50]. This method is a development of finite element analysis in hydrodynamic problems. In the calculation equations describing the physical process, the NURBS surface model is used directly, without switching to a simplified triangulation model of the surface as in traditional finite element analysis. That is, the term "Isogeometric Analysis" means that in the CAE system, the same geometric model that was modeled in the CAD system is used in the calculations of the physical process. We also note the property of the NURBS surface to preserve the exact model during adaptive local compaction of the NURBS surface to improve the accuracy of physical calculations. The problem of isogeometric approximation of a function by splines was formulated by Grebennikov A.I. [48]. Introduced by Grebennikov A.I. the concepts and definitions of isogeometric approximation of functions allow us to formalize the definition of stable shaping and abandon such fuzzy concepts as “shape preserving approximation.”

In the works of the authors [51] terms are introduced for the analysis of the isogeometric shaping of spline curves

A regular curve C^k admits a parametrization of the form r(t) at any point of the curve with continuous derivatives $r'(t), r^2(t), \dots r^k(t)$. For $r'(t)$ not equal to 0, [52], [53] these are ordinary points of a regular curve; otherwise, we are dealing with singular points of a regular curve.

Sometimes a regular curve is defined as a C^1 curve for $r'(t)$ not equal to 0, [53]. A parametrization of the form $r'(t)$ with continuous derivatives $r'(t), r^2(t), \dots r^k(t)$ ($r'(t)$ not equal to 0) defines a curve of smoothness class C^k [53].

In applied geometry, a designer models a regular curve using a geometric determinant. The curve's determinant can be treated as a control polyline. A control polyline can take the form of a base polyline, a tangent polyline, or an S-polygon of a NURBS curve.

Let us introduce definitions characterizing the configuration of the control polyline and definitions relating the configuration of the control polyline to the geometric structure of a regular curve.

A polyline $\{P_i\}$, $i = 0,...,k$ is properly inscribed in the curve $r(t)$ [51] if the sequence of preimages of the vertices on the curve follows the same order as on the polyline.

We say that the control polygonal line correctly (regularly) structures the regular curve $r(t)$ if the sequence of points of the curve $r(t_i)$, $i = 0,...,k$, closest to the vertices of the polyline follows the same order as the vertices of the polyline $\{P_i\}$, $i = 0,...,k$, and the distance from the point $r(t_i)$ to the corresponding vertex P_i is less than to two adjacent vertices P_{i-1}, P_{i+1} of the polyline, and the angles between adjacent segments of the polyline are obtuse. Such a polygonal line associated with a regular curve is called a polygonal line of regular form.

To analyze the shape of a polyline, it is more convenient to use central differences due to the symmetry of the definition of central differences relative to the current vertex of the polyline. Using first- and second-order central divided differences, discrete approximations of tangent vectors and curvature vectors at points of the polyline are determined. To control the shape of a polyline on a projection, we introduce the following definitions characterizing the shape of a planar polyline.

The shape or orientation of a polyline is defined as the law of sign change of the discrete approximation of curvature.

A polyline is locally convex if the signs of the discrete approximation of curvature are the same at all points.

A strictly convex polyline is a polyline whose closure defines a convex polygon.

A polyline with two locally convex sections of different orientations is called an S-shaped polyline.

A strictly S-shaped polyline is an S-shaped polyline consisting of two strictly convex segments of different shapes.

A regular polyline of order *m* is a polyline of arbitrary shape with the following constraint: any local part of m segments must be strictly convex or strictly S-shaped.

A characteristic polyline is a regular polyline associated with a curve whose shape is similar to the curve's shape. The number of locally convex sites of the polyline coincides with the number of locally convex sites of the curve.

NURBS curves are the standard for representing curves in CAD systems. The authors propose **two geometrically oriented methods**: 1) approximation using a cubic rational Bézier spline curve, 2) approximation using a B-spline curve of high even degree m ($m = 4, 6, 8, 10$).

4.1 Isogeometric approximation by means of NURBzS curve

The construction of the spline basis is completed by constructing conical curves of double osculating. It is absolutely obvious that the approximating curve must pass through the "lens" region (see Fig. 1, 2). Here the developers face a trap. The first thing that comes to mind is to introduce a local coordinate system [44]. Imagine the first of the form $f_1(x)$, the second of the form $f_2(x)$, $0<x<1$. Then the resulting curve $F(x) = (1. - x)*f_1(x) + x* f_2(x)$. Elegant, but incorrect. With sharp changes in curvature, the resulting curve will have an oscillating shape.

Both curves are convex, therefore the solution must also be sought in the form of a convex curve. We proposed such a method in [54]. The approximation scheme is realized as follows:

- Segments of conic curves are transformed into quadratic Bézier curves.
- The degrees of the Bézier curves are raised to the 3rd degree.
- B-polygons are averaged while maintaining the convex shape.
- A rational cubic Bézier curve is formed on the averaged *B*-polygon with curvature at the start point from the first curve and with curvature at the end point from the second curve.

When the curvature is zero or very small at the endpoints of the segments, the shape of the B-polygon is formed specially, [54].

F-curves constructed on the V-curve and approximated by means of NURBzS curves, we will call F-NURBzS curves.

Метод идеально подходит для **аппроксимации аналитических кривых**. На аналитических кривых подготавливаются ГО Эрмита второго порядка фиксации. Схема построения аппроксимирующей кривой использует ту же схему (см. Рис. 3), но абстрагирована от определения *V*-кривой. Исходные конические кривые предварительно строятся в соприкасающемся треугольнике по значения кривизны в начальной точке для первой кривой и по кривизне во второй точке для второй кривой. На рис. 3 показан пример аппроксимации клотоиды на нарочито редких точках.

The method is ideal for approximating analytical curves. Second-order Hermite GD are prepared on the analytical curves. The scheme for constructing the approximating curve uses the same approach (see Fig. 3), but abstracts from the definition of a V-curve. The initial conical curves are first constructed in the osculating triangle based on the curvature at the start point for the first curve and on the curvature at the second point for the second curve. Figure 3 shows an example of approximating a clothoid using deliberately sparse points.

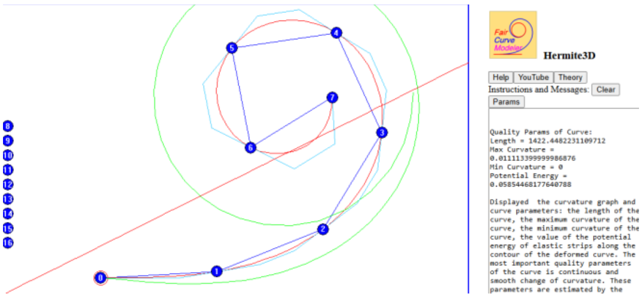

Fig. 3. Clothoid approximation using second-order Hermite GD. Example prepared in Web FairCurveModeler.

Note how the quality of the original curve is preserved in the form of a linear curvature graph in red.

4.2 Isogeometric approximation by means of a B-spline curve

Let us consider the construction of a B-spline curve of degree m. Let the number of links of the support polygon be n. Then the number of vertices of the S-polygon is equal to $k = n + m$ [37]. The method uses an even degree m. Additional vertices *m/2* before the first support point P_1 and additional vertices *m/2* after the end point P_n are calculated using the boundary conditions.

For example, let us assume a first-order Hermite GD with 3 support polygon segments (Fig. 4). Then, for $n = 3$ and degree $m = 6$, the number of vertices of the S-polygon is $k = 9$, and the number of additional vertices is $3 + 3$.

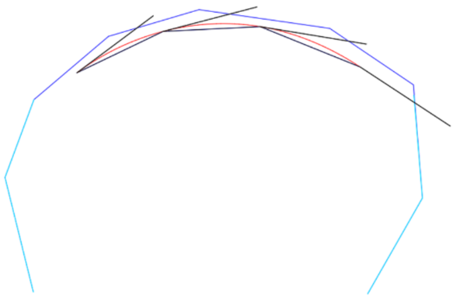

Fig. 4. Scheme of isogeometric construction of B-spline curve. The initial tangents define the directions of the links of the S-polygon.

Basic Algorithm

Just like the method for determining a nonlinear spline on double-osculating conical curves, the method uses Tikhonov's scheme [46] with parametric approach to solving an ill-posed problem.

The initial geometric determinant is defined by a first-order Hermite geometric determinant—a set of base points $\{P_i\}$ and a set of tangent lines $\{T_i\}$ at the base points. The fixed tangents can be obtained 1) on the spline basis of the V-curve, 2) as tangents to a fixed analytic curve, or 3) arbitrarily specified. The directions of the tangents at the vertices of the polyline must not contradict the shape of the polyline.

The solution is sought in the form of an S-polygon, isogeometric to the original base polyline with fixed tangents and defining a spline of a given degree passing through the vertices of the base polyline. The spline's nodal points are not required to coincide with the base points.

Isogeometricity is ensured by the fact that the links of the S-polygon are parallel to the original tangents.

The initial S-polygon is defined as the polyline of the intersection points of the tangents. When defining a spatial polyline, the S-polygon is defined as the polyline of the "quasi-intersection" points of the intersecting tangent lines (the quasi-intersection point lies at the same distance from the tangent lines).

In the iterative approximation scheme, at the current approximation step m, a new S-polygon $\{V_i\}_m$ is determined by the residual reduction operation as follows. A spline of a given degree is calculated on the S-polygon and the distances $\{\delta_i\}$ from the spline to the support points $\{P_i\}$ are determined. The closest spline point to a support point does not coincide with a nodal point of the spline! New links of the S-polygon $\{V_i\}_m$ are determined as follows. The tangent lines containing the links of the current S-polygon $\{V_i\}_{m-1}$ by parallel displacement by values $\{-\delta_i\}$ define a new set of tangent lines $\{T_i\}_m$. The S-polygon $\{V_i\}_m$ is defined as a new polygonal line of the intersection points of the displaced tangent lines $\{T_i\}_m$.

Boundary Conditions

Additional points of the S-polygon are determined by boundary conditions.

Geometrically clear universal symmetry conditions and geometric procedures invariant with respect to the degree of the spline curve are used as boundary conditions:

1) Condition of closeness of the spatial curve.
2) Condition of spatial symmetry of additional vertices with respect to the plane passing through the starting (ending) point of the curve perpendicular to the first (last) tangent of the curve.
3) Condition of spatial central symmetry of additional vertices with respect to the first (last) point of the curve.
4) Fixed curvature values at the endpoints of the curve.
5) Condition of smooth monotonic continuation of the curvature at the endsites.

Let's take a closer look at the types of boundary conditions and the procedures for determining additional vertices:

1) Condition of spatial curve closure

The additional vertices of the initial section of the S-polygon coincide with the end vertices of the main section of the S-polygon.

The additional vertices of the end section of the S-polygon coincide with the initial vertices of the main section of the S-polygon.

2) Condition of spatial symmetry

The additional points are mirror images of the points with respect to the plane passing through the endpoint normal to the tangent vector.

3) Condition of spatial central symmetry

The additional points are centrally symmetric to the points of the main section with respect to the endpoint.

4) Fixed curvature value at the end points of the curve

The method ensures a fixed curvature at the endpoint with a harmonic spline shape at the end section.

The method utilizes the following key property of the open S-polygon of a B-spline curve. The key to high curve quality (including at the boundary sections) is the harmonious shape of the open S-polygon.

To ensure the harmonious shape of the S-polygon, the following procedure for determining boundary conditions is used.

Two limiting configurations of the open S-polygon are defined: a symmetrical configuration and a centrally symmetrical configuration.

The intermediate configuration is defined as a linear combination of the angle between the links of the limiting configurations.

An intermediate configuration with an exact fixed curvature value is found using the bisection method.

All configurations (limit and intermediate) have the same lengths of the corresponding links.

This method allows one to obtain an arbitrary curvature value (from zero for the centrally symmetrical configuration to the curvature value of the symmetrical configuration) with good end sections of the B-spline curve. A limitation of the method is that it is impossible to specify a fixed curvature value greater than the curvature value for the symmetrical configuration.

5) Condition for Smooth Continuation of an S-Polygon

In [55] [56], [57], a method for forming a B-polygon to ensure a monotonic change in the curvature of a Bézier curve is proposed. The essence of the method lies in constructing a Bézier curve polygon with a monotonic change in the length of links with a fixed elongation coefficient.

Naturally, the Meunier-Farin configuration is more correctly applied to an S-polygon in an float format for modeling a B-spline curve with a monotonic change in curvature [56].

This approach is the key to constructing the end section of an S-polygon while maintaining monotonic change in curvature. Additional end sites of S-polygons are defined as Meunier-Farin configurations.

Correction of tangents at endpoints and inflection points

In the main algorithm, the first approximation is based on the parallelism of the S-polygon links to the tangent vectors of the Hermite GD.

After defining the S-polygon and calculating the B-spline curve, the tangent vectors at the endpoints and inflection points of the B-spline curve will differ from the fixed tangents at the endpoints and inflection points of the Hermite GD.

To maintain fixed values of the tangents at the end points and at the inflection points of the Hermite GO in the values of the tangent vectors at the end points and at the inflection points of the B-spline curve, at each iteration step, the tangent lines for determining the directions of the links of the S-polygon, in addition to moving, are also rotated by a correcting angle -δ, where δ is the angle of misalignment of the tangent vectors of the B-spline curve with the fixed tangents of the Hermite GD.

After defining a B-spline curve with a given accuracy, the transition from an open (float) S-polygon to a closed (clamped) S-polygon is performed using well-known algorithms [59], [60].

This method radically changes the approach to constructing global polynomial splines. An infinite number of global polynomial splines can be constructed on a given base polyline. Spline optimization can be directed toward harmonizing the S-polygon configuration defined by the initial tangent polyline.

The method is invariant with respect to spline degree. Boundary parameters are also defined geometrically and implemented in geometrically oriented algorithms.

The algorithm can be easily modified to construct splines of odd degree [61].

4.3 Isogeometric modeling of NURBS by S-polygon

The first study of the form of a cubic parametric polynomial given by points and vectors of first derivatives was carried out by Forrest F.R.

In [9], the results of Forrest's analysis are considered using a more convenient for analysis equivalent representation of a cubic Bezier curve by a B-polygon. The first link of the B-polygon coincides with the vector of the first derivative at the initial point of the segment, and the third link coincides with the vector of the first derivative at the end point of the segment.

To prevent the curve from becoming looped form, Forrest recommends limiting the vector lengths and the chord length.

Note that these constraints result in two mandatory inflection points on the looped configuration of the control polyline.

In our opinion, a looped B-polygon should define a looped curve, if we follow the logic of the isogeometric shape of the B-polygon and the curve. Conversely, a designer won't be thrilled if, instead of the expected looped curve on a looped B-polygon, they get a curve with two unexpected inflection points.

In any case, the B-polygon Bézier curve shape analysis function should recognize such situations.

The exact equation relating the B-polygon configuration of a plane cubic Bézier curve to its inflection points is defined in our work, [60]. Analysis of this equation allows us to determine visually controllable zones of admissible B-polygon configurations. Solving the equation provides an exact answer about the curve shape, and for an S-shaped configuration, the exact coordinates of the inflection point. Admissible configuration zones are: S-shaped B-polygon, strictly convex B-polygon.

Note that a similar result was obtained for the S-polygon of a cubic B-spline curve [61].

In our work [64] the properties of the algorithm for calculating and subdividing the specification of a rational Bézier-Bernstein curve are used to analyze the stability of constructing a Bézier curve of arbitrary degree. It is proved that a B-polygon of regular shape of order *m* is a characteristic polygon.

The proof is based on the fact that the algorithm reduces to a sequence of operations: "taking a point" strictly on the segment and "cutting off the vertex" of the control polyline. These operations do not change the shape of the original regular polyline.

.

This approach is used by the authors to analyze the stability of the formation of a B-spline curve of arbitrary degree. The theorem is proved:
- An S-polygon of a regular shape of order *m* is a characteristic polygon (isogeometrically, up to the sign of curvature, it determines the shape) of a B-spline curve of degree *m*. Moreover, the number of sign changes in the orientations of the S-polygon links exactly coincides with the number of sign changes in the curvature of the B-spline curve.

5. Advantages of FairCurveModeler

5.1 Edvantages

F-curves vs A-curves

Class F curves (Fig. 5) Isogeometrically and exactly approximated the conics on polylines.

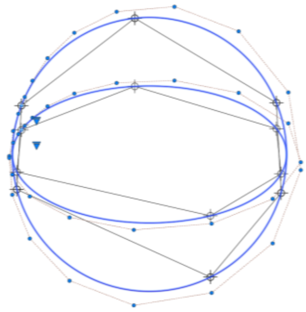

Fig. 5. Class F curves (FairCurveModeler).

Class A curves (Fig.6) do not provide a desired performance.

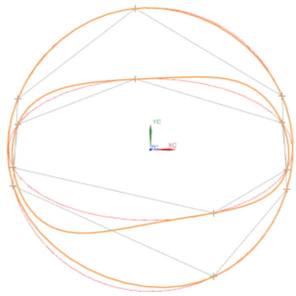

Fig. 6. Class A curves (NX).

In the given pictures, a generated B-spline curve from the given set of points on a circle by using Siemens NX software (Fig. 6) and C3D FairCurveModeler (Fig. 5) illustrates the difference between class A and class F curves.

As can be seen from this example, class A curves do not provide a desired performance.

Fixing tangents at base points when constructing a global spline

How does it work?

It is implemented as follows: we enter auxiliary "non-native" points on the polyline, and fix the required directions in the native points.

The algorithm changes the position of the "non-native" points to ensure the required direction of the tangents and builds a high-quality global spline that passes exactly through the "native" points with fixation of the tangents (Fig.7).

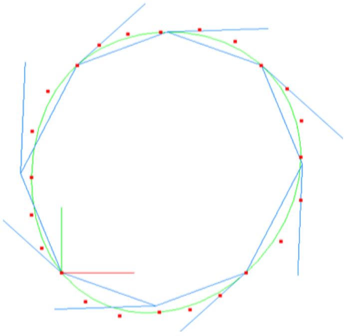

Fig. 7. Fixing tangents on a global spline

Geometric exactly modeling of conical curves

The F-curve models geometrically exactly any conic curve (Fig. 8).

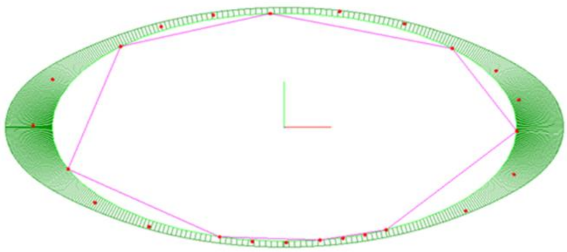

Fig. 8. F-curve in NURBzS curve format on ellipse points. Geometrically accurate ellipse. Executed in C3D FairCurveModeler.

Modeling fair spatial curves

KnowHow, [65], is the construction of an F-curve in NURBzS curve format on a spatial polyline (Fig. 9).

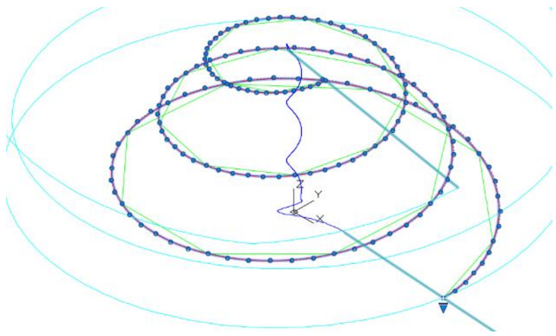

Fig. 9. F-curve in NURBzS curve format on the spatial polyline taken from the _Helix primitive. Executed in the FairCurveModeler app, nanoCAD/ZWCAD/AutoCAD.

Note the perfect shape of the evoluta graph (except for the end sections) (see Fig. 3, evoluta in blue color).

Analytical curves with monotonic curvature

Commands for creating analytic curves with monotone curvature.

Two remarkable analytical curves are introduced directly as construction commands into the FairCurveModeler system:

- Clothoid. A curve with monotone linear curvature (Fig. 10).

- Maclaurin sectrix. A curve with monotone curvature defined in an osculating triangle (Fig. 11)

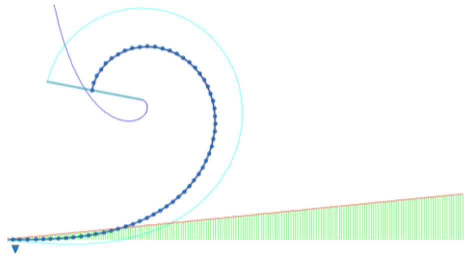

Fig. 10. Clothoid.

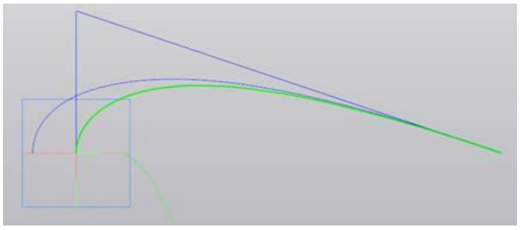

Fig. 11. Maclaurin sectrix.

5.2 F-curves vs physical spline

Elastic bars (physical splines) were used to model transverse spars, buttresses, and horizontals in the design and construction of marine vessels, and later in the manufacture of automobiles and airplanes. Many experts believe that physical splines are optimal for modeling functional curves.

Mehlum on the quality of the physical spline

The KURGLA curve modeling program for the AUTOKON shipbuilding system uses mathematically accurate modeling of a physical spline line, [66], [67].

In one of the KURGLA algorithms, the virtual physical spline is represented by segments of the clothoid.

According to, [66], [67], the curvature between fixed points of the physical spline varies linearly, as in the case of a clothoid.

Since Mehlum believes that the physical spline is accurately modeled by clothoid segments with a linear change in curvature, it turns out that the curvature graph of the physical spline is not fair, but piecewise linear. This is not an F-class curve.

Let us turn to Levien's work, [13], where he compares the ability of splines to model “roundness”. The work compares different types of splines.

In particular, one of the participants in the spline competition is Minimum-Energy Curves (MEC). In the work, the MEC is defined as a mathematical idealization of an elastic bar. The MEC in this work is defined as the curve that minimizes the energy functional of bending. Moreover, Levien clearly states the disadvantage of MEC. It is obvious that a circle is the fairest possible curve on three points, but the MEC spline deviates rather significantly from a circle (Fig. 12).

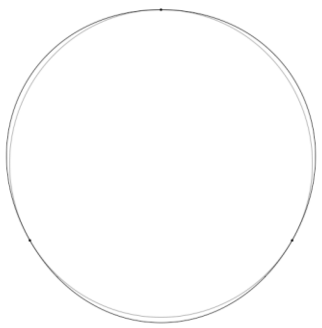

Fig. 12. Roundness failure of the minimum energy curve (MEC), [14]

It is also obvious that the circle has less potential energy than any other curve passing through three points. That is, MEC has a greater value of potential energy! Paradox!

Perhaps the following statement is true here: among curves of the same order of smoothness, the fairer curve passing through the same points has less potential energy.

It remains to admit that this conclusion is correct not only when analyzing roundness, but in general when analyzing the smoothness of the curve shape. That is, the fairer the curve, the smaller its potential energy.

Experiment F-curve vs. physical spline

Moreover, we will show that FairCurveModeler methods construct curves with lower potential energy than the method using a physical spline.

Let's compare FairCurveModeler methods and the "physical spline" method on the "Hamburg score". We will use FairCurveModeler methods implemented in the FairCurveModeler app ZWCAD / BricsCAD / AutoCAD, [14].

We will demonstrate this using three examples with different numbers of points.

Example 1. Four base points

A physical spline in the form of an elastic metal ruler is placed on an edge and deformed. The shape is fixed with weights in the form of cubes so that the ruler precisely contacts the cubes at points {(0,0), (100,130), (300,170), (390,0)} (Fig. 13).

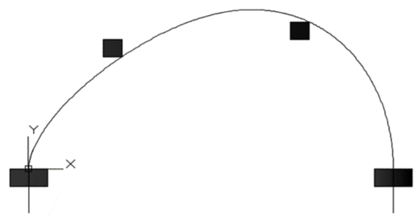

Fig. 13. Flexible ruler deformed by cube-shaped weights.

At the endpoints, the ruler is clamped vertically by the flat faces of two cubes. In order for the ruler to have a minimum length in the section between the clamping points, clamping is performed last after the ruler's shape has been established. During the process of establishing the shape, the ruler at the end points must pass freely between the weights.

The potential energy of the physical spline line is calculated indirectly by constructing a curve on the points taken from the physical spline line.

The line of the deformed ruler is outlined with a thin line. The points of contact with the cubes are marked on the image of the line. The coordinates of the points of contact of the ruler with the cubes are taken. Additional points on the physical spline are added to the original points of contact with the cubes. These points are taken approximately at the midpoints of the segments of the physical spline between the original points of the base polyline. The additional and original points are combined into one extended polyline with the coordinates presented in Table 1.

TABLE I. COORDINATES X,Y AND TANGENTS DX,DY

x	y	dx	dy
0	**0**	**0**	**1**
27	67.5		
100	**130**		
210	181.5		
300	**170**		
367	101		
390	**0**	**0**	**-1**

On the extended array of points (Table 1), a curve is determined using the FairCurveModeler program.

We will determine the parameters of the physical spline on the approximating curve: curvature and evoluta graphs (Fig. 14)

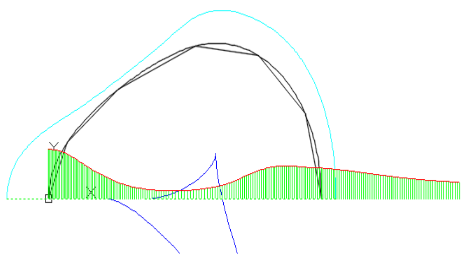

Fig. 14. Approximation of a physical spline. The graphs of curvature, evoluta, and the graph of the curvature function as a function F(x) are displayed.

Physical spline macroparameters:
"Real Length = " 585.573
"Approximated Length = " 585.468
"Potential Energy = " **0.0218344**
"Min Curvature = " 0.00249702
"Max Curvature = " 0.0245549

Then, on the original polyline {(0,0), (100,130), (300,170), (390,0)}, the vertices of which correspond to the contact points of the physical spline with the cubes, we construct a V-curve with approximation by means of the NURBzS curve. We test the quality of the curve (Fig. 15).

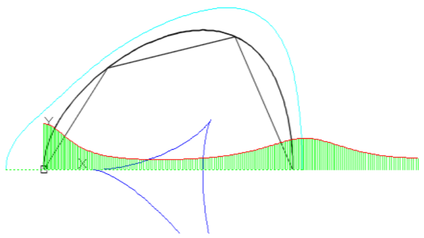

Fig. 15. V-curve on the original polyline of a physical spline. The curvature graphs, curve evoluta, and the curvature function graph as a function of F(x) are displayed.

V-curve macro parameters:
"Real Length = " 585.818
"Approximated Length = " 585.393
"Potential Energy = " **0.0205987**
"Min Curvature = " 0.00282694
"Max Curvature = " 0.0149465

Let us construct a V-curve with approximation by means of a B-spline curve of the 8th degree (Fig. 16).

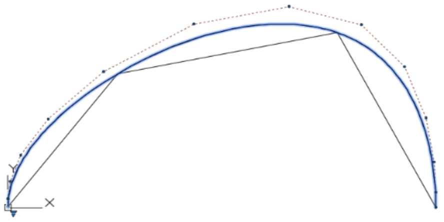

Fig. 16. B-spline curve of the 8th degree.

B-spline curve parameters:
"Real Length = " 586.083
"Approximated Length = " 585.976
"Potential Energy = " **0.0204288**
"Min Curvature = " 0.00283567
"Max Curvature = " 0.0129598

Note the high quality of the V-curve approximation both by the NURBzS curve and by the 8th degree B-spline curve.

Example 2. Three base points

The shape is fixed with weights at points {(0,0), (299,219), (550,0)}.

At the endpoints, the ruler is clamped vertically.

The additional and original points are combined into one extended polyline with the coordinates presented in Table II.

TABLE II. COORDINATES X,Y AND TANGENTS DX,DY

x	y	dx	dy
0	**0**	**0**	**1**
80	150		
299	**219**		
471	150		
550	**0**	**0**	**-1**

On the extended array of points (Table II), a curve is determined using the FairCurveModeler program.

We will determine the parameters of the physical spline on the approximating curve: curvature and evoluta graphs (Fig. 17).

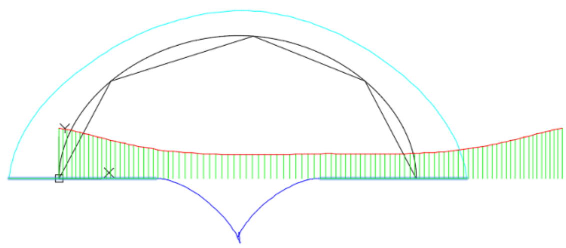

Fig. 17. Approximation of a physical spline. The graphs of curvature and evoluta are displayed.

Physical spline macroparameters:
"Approximated Length = " 773.991
"Potential Energy = " **0.0136505**
"Min Curvature = " 0.00313575
"Max Curvature = " 0.00673551

Then, on the original polyline {(0,0), (299,219), (550,0), the vertices of which correspond to the contact points of the physical spline with the cubes, we construct a V-curve with approximation by means of the NURBzS curve. We test the quality of the curve (Fig. 18).

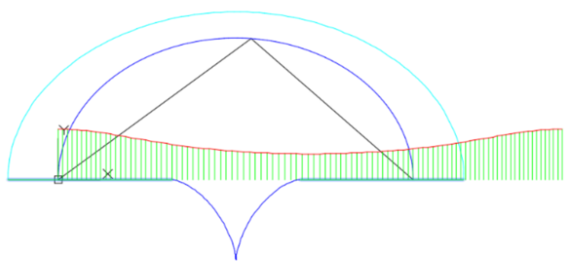

Fig. 18. V-curve on the original polyline of a physical spline. The curvature graphs and curve evoluta are displayed.

V-curve macroparameters:
"Approximated Length = " 779.675
"Potential Energy = " **0.013383**
"Min Curvature = " 0.00290712
"Max Curvature = " 0.00569015

Example 3. Five base points

The shape is fixed with weights at points {(0,0), (70, 240), (330,359), (600 389)}.

At the endpoints, the ruler is clamped vertically.

The additional and original points are combined into one extended polyline with the coordinates presented in Table III.

TABLE III. COORDINATES X,Y AND TANGENTS DX,DY

x	**y**	**dx**	**dy**
0	**0**	**0**	**1**
18	130		
70	**240**		
192	315		
339	**359**		
473	3		
600	**389**	**1**	**0**

On the extended array of points (Table II), a curve is determined using the FairCurveModeler program.

We will determine the parameters of the physical spline on the approximating curve: curva1ure and evoluta graphs (Fig. 19)

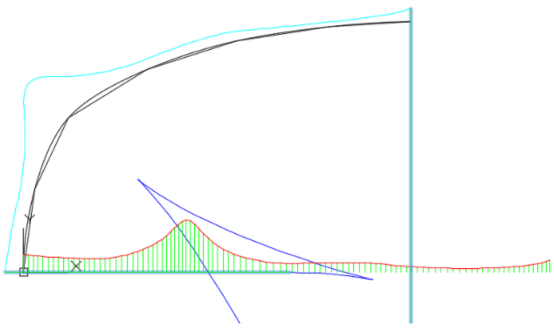

Fig. 19. Approximation of a physical spline. The graphs of curvature and evoluta are displayed.

Physical spline macroparameters:
"Approximated Length = " 815.686
"Potential Energy = " **0.0048364**
"Min Curvature = " 0.000540933
"Max Curvature = " 0.00696452

Then, on the original polyline {(0, 0), (299, 219), (550, 0), the vertices of which correspond to the contact points of the physical spline with the cubes, we construct a V-curve with approximation by means of the NURBzS curve. We test the quality of the curve (Fig. 20).

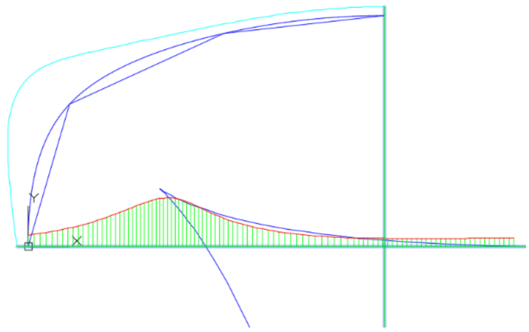

Fig. 20. V-curve on the original polyline of a physical spline. The curvature graphs and curve evoluta are displayed.

V-curve macroparameters:
"Approximated Length = " 818.168
"Potential Energy = " **0.00439636**
"Min Curvature = " 0.000801051
"Max Curvature = " 0.00479479

The curve constructed on the extended points of the physical spline has good qualities. But, as can be seen from the macroparameters, the curve constructed by the FairCurveModeler methods on the same initial polyline as the physical spline has a smaller curvature variation and a lower energy value. That is, it undoubtedly has better qualities in terms of fairness criteria than the curve drawn along the contour of the physical spline.

This is the hit of the FairCurveModeler system. This fact will undoubtedly cause cognitive dissonance in all fans of the physical spline. This fact can be explained as follows. A flexible bar clamped at both ends really takes a shape with minimal potential energy. This is elastic with a smooth curvature graph, [66]. But a flexible lath, additionally deformed at intermediate points, is no longer an elastica. Yes, the segments of a physical spline are elastic individually. But the profile of a physical spline according to [66], is a curve of not very high quality (with a piecewise linear curvature graph). And, as the experiment shows, a high-order F-curve with smooth curvature has less potential energy than a physical spline.

C. V-curve on Elastica

The development of the global parametrizited (GP) spline method can be aimed at using various types of analytical basis curves and/or increasing the parametric number of the curve. The GP spline method on a conic basis ensures high-quality modeled curves. The method ensures geometrically accurate modeling of circles and circular arcs if the support points of the polyline allow this.

However, in the case of a base polyline configuration with inflection points, the method fixes the inflection points and splits the polyline into locally convex sites. Although the locally convex sites are V-curves of class C5, the composite curve at the inflection points retains only second-order smoothness.

The authors suggest that the use of a basic 5-parameter curve with an inflection point will allow modeling V-curves of class C^5 on the entire curve with inflection points

Which curve to choose?

Let's consider the capabilities of a method based on Euler elastics. An elastic is a 5-parameter curve defined by two points, two tangents, and a segment length. An elastic geometrically accurately represents a circle and, under certain parameters, reliably defines a convex region and a region with a single inflection point [69]. The following challenges are facing researchers and developers.

First of all, all the equations of the listed stable configurations of the Euler elastica [69] must be reduced to working formulas for calculating points and derivatives on the interval of definition of the elastica between the end points.

Next, it is necessary to develop lower-level algorithms for solving the following geometric problems using the Euler elastic model:

- Calculating the tangent to the elastic model at a fixed point.
- Calculating the point of contact between the elastic model and a fixed tangent.
- Calculating the coordinates of an arbitrary point, the tangent vector, and the curvature vector of the elastic model based on the value of an internal parameter within the elastic segment definition.

When using a 5-parameter elastic model, an algorithm for constructing a basis for 5-parameter conical curves and an algorithm for generating V-curve points on locally convex polygons can be used.

Elastic models with cantilevered attachments (with non-zero curvature) at the ends must be used as base curvature segments.

For polylines with inflection points, the algorithms need to be refined. It is necessary to:

- Modify the algorithm for determining the spline basis from the set of double-contact elastics with floating inflection points.
- Modify the algorithm for determining the spline basis from the set of double-contact elastics with fixed inflection points. - For the end section of an unloaded spline (or for a segment with an inflection point at the segment endpoint), use a convex form of the Euler elastic with zero curvature at the endpoint.

6. Innovations of Surface modeling

6.1 Frame-kinematic scheme

The frame-kinematic scheme of construction allows for reducing the procedure of surface construction to two stages, [70]:

- Construction of the frame of generators of F-NURBS-curves on a uniform grid.
- Construction of the frame of guides of F-NURBS-curves on the frame of control spline S-polygons.

The advantages of the methods of isogeometric construction of F-curves are generalized to the methods of surface construction.

Isogeometrical creation of a B-spline surface on a set of polylines is shown, (Fig. 21, and Fig. 22).

.

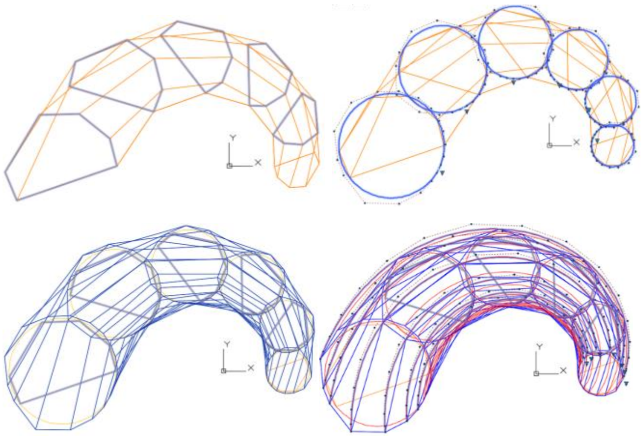

Fig. 21. S-frame of a B-spline surface.

The frame-kinematic scheme of construction allows for reducing the procedure of surface construction to two stages: construction of the frame of forming F-NURBS-curves on a uniform grid; construction of the frame of guiding F-NURBS-curves on the frame of control spline S-polygons.

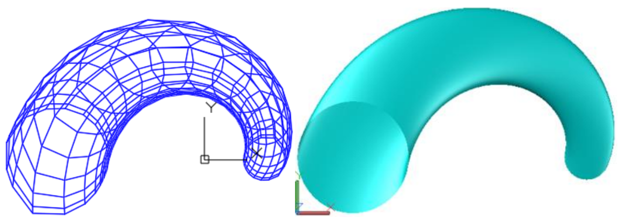

Fig. 22. B-spline surface of degrees (8 * 8) models a surface with high accuracy.

S-polygons of guiding F-NURBS-curves are united to the S-frame of the B-Spline surface.

Advantages of methods of isogeometric modeling of F-curves are generalized to methods of construction of surfaces.

6.2 Isogeometric modeling NURBS surfaces

There are various ways to evaluate the shape, aesthetics, and functional qualities of a surface. These include analyzing the shape of the framework of flat sections, analyzing maps of constant Gaussian curvature values, evaluating the shape and quality using a tinted image, studying a full-scale sample, and testing in a CAE system.

However, it is first necessary to ensure that the shape of the spline surface's control frame is isogeometric with the shape of the surface's isoparametric line family. Oscillating shapes of isoparametric lines, especially in the direction of fluid flow, are unacceptable for dynamic surfaces.

Due to the affine invariance of the S-frame projection of a B-spline surface, it is possible to construct the surface and evaluate its shape on flat projections.

A surface is isogeometrically defined on a projection if, based on the configuration of the S-frame projection, it is possible to unambiguously judge the shape of the isoparametric lines of the surface on the projection.

Let's assume that the initial geometric determinant of the S-frame of a surface is represented on the projection as a network of points, and the lines of the network represent a framework of convex polylines-rows. Clearly, the designer expects to see a family of convex isoparametric lines on the surface. The appearance of an oscillating isoparametric line is undesirable and indicates a flaw in the modeling method. Conversely, if the isoparametric lines follow the shape of the rows and columns of the isoparametric line, then the isoparametric lines are considered to isogeometrically define the surface shape.

In our work [64] we propose rules for constructing the S-frame of a B-spline surface that ensure isogeometricity.

A definition **of similarity in shape between two plane polylines** is introduced. It is proved that, given an affine projection of a compartment of a B-spline surface of degree *m* onto an arbitrary plane, and the projections of the polygonal lines of a local S-polytope are pairwise similar regular polygonal lines of order *m*, then the projections of the isolines of the lines will have the same shape.

Let's consider another configuration case: adjacent polylines of the GD have different forms on the projection. Ideally, the family of isoparametric lines should consist of two families of different forms, separated by a straight line. In any other case, the appearance of S-shaped isoparametric lines is inevitable. Therefore, the method should allow control over the shape of the intermediate isoparametric line (the amplitude and position of the inflection point of the S-shaped line) up to straightening. In other words, the shape of the transitional isoparametric lines should be editable using the modeling method.

A method for editing the S-frame of a B-spline surface is proposed to ensure the transition from one form of isoparametric lines of a NURBS surface section to another form without oscillation of the isoparametric lines [64].

6.3 Complex Topology Surfaces

When modeling the integral surface of a product, it is proposed to use a “mosaic” composed of sections of B-spline surfaces. In this case, adjacent sections have common subarrays of S-frames in float format to ensure overall smoothness (Fig. 23).

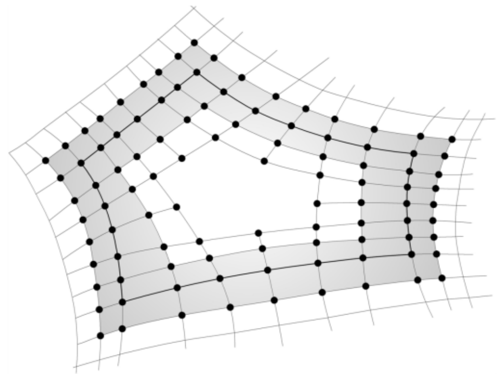

Fig. 23. Scheme of combining open (float) S-frames of B-spline surfaces.

When creating a mosaic, irregular areas inevitably arise that cannot be covered with a “rectangular patch” (Fig. 24).

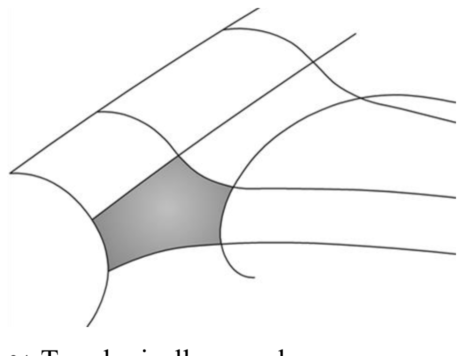

Fig. 24. Topologically complex areas.

For constructing such sections, [71], proposed a method, the essence of which is that pentagonal and triangular "holes" are patched with rectangular "patches" of B-spline surfaces defined by S-frames. In this case, the S-frames of the "patches" in the open format have common subarrays (see Fig. 23). The procedure for constructing a section is a recursive scheme of sequential "embedding" of sections of B-spline surfaces into a topologically complex section (Fig. 25).

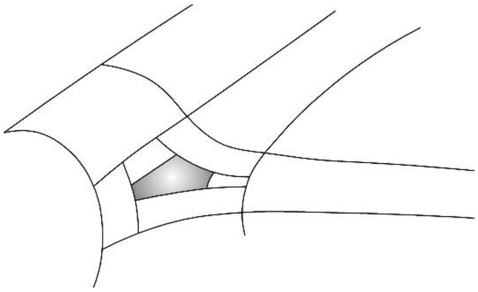

Fig. 25. Recursive scheme for "patching" a topologically complex area.

The "patching" procedure must be performed until the "hole" is reduced to a negligible size.

The following simple example demonstrates the possibility of constructing topologically complex surfaces composed of B-surface sections in the FairCurveModeler system.

The validity of this approach for B-surfaces is demonstrated in the scheme for constructing a one-sided Klein bottle surface (Fig. 26). The S-frame is joined to itself along the end lines r(u, v0) and r(u, vk) with the provision of the 4th order of smoothness. The S-frame rotates and glues to itself as a Mobius strip (Fig. 26, and Fig. 27).

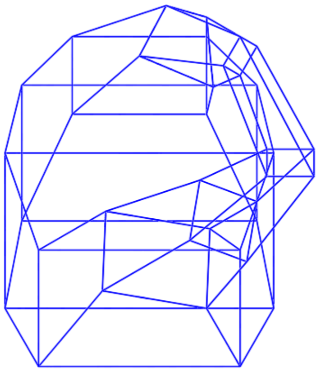

Fig. 26. S-frame of a one-sided surface.

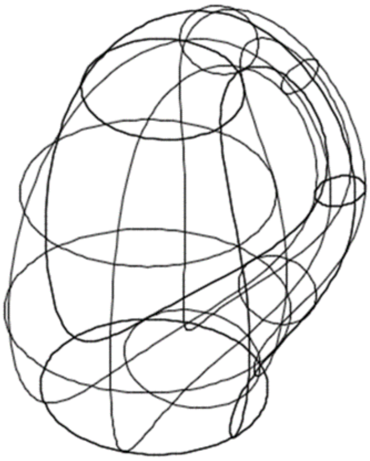

Fig. 27. Network of isoparametric curves of a one-sided surface.

Surface quality analysis is performed using the "Zebra" option (Fig. 28).

Fig. 28. Testing surface quality using the "Zebra" option.

Smoothness of the 4th order is ensured at any point on the surface. Including on the "seam" section (the line where the edges of the surface meet).

7. Plans

1) Development of a geometrically parameterized spline based on basic analytical curves called Elasticas. The V-curve is formed based on five-parameter analytical curves—an elastica of double tangency. The recurrence formula for constructing the V-curve itself remains unchanged. It is only necessary to develop a library of basic programs:

- Constructing an elastic in an osculating triangle at a given point.
-Determining a tangent on an elastica in an osculating triangle at a given point.
- Generating a point in a lens of osculating elasticas.

The authors expect a stunning result. The V-curve on elastics will be a class C^5 curve with an arbitrary shape (with inflection points) and will actually possess a minimum potential energy value.

Section "*C. V-curve on Elastica*" describes in sufficient detail a new method that the authors will patent in the near future.

2) Refinement for practical application and implementation of the proposed methods for modeling topologically complex surfaces in the actual design of ship surfaces.

The complex integral ship surface, comprising a complex bulbous form, a cylindrical hull part, and a flat bottom, will have a super-smoothness of orders of magnitude (7 x 7). CAM system testing will demonstrate high seaworthiness, dependent on the smoothness of the ship's contours.

8. Conclusions

The requirements for the quality of functional curves that directly determine the functionality and consumer properties of products are formulated and substantiated.

The article discusses the innovative characteristics of the software and methodological complex (SMC) FairCurveModeler for modeling high-quality functional curves according to fairness criteria.

The main innovations of the system are described - a parametric approach to constructing splines: constructing a spline basis as a set of 5-parameter conical curves of double contact, constructing a virtual curve of class C5 on the spline basis. Isogeometric approximation of a virtual curve using NURBS curves.

The system advantages of FairCurveModeler are shown. A unique variety of tools, various aspects of flexibility, the possibility of isogeometric approximation of analytical curves. The advantage of F-curves over class A curves, over a physical spline and over splines of their approximation is substantiated.

Innovative methods of surface construction are described: the frame-kinematic method of constructing a spline surface, the method of constructing a topologically complex surface.

References

[1] C. Zwikker, The Advanced Geometry of Plane Curves and Their Applications. New York: Dover Publications, Inc., 1963. 299 p. ISBN 0486610780. ISBN 9780486610788.

[2] Dome of St. Paul's Cathedral in London //https://architecturalidea.com/architecture-history/kupola-sobora-sv-pavla-v-londone

[3] Catenary line. A perfect arch from nature.https://artelectronics.ru/blogs/tsepnaya-liniya-idealnaya-arka-ot-prirody

[4] V. E. Pavlov, Brachistochrone as applied to a marshalling yard. Application of modern mathematical methods in railway operation // Coll. scientific. t. LIIZHT. 1969 (300). pp. 138-146.

[5] A.P. Rozhkov Valve drive cam. Author's certificate No. 1237778, priority from September 5, 1983, registered

[6] Neamtu M., Pottmann H., Schumaker L.L. Designing NURBS cam profiles using trigonometric splines. Journal of Mechanical Design 120 (2), pp. 175–180.

[7] N.I. Karyakin, K.N.Bystrov, P.S. Kireev, Brief reference book on physics. - M.: Higher School, 1969. - 600 p.

[8] Handbook of Physics for Engineers and University Students / B. M. Yavorsky, A. A. Detlaf, A. K. Lebedev. – M.: OOO "Izdatelstvo Onyx", 2006. – 1056 p.

[9] Fox, M. Pratt Computational Geometry. Application in design and production: Trans. from English. Moscow: Mir, 1982. 304 p., ill.

[10] O. V. Andreev, V. F. Babkov Handbook of road engineer. 2nd edition, revised and enlarged. Moscow: Transport, 1969. 552 p.

[11] Reference encyclopedia of road engineer / Ministry of Transport of Russia, Rosavtodor. Vol. 5. Design of highways // Ed. by G. A. Fedotov, P. I. Pospelov. Moscow: Informavtodor, 2007. 815 p.

[12] V.G. Mufteev, S.G.Mudarisov, I.M. Farkhutdinov et al. Justification for the selection of the optimal form of the functional curve of the dynamic surface of a technical product // Bulletin of the International Academy of Agrarian Education, 2013. Issue 17. P. 9093.

[13] Interpolating Splines: Which is the fairest of them all? / Raph Levien and Carlo H. Séquin. Computer-Aided Design and Applications © 2009 CAD Solutions, LLC. http://www.cadanda.com2009.

[14] APPLIED CAD AND APPLICATIONS BASED ON THE C3D GEOMETRIC KERNELS FOR DESIGNING PRODUCTS WITH FUNCTIONAL CURVES / Mufteev V.G., Maksimenko A.E., Akhmetshin R.I., Zelev A.P., Ziatdinov R.A., Ziganshina F.T., Nabiev R.I., Farkhutdinov I.M., Fattakhov M.M. // Proceedings of the International Conference on Computer Graphics and Vision "Graphicon". 2021. No. 31. P. 75-87.

[15] G. Farin Class A Bézier curves // Computer Aided Geometric Design. October 2006 23(7) P. 573-581. DOI: 10.1016/j.cagd.2006.03.004

[16] V.A. Zalgaller Theory of envelopes. -M.: Nauka, 1975. -104 p.

[17] Mathematics and CAD. In 2 books. / translated from French Moscow: Mir, 1980. 204 p. Book 1. ISBN 5-03-000417-3.

[18] Mathematics and CAD. In 2 books. / translated from French Moscow: Mir, 1980. 204 p. Book 2. ISBN 5-03-000465-3.

[19] Bernshtein S.N. Collected Works. - M., 1952. - T. 1. - P. 105-106.

[20] Bernshtein S.N. Collected Works. - M., 1954. - T. 3. - P. 310-348.

[21] Gordon, W.J. and Riesenfeld, R.F. (1974), 'B-spline curves and surfaces', in Computer Aided Geometric Design (Barnhill, R.E. andRiesenfeld, R.F., eds), Academic Press (New York), 95–126.CrossRef | Google Scholar

[22] De Boor C. A Practical Guide to Splines / Translated from English, edited by V.I.Skurykhin. - Moscow: Radio i Svyaz, 1985.- 304 p.

[23] Cox M. G. The Numerical Evaluation of B-Splines, J. Inst. Maths. Applies. 1972, 10, 134-149.

[24] Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, Quart. Appl. Math., vol. 4, pp. 45–99 and 112—141, 1946.]

[25] N.F. Chetverukhin On the parametrization of curved lines and surfaces and its importance in the educational process // Mathematics in school, 1964, No. 5, pp. 29-33.

[26] N.S. Chetverukhin, L.A. Yatskevich Parametrization and its application in geometry // Mathematics in school, 1965, No. 5, pp. 15-23.

[27] Kotov I.I. Approximation of contours by five- and six-parameter curves. Thematic collection of scientific papers/Moscow Aviation Institute, 1975, issue 331, pp. 39-49.

[28] Kotov I.I. et al. Algorithms of machine graphics / I.I.Kotov, V.S.Polozov, L.V.Shirokova. -M.: Mechanical Engineering, 1977. -271 p.

[29] Pro/ENGINEER. https://www.itweek.ru/themes/detail.php?ID=77145

[30] Baikalova S.M. Multiparameter curves and contours / Cybernetics of graphics and applied geometry of surfaces. Moscow: MAI. 1973. - Issue X. - p..

[31] Osipov V.A. Theoretical foundations of automation of geometric calculations and machine graphics (automated system of geometry and graphics). - M: Voenizdat, 1985. - 80 p.

[32] Computer-aided design. Geometric and graphical problems / V.O. Polozov, O.A. Budekov, S.I. Rotkov, L.V. Shirokova. -M.: Mashinostroenie, 1983. -280 p.

[33] Klevensky A.E. Modeling of geometric concepts and design technology [Text] / Ed. by Cand. of Technical Sciences O. I. Semenkov; Academy of Sciences of the BSSR. Institute of Technical Cybernetics. - Minsk: Science and Technology, 1973. - 125 p.

[34] J. Ahlberg et al. Spline Theory and Applications / J. Ahlberg, E. Nielson, J. Walsh. -M.: 1972. -316 p

[35] Yu. S. Zavyalov et al. Spline Function Methods / Zavyalov Yu. S. Kvasov B. I., Miroshnichenko V. L. -M.: Nauka, 1980, -352 p.

[36] Zavyalov Yu. S. Splines in engineering geometry. Moscow: Mashinostroenie, 1985. 224 c.

[37] Stechkin S. B., Subbotin Yu. N. Splines in computational mathematics. - Moscow: Nauka, 1976.

[38] Bor K. Practical guide to splines / translated from English by V. K. Galitsky, S. A. Shestakov; edited by V. I. Skurikhin. Moscow: Radio i svyaz, 1985. 304 c.

[39] Piegl L., Tiller W. The NURBS Book. 2nd ed. Springer Verlag, 1997. 660 p.

[40] Rogers D., Adams J. Mathematical foundations of computer graphics / translated from English by P. A. Monakhova. Moscow: Mir, 2001. 604 c. ISBN 5-03-002143-4.

[41] Mufteev V.G. Construction of plane curves by the envelope method // News of Higher Education Institutions. Aviation Technology, 1980

[42] Kotov I.I. Descriptive Geometry (based on the principles of programmed learning). Moscow: Higher School, 1970. 384 p.

[43] J.M. Chaikin Shot note on algorithm for high-speed curve generation // Computer graphics and image processing. -1974. –N3/ -p/346-349.

[44] V.A. Leus Smooth circular interpolation of curves / Computing systems, -Novosibirsk: IM SO AN SSSR, 1970. -pp.102-127.

[45] S.L. Pevzner. Projective geometry. - Moscow: "Prosveshchenie", 1980. - P. 68-69.

[46] Tikhonov A.I. and others. Regularizing algorithms and a priori information / A.I. Tikhonov, A.V. Goncharsky, V.V. Stepanov, A.G. Yagola. –M.: Nauka, 1983. -200 p.

[47] R.W. Soanes. Thrice differentiable affine conic spline interpolation. U.S. Army ARDEC Technical Report ARCCB-TR-95038, Benet Laboratories, Watervliet, NY, September 1995.

[48] A.I. Grebennikov The spline method and the solution of ill-posed problems of approximation theory. -M.: Publishing house of Moscow. University, 1983. -208 p.

[49]]Kvasov B. I. Methods of isogeometric approximation by splines. Moscow: Physics and Mathematics Literature. 2006. 416 c. ISBN 978-5-93972-416-6.

[50] T.J.R. Hughes, J.A. Cottrell, and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry, and mesh refinement. Computer Methods in Applied Mechanics and Engineering, 194:4135–4195, 2005.

[51] Mufteev V.G., Romanyuk A.N. Geometrically stable modeling of b-spline curves and surfaces of arbitrary degree // Wspotczesne problemy informatyki. Nowe kierunki w badan w informatyce.Pod redakcja Swietlany Lebiediewej I Arkadiusza Libera. Seria: monografie i prace naukowe. Wydawnictwo Wyższej Szkoły Menedżerskiej. Legnica 2007. C.209 – 232.

[52] Pogorelov A.V. Geometry. –M.: Nauka, 1983, -288 p.

[53] Zalgaller V.A. Theory of envelopes. -M.: Nauka, 1975. -104 p.

[54] Mufteev V.G. Modeling of high-quality curves based on v-curves // Applied Geometry. Applied Geometry [Electronic resource]: scientific journal / Moscow Aviation Institute (State Technical University) "MAI". - Electronic journal. - Moscow: MAI, 2007. - No. 19; issue 9, – Access mode to the journal: http://www.mai.ru. – Title with the title screen. - pp. 25-74.

[55] Mineur, Y., Lichah, T., Castelain, J.M., Giaume, H., 1998. A shape controlled fitting method for Bézier curves. Computer Aided Geometric Design 15 (9), 879–891.

[56] Yves Mineur. A Shape Constrained Curve Approximation Method for Styling Surfaces Modeling.University of Valenciennes. CNRS, UMR 8530 LAMIH France BP311 59313 Valenciennes Cedex9. yves.mineur@univ-valenciennes.fr

[57] Farin, G. Class A Bézier curves // Computer Aided Geometric Design 23 (2006) 573–581.

[58] Techniques for modeling a high-quality B-spline curves by S-polygons in a float format / Ziatdinov R., Muftejev V., Nabiyev R., Mardanov A., Akhmetshin R. // Proceedings of the International Conference on Computer Graphics and Vision "Graphicon". 2018. No. 28. Pp. 324-327.

[59] Sabloniere P. Spline and Bezier polygons assotiated with a polynomial spline curve // Computer aided design. -1978. –v.10. –N4. – p.257-261.

[60] Boehm. Cubic B-spline curves and surfaces in computer-aided geometric design// Computing 19(1977) –p.29-34.

[61] C3D ToolKit. https://c3dlabs.ru/products/c3d-toolkit/

[62] Mufteev V.G. et al. Design of plane contours using Bezier-Bernstein curves // Mufteev V.G., Lukmanov F.F., Spiridonova E.B., Syrtlanov R.V.

/ News of Higher Education Institutions. Aviation Engineering, 1983, No. 2. pp. 106-108

[63] Муфтеев В.Г. Моделирование кривых с помощью b-сплайнов третьей степени с учетом ограничений, накладываемых на форму кривой // Прикладная геометрия и инженерная графика. -Киев: Будівельник, 1982. Вып. 35. С.111-116

[64] Mufteev V.G., Romanyuk A.N., Mardanov A.R., Farkhutdinov I.M. Geometrically stable modeling of NURBS curves and surfaces of arbitrary degrees // Applied Geometry. Applied Geometry [Electronic resource]: scientific. journal / Moscow Aviation Institute (State Technical University) "MAI". - Electronic. journal. - Moscow: MAI, 2009. - No. 22; issue 11, - pp. 19-77. - Access mode to the journal: http://www.mai.ru . - Title with the title screen. - State registration No. 019164.

[65] Qualitative modification of geometrically oriented methods for constructing spatial curves in C3D FairCurveModeler / Mufteev V.G., Ziganshina F.T., Gumerov V.I., Nabiev R.I., Efimova G.M. // Omsk Scientific Bulletin. 2024. No. 3 (191). P. 48-55.

[66] E. Mehlum, Nonlinear splines // Computer Aided Geometric Design. Academic Press, London, 173-207, 1974

[67] Mehlum E. and Sorenson P. F. Example of an Existing System in the Shipbuilding Industry: the AUTOCON System, Proc. Roy. Soc. Lond. A. 321, 219-233.

[68] Euler L. Method of Finding Curved Lines with Maximum or Minimum Properties, or Solution of the Isoperimetric Problem Taken in the Broadest Sense, by Leonhard Euler, Royal Professor and Member of the Imperial St. Petersburg Academy of Sciences. Appendix I. "On Elastic Curves". Moscow Leningrad: GTTI, 1934. Pp. 447-572.

[69] Sachkov Yu.L., Levyakov S.V. Stability of inflectional elasticae centered at vertices or inflection points / PROCEEDINGS OF THE STEKLOV MATHEMATICAL INSTITUTE, 2010, v. 271, pp. 187–203

[70] Osipov V.A. Design of continuous frameworks of surfaces with predetermined differential properties / Proceedings of MIREA. - M: MIREA, 1972, issue 63. - pp. 47-53.

[71] A.R. Forrest A unified approach to geometric modelling //SIGGRAPH’78. – NewYork:ACM, 1978. – p.264-270.