Numerical Inference on the Generalized Gamma Distribution Parameters

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Abstract: - In life data analysis, there are many methods for estimating distribution parameters. However, most of them are less efficient than Bayes' method based on the informative prior. Therefore, the main objective of this study is to introduce optimal numerical iteration techniques, such as the Picard, Runge-Kutta and Adams methods for estimating the generalize life model parameters and compare them with the Bayes' method based on the informative gamma and kernel priors. A comparison between these methods is provided using an extensive Monte Carlo simulation study based on two criteria, namely, the absolute bias and the mean squared error. The simulation results indicated that the numerical methods are highly favourable, which provide better estimates and outperform the Bayes' method based on the generalized progressive hybrid censoring scheme. Finally, two real datasets analyses are presented to illustrate the efficiency of the proposed methods.

Key-Words: - Bayes estimation; Adams's method; Picard's method; Runge-Kutta method; generalized progressive hybrid censoring scheme; informative prior; kernel prior

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1 Introduction

In statistical inference, many methods have been used for estimating the distribution parameters. However, the most usable method in reliability analysis is the Bayes' method, which is unfortunately, subjective to prior information other than data. Thus, in this study, we present optimal estimation methods using numerical iteration techniques, such as the Picard, Runge-Kutta and Adams methods for estimating the distribution parameters, which are more efficient than the Bayes' method based on the informative and kernel priors. To illustrate that, we employed the proposed methods on one of the employed lifetime distributions and reliability theory, the generalize life model (GLM) with application to the three-parameter generalized gamma distribution (GGD). This distribution has been introduced in the literature that has utility in survivorship applications. [2] and [3] showed that if one chooses the parameters of the GLM appropriately, it will cover a large proportion of curve-shaped characteristics of types I, IV, and VI in the Pearson family of distributions. The cumulative distribution function and the probability density function for the GLM, which belongs to the shape-scale family are given, respectively as:

$$\mathrm{F(x)} = 1 - [1 + \beta g^{\alpha}(\mathrm{x})]^{-\gamma}, \quad 0 \le \mathrm{x}, \quad \alpha, \beta, \gamma > 0.$$

$$\mathrm{f(x)} = \alpha\beta\gamma g^{\alpha-1}(x)\mathrm{g}'(\mathrm{x})[1 + \beta g^{\alpha}(\mathrm{x})]^{-\gamma-1},$$
$$0 \le \mathrm{x}, \quad \alpha, \beta, \gamma > 0.$$

The parameters $\alpha, \gamma > 0$ are the shape parameters and $\beta > 0$ is the scale parameter.
For convenience we assume $g(x)$ to be differentiable as well as strictly increasing function of x, $\mathrm{g}(0^{+}) = 0$ and $g(x) \to \infty \quad as \quad x \to \infty$.

This family includes among others the most popular parameteric models in lifetime distributions such as the generalized Weibull distribution, the generalized gamma distribution, Pareto distribution, Burr-type-XII distribution, Lamox distribution and the Generalized Pareto distribution according to the values of $g(x)$. Some important members of this family are shown in Table 1.

As application for the generalized shape-scale

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family, the generalized gamma distribution (GGD) is the one that has a great part of applications based on progressive order statistics for its flexibility to describe the life test data. The GGD with the (cdf) F(x) and the (pdf) f(x), which are given respectively as follows:

$$\boldsymbol{F(x) = 1 - (1 + e^{\beta x^{\alpha}})^{-\gamma},\ x \geq 0, \alpha, \beta, \gamma > 0.} \quad (1)$$

$$\boldsymbol{f(x) = \alpha\beta\gamma x^{\alpha-1} e^{\beta x^{\alpha}} (1 + e^{\beta x^{\alpha}})^{-\gamma-1},} \quad \boldsymbol{x \geq 0, \alpha, \beta, \gamma > 0} \quad (2)$$

where α and γ are form factors and beta is the scale factor.

Table 1: Some important lifetime distributions

No.	$g(x)$	F(x)	Distribution
1	x	$1-(1+\beta x^{\alpha})^{-\gamma}$	3P-Burr-XII
2	x	$1-(1+x^{\alpha})^{-\gamma}$, $B=1$	Burr -Type XII
3	x	$1-(1+\beta x)^{-\gamma}$, $\alpha=1$	Lomax
4	$-x$	$1-(1-\beta x)^{-\gamma}$, $\alpha=1$	Generalized Pareto
5	x	$1-(1+x^{\alpha}/\gamma)^{-\gamma}$, $\beta=\frac{1}{\gamma}$	Generalized Weibull
6	$e^{\alpha x^{\theta}}$	$1-(1+e^{\alpha x^{\theta}})^{-\gamma}$, $\beta=1$	Generalized Gamma

In applied statistics, modelling and analysing of real-life data are necessary to understand the important features of random and non-random phenomena to draw appropriate conclusions. This requires the selection of statistical models based on probability distributions. Therefore, analysis of recent data in applied sciences (medical sciences, environmental sciences, engineering, finance, etc.) has shown the GGD is an appropriate model that allows revealing some important details from these datasets. In this study, we introduce improved estimation methods, such as the Picard. Runge-Kutta and Adams for estimating the GGD parameters compared with the Bayes' method based on the informative and kernel priors. The statistical analysis of the GGD based on complete and censored samples has been studied by several authors for analysing data in applied sciences, such as [32] derived the estimation of the GGD parameters, [13] derived the asymptotic maximum likelihood estimators (MLEs) for the GGD parameters, [15] applied the conditional inference for the GGD and log gamma parameters, [9] applied the approximate inference for the GGD parameters, [34] derived the MLEs of the GGD parameters, [35] derived of the MLEs of the three GGD parameters. [16] derived the conditional and structural confidence intervals for the GGD parameters, and [17] derived the structural confidence intervals based on the type-II progressively censored samples from the three-parameter GGD. [14] derived the moment estimates of the GGD parameters using its characterization. [8] introduced some concepts of the GGD based on the information theory. [11] described several difficulties in estimating the parameters and the complexity of the MLE method, and [31] used their perception in developing inference procedures with the GGD, especially when the MLE is used to justify the model in terms of a simpler alternative form. [10] suggested a new re-parameterization of the GGD to maintain the numerical stability for the MLE based on type-II progressively censored samples. [28] presented a Bayesian study comparing the GGD with its components. [3] obtained a new discrete distribution related to the GGD and discussed some of its properties. [29] introduced comparative studies on the modeling of lifetime data using the three-parameter GGD.

In reliability analysis, the progressive type-II censoring scheme is the most applicable in life test experiments, it is useful for both industrial life test applications and clinical trials and allows removing some of the surviving experimental units at different stages before testing is terminated. However, the trial time can be quite long due to some highly reliable units. Thus, [5-7], and [12] proposed the generalized progressive hybrid censoring scheme (GPHCS) that modifies the progressive hybrid censoring scheme, see [4]. Let $\underline{X} = (x_1, x_2, \dots, x_n$, be the number of observations, which are subjected under the experiment that allows to continue beyond defined time **T** to observe at least **k** failures, if the number of failures is less than $\mathbf{m}$, where $k < \mathrm{m}$. Let $R_1, R_2, \dots, R_m$ are the random removal units that are fixed at the beginning of the experiment such that $n = m + \sum_{i=1}^{m} R_I$, and D denote the number of observed failures up to the time T. Generally, at the time of the i_{th} failure, R_I units are randomly removed from the remaining surviving units $S_i = n - i - \sum_{j=1}^{i-1} R_j$, where $i = 1,2,\dots,m$. This process continues until, immediately following the terminated time $T^* = \max\{X_{k:m:n}, \min\{X_{m:n:n}, T\}\}$, where at this time all the remaining surviving units $\mathbf{R_T}^*$are removed from the experiment. The algorithm of the GPHCS has been described in [19-20].

Thus, given a GPHCS, the likelihood function can be written in a unified form as follows:

$$L(\bar{X}; \theta) = C \prod_{i=1}^{n} f(x_i) \, [1 - F(x_i)]^{R_i} [1 - F(T)]^{\delta R_T^*}, \quad (3)$$

$$n = \begin{cases} m, & \delta = 0 \quad , \ if \quad X_{k:m:n} \le X_{m:m:n} < T \\ k, & \delta = 0 \quad , \ if \quad T < X_{K:m:n} \le \quad X_{m:m:n} \, , \\ D, & \delta = 1 \quad , \ if \quad X_{k:m:n} < \quad T < \quad X_{m:m:n} \end{cases}$$

The GPHCS has been applied for some distributions, see [5-7], [12], and [19-25].

2 Estimation Methods

2.1 Numerical Methods

2.1.1 Runge-Kutta Method

The MLE $\hat{\theta} = \hat{\theta}(\bar{x})$ of θ is the solution of the stationary equation, $\frac{\partial H(\underline{X};\theta)}{\partial \theta}|_{\theta=\hat{\theta}} = 0$, which is a function of $\underline{X}$ and $\hat{\theta}(\bar{x})$, where $H(\underline{X};\bar{\theta})$ is the log-likelihood function that depends on the unknown parameter $\hat{\theta} = (\alpha, \beta)$ and the data $\underline{X} = (x_1, x_2, \dots, x_n)$. Applying the implicit function theorem to the stationary equation with considering all partial derivatives as well as the total derivatives are assumed to be evaluated at some known value of $\hat{\theta}(x) = \theta_0$, say. Taking the total derivative for the stationary equation with respect to any $x \in \underline{X}$, we obtain.

$$\frac{d}{dx}\left(\frac{\partial H(\underline{X};\theta)}{\partial \theta}\right)|_{\theta=\hat{\theta}} = \frac{\partial^2 H(\underline{X};\theta)}{\partial \theta\, \partial x}|_{\theta=\hat{\theta}} \cdot + \frac{\partial^2 H(\underline{X};\theta)}{\partial \theta^2}|_{\theta=\hat{\theta}} \frac{d\hat{\theta}}{dx} = 0. \quad (4)$$

Solving (4) we obtain the first derivative with respect to x for $\hat{\theta}$ at $\theta = \hat{\theta}$ as follows:

$$\frac{d\hat{\theta}}{dx} = f(x, \hat{\theta}), \quad \hat{\theta}(x_0) = \theta_0, \quad (5)$$

where

$$f(x, \hat{\theta}) = -\left(\frac{\partial^2 H(\underline{X};\theta)}{\partial \theta^2}|_{\theta=\hat{\theta}}\right)^{-1} \frac{\partial^2 H(\underline{X};\theta)}{\partial \theta\, \partial x}|_{\theta=\hat{\theta}} \, .$$

Here $f(x, \hat{\theta})$ and $\frac{df(x,\hat{\theta})}{d\hat{\theta}}$ are defined and continuous functions at all points $(\underline{X}, \hat{\theta})$, which ensures the existence of a unique solution for (5). If the initial conditions are unavailable, they must be appended to the MLE $\hat{\theta}$ as quantities with respect to which the fit is optimized.

Thus, the recurrence relation for the Runge-Kutta (R-K) method for (5) can be obtained as follows:

$$\theta^*_{n+1} = \theta^*_n + \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}, \quad (6)$$

for $n = 0,1,2, \dots,$

where

$$K_1 = hf(x_n, \theta^*_n), \ K_2 = hf\left(x_n + \frac{h}{2}, \theta^*_n + \frac{K_1}{2}\right),$$

$$K_3 = hf\left(x_n + \frac{h}{2}, \theta^*_n + \frac{K_2}{2}\right),$$

$$K_4 = hf(x_n + h, \theta^*_n + K_3).$$

It is important to note that the R-K method has been obtained for some distribution parameters, see [19-21].

2.1.2 Adam's Method

For the initial differential equation (5), the recurrence relation for Adams' method can be derived as follows:

$$\theta^*_{n+1} = \theta^*_n + h[23f(x_n, \theta^*) - 16f(x_{n-1}, \theta^*) + 5f(x_{n-2}, \theta^*)]/12, \quad (7)$$

In (6) and (7) h is a small known value (say, 0.01) and $\theta^*_0 = \theta_0$, is the initial value for θ^*.

This method has been applied on some distribution, see [26-27].

2.1.3 Picard's Method

The recurrence relation for Picard's method for the parameter α (say) can be obtained by writing (5) as follows:

$$\frac{d\hat{\alpha}}{dx} = f(x, \hat{\alpha}, \hat{\beta}, \hat{\gamma}), \quad \hat{\alpha}(x_0) = \alpha_0, \quad (8)$$

Using Picard's method, we can find the approximate solution for the parameter given a trial set of values for the first order differential equations (8), we can write the general iteration rule for Picard's method by integrating (8) with respect to x from x_0 to x as follows:

$$\hat{\alpha}_{n+1}(x) = \hat{\alpha}(x_0) + \int_{x_0}^{x} f(x, \hat{\alpha}_n, \hat{\beta}, \hat{\gamma}) dx$$

$$= \alpha_0 + \int_{\hat{\beta}(x_0)}^{\hat{\beta}(x^*)} f(x, \hat{\alpha}_n, \hat{\beta}, \hat{\gamma}) \frac{dx}{d\hat{\beta}} d\hat{\beta}$$

From the differential equation for the parameter β we get $\frac{d\hat{\beta}}{dx} = g(x, \hat{\alpha}, \hat{\beta}_0, \hat{\gamma})$, thus

$$\hat{\alpha}_{n+1}(x) = \alpha_0 + \int_{\beta_0}^{\hat{\beta}(x^*)} \left[\frac{f(x, \hat{\alpha}_n, \hat{\beta}, \hat{\gamma})}{g(x, \hat{\alpha}_n, \hat{\beta}_0, \hat{\gamma})}\right] d\hat{\beta}\, , \quad (9)$$

where α_0 is the initial point and $\hat{\alpha}(x^*)$ is the value, which the desired solution should be optimized. The iterative process for the numerical methods based on

(6), (7), (8) and (9) are continued until two consecutive numerical solutions are almost the same, that is if $\left|\hat{\theta}_{n+1}-\hat{\theta}_n\right| < 1E-05$, for $n = 0,1,2,3,..$ Similarly, we can find the Picard estimates for the parameters β and γ.
This method has been applied for some distributions, see [24, 26, 27].

2.2 Bayes' Method

In this section, the Bayes estimators for the parameters will be derived using the informative gamma and kernel prior distributions based on the squared error loss function:

2.2.1 The Informative Gamma Prior

We consider the unknown parameters α, β and γ have independent gamma prior distributions with the joint probability density function, which is given by:

$$H(\alpha,\beta,\gamma) \propto \alpha^{a-1}\beta^{c-1}\gamma^{e-1}e^{-b\alpha-d\beta-f\gamma},, \quad (10)$$

where the hyper-parameter a, b, c, d, e and f are assumed to be known non-negative real numbers and chosen to reflect the prior belief about the unknown parameters.

2.2.2 The Kernel Prior

For deriving the kernel prior, we introduce the trivariate kernel density estimator for the unknown probability density function $\boldsymbol{g(\alpha,\beta,\gamma)}$ with support on $(\mathbf{0},\infty)$, which is defined as

$$\hat{\mathbf{g}}(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = \frac{\mathbf{1}}{\mathbf{nh_1h_2h_3}}\sum_{\mathbf{i=1}}^{\boldsymbol{n}}\mathbf{K}\left(\frac{\boldsymbol{\alpha}-\boldsymbol{\alpha}_\mathbf{i}}{\mathbf{h_1}},\frac{\boldsymbol{\beta}-\boldsymbol{\beta}_\mathbf{i}}{\mathbf{h_2}},\frac{\boldsymbol{\gamma}-\boldsymbol{\gamma}_\mathbf{i}}{\mathbf{h_3}}\right), \quad (11)$$

$\boldsymbol{h_i}, \boldsymbol{i}=\mathbf{1,2,3}$ are called the bandwidths or smoothing parameters, which chosen such that $\boldsymbol{h_i}\to\mathbf{0}$ and $\boldsymbol{nh_i}\to\infty$ as $\boldsymbol{n}\to\infty$, where $\boldsymbol{n}$ is the sample size. The influence of the smoothing parameter $\boldsymbol{h}$ is critical because it determines the amount of smoothing. However, the optimal choice for $\mathbf{h_I}$ that minimizes mean squared errors is given by $\mathbf{h_i} = \mathbf{1.06\hat{S}_i n^{-0.2}}$, where $\mathbf{S_I}$ the sample standard deviation. The optimal choice for the kernel function $\boldsymbol{K}(.,.)$ can be used as the trivariate standard normal distribution for the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$.
The algorithm for deriving the kernel prior has been developed in [1] and [18].

Thus, using the joint priors (10) and (11) with the likelihood function of the GPHCS (3) the posterior density for the parameters α, β and γ can be written in a unified form as follows:

$$f(\alpha,\beta,\gamma|\underline{x}) = Kl(\alpha,\beta,\gamma)L(\bar{X};\theta), \text{ where}$$

$$l(\alpha,\beta,\gamma) = h(\alpha,\beta,\gamma)\hat{g}(\alpha,\beta,\gamma)$$

$$\propto \alpha^{a-1}\beta^{c-1}\gamma^{e-1}e^{-b\alpha-d\beta-f\gamma}\hat{g}_1^{p_1}(\alpha)\,\hat{g}_2^{p_2}(\beta)\,\hat{g}_3^{p_3}(\gamma),$$

is the general prior distribution function with
i) $P_1 = p_2 = p_3 = 0$ for the informative prior (10),
ii) $p_1 = p_2 = p_3 = 1$, $a = c = e = 1$, and $b = d = f = 0$ for the kernel prior (11).

Thus, the log posterior density can be written as

$$\begin{aligned}H(\alpha,\beta,\gamma|\underline{x}) &= [p_1\ln\hat{g}_1(\alpha) + p_2\ln\hat{g}_2(\beta) + p_3\ln\hat{g}_3(\gamma)\\
&+(n+a-1)\ln\alpha + (n+c-1)\ln\beta\\
&+(n+e-1)\ln\gamma - \alpha b - \gamma f\\
&+(\alpha-1)\sum_{i=1}^{n}\ln x - \beta\left(d-\sum_{i=1}^{N}x_i^{\alpha}\right) - \sum_{i=1}^{n}ln\left(1+e^{\beta x_i^{\alpha}}\right)\\
&-\gamma\left[\sum_{i=1}^{n}(1+R_i))ln\left(1+e^{\beta x_i^{\alpha}}\right)+\delta R_T^*\ln\left(1+e^{\beta T^{\alpha}}\right)\right].\end{aligned} \quad (12)$$

Thus, based on (12), we can use the Tierney-Kadane approximation method to approximate all the Bayes estimators for the unknown parameters. [33] introduced an easily computable approximation for the posterior mean and variance of a non-negative parameter or more generally, of a smooth function of the parameter that is non-zero on the interior of the parameter space. For detail, let $w(\alpha,\beta,\gamma)$ be a smooth, positive function on the parameter space. The posterior expectation of $w(\alpha,\beta,\gamma)$ can be obtained as

$$w^* = E(w(\alpha,\beta,\gamma)|\bar{X})$$

$$= \frac{\int_0^\infty\int_0^\infty\int_0^\infty e^{nH^*(\alpha,\beta,\gamma)}d\alpha d\beta d\gamma}{\int_0^\infty\int_0^\infty\int_0^\infty e^{nH(\alpha,\beta,\gamma)}d\alpha d\beta d\gamma},$$

where $H = \dfrac{\ln f(\alpha,\beta,\gamma|x)}{n}$, and

$$H^* = H + \frac{\ln w(\alpha,\beta,\gamma)}{n}.$$

For (α,β,γ) the Bayes estimator using Tierney-Kadane approximation for $w(\alpha,\beta,\gamma)$ can be obtained as

$$w^* = \sqrt{|\Sigma^*|/|\Sigma|}\exp[\,n[H^*(\alpha,\beta,\gamma) - H(\alpha,\beta,\gamma)],$$

where $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$ maximize the $H(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $H^*(\hat{\alpha}^*,\hat{\beta}^*,\hat{\gamma}^*)$, respectively. Let

$$|\Sigma| = \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix}^{-1} \text{ and}$$

$$|\Sigma*| = \begin{vmatrix} H_{11}^{*} & H_{12}^{*} & H_{13}^{*} \\ H_{21}^{*} & H_{22}^{*} & H_{23}^{*} \\ H_{31}^{*} & H_{32}^{*} & H_{33}^{*} \end{vmatrix}^{-1}$$

denote the minus of inverse of Hessians of $H(\alpha,\beta,\gamma)$ and $H^{*}(\alpha,\beta,\gamma)$ at $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and $(\hat{\alpha}^{*},\hat{\beta}^{*},\hat{\gamma}^{*})$ respectively. The derivatives of $H(\alpha,\beta,\gamma)$ and $H^{*}(\alpha,\beta,\gamma)$ have been derived in the Appendix A.

3. Real Data Analysis

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the three-parameter GGD in practice and to illustrate that this distribution can be considered as a good lifetime model for some new area of applications, comparing with many known distributions such as the Weibull distribution. We have fitted these data sets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-darling (A-D) and Chi-Square (CH2) tests for significance level test equals 0.05.

3.1 The Reactor Pumps Data

In this section, a real data set for secondary nuclear pump has been analyzed to illustrate the proposed methods. One of the most severe accidents in nuclear power generation is the loss of coolant, where the re-circulating coolant of the pressurized water reactor may flash into steam. Under such conditions, the reactor cooling pumps become unable to generate the same head as that of the single-phase flow case. Thus, the secondary reactor pump is a feedwater pump that takes from the desecrator storage tank feedwater pressured up by the booster pump and pushes it into the steam generator through the high-pressure heater. Accordingly, the main feed pump must be high temperature and high-pressure pump since it requires a head larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following data set represents the times between the failure of the secondary reactor pumps. [29, 30] have discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this data set. The times between failures of 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

We found the GGD is a good fit for this dataset as shown in Figure (1a). For studying the reliability of these reactor pumps based on this dataset we find the estimates for the parameters which represent the shape of the failures between pumps using our model to determine the behavior of the failure pumps. We noticed that the Picard and Bayes estimates for α lies in the interval [0.5,0.6] and for γ lies in the interval [0.9, 1.0]. For the scale parameter β lies in the interval [0.01,0.02], which is close to zero. Thus, these values indicating that the above dataset is heavily right-skewed and that means the failure rate decreases with increasing the time, see Figure (1b). Thus, the reliability of safety mechanism will decrease with increasing time, which indicate lacking safety.

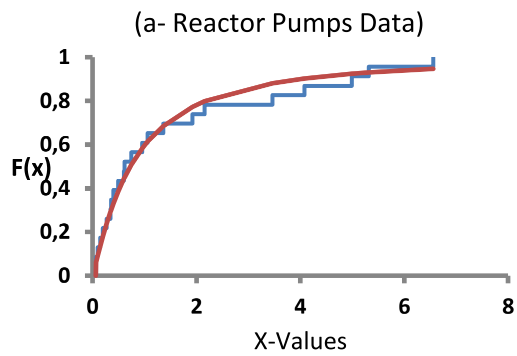

Fig. 1: a) The Empirical CDF and the fitted CDF.

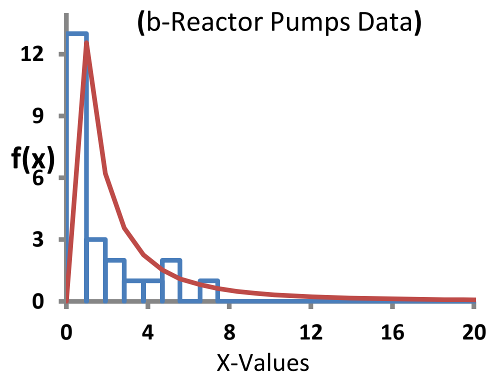

Fig. b) The Histogram and the fitted PDF.

3.2 The Vinyl Chloride Data

As vinyl chloride is a known human carcinogen, exposure to this compound should be avoided as far as practicable, and levels should be kept as low as technically feasible. Where it is known that a concentration of vinyl chloride in drinking water of

0.5 mg/liter was calculated as being associated with an excess risk of liver and brain tumors for exposure beginning at adulthood and it would double cancer risk for continuous exposure from birth. Therefore, we considerd the dataset used by [2], which represents 34 data points in mg/L from the vinyl chloride obtained from clean upgrade monitoring wells as follows:
5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3,3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.
We found the GGD model is a very good fit for this dataset as shown in Figure (2a). For studying the concentration of the vinyl chloride in the water of these wells based on this dataset we find the estimates for the parameters, which represent the shape of the concentration using our model to determine the average concentration in the water. We noticed that the numerical and Bayes estimates for α lie in the interval [0.7, 0.8] and for γ lie in the interval [1.2, 1.3] indicating that the above dataset is moderately right skewed and that means the concentration decreases with increasing time, see Figures (2b). Also, the numerical and Bayes estimates for β lie in the interval [0.09, 0.12], which is less than one and that mean this dataset is right-skewed and the vinyl chloride concentration will decrease with increasing time, therefore, monitoring these wells is very significant.

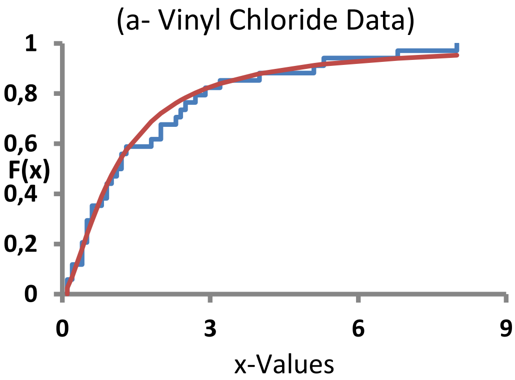

Fig. 2: a) The Empirical CDF and the fitted CDF.

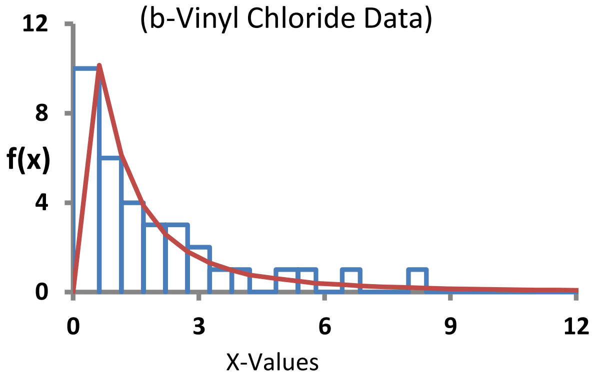

Fig. 2 b) The Histogram and the fitted PDF.

4. Simulation Study

The purpose of the simulation study is to compare the performance of the estimates using the numerical and Bayes methods based on the informative gamma and the informative kernel priors, through two criteria the average bias (AVB) and the mean squared error (MSE) as given respectively by:

$\mathbf{AVB} = \frac{1}{L}\sum_{i=1}^{L} |\hat{\theta}_i - \theta\,|, \quad \mathrm{MSE} = \sum_{i=1}^{L}\left(\hat{\theta}_i - \theta\right)^2 /L$

$\hat{\theta}$ is the estimate of θ and L is the number of replications.

In our simulation study we choose the hyperparameters for informative gama prior for α, β and γ as follows:

$a = c = e = 2, b = d = f = 5$ and the values for the parameters are $\alpha = (1,2)$, $\beta = (0.75,\ 1.75)$ and $\gamma = (2,\ 3)$ respectively. Using the parameter values for generating different samples from the GGD with sizes $N = 20, 40$ and 60 to represent small, moderate and large sizes. To assess the performance of these estimates, the average Bias (AVB) and the MSEs for each were calculated using 1000 replicates. An algorithm for generating the generalized progressive hybrid censoring scheme has been written in [18, 19].

From the simulation results in Tables 3, 4, 5, 6, 7 and 8, some of the points are quite clear based on these estimates and the others have been summarized in the following main points:

1. It is clear that generally for the parameters α, β and γ the AVB and the MSEs values based on the numerical methods outperform the corresponding values based on the Bayes' method for the different priors.
2. It is evident that the estimated AVB and MSEs values increase with increasing the parameters α, γ, while decrease as the values of the parameter β increase.
3. The estimated AVB and MSEs values decrease with increasing the termination time of the

experiment T, and the sample sizes as expected for all methods.

4. In general, the estimated MSE values for the Bayes' method based on the kernel prior are less than those based on the informative gamma prior.

In conclusion, the Picard, R-K, and Adams methods compete and outperform the Bayes method based on the informative gamma and kernel priors.

5 Conclusions

In this study, it is found that based on the generalized progressive hybrid censored data, the Picard, R-K, and Adams methods are strongly unbiased and much more efficient than the Bayes' method based on the informative gamma and kernel priors. The Bayes estimates based on the nonparametric kernel prior are more efficient than those based on the informative gamma prior, and they are very close to those based on the numerical methods. Thus, the statistical significance of the numerical methods is their efficiency than most estimation methods, beside they are reliable and easy to apply, especially for social sciences and psychology researchers.

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Contribution of the Author

I completely did this work.

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Appendix A.

$$H(\alpha,\beta,\gamma|\underline{x}) = [p_1 \ln\hat{g}_1(\alpha) + p_2 \ln\hat{g}_2(\beta) + p_3 \ln\hat{g}_3(\gamma)$$

$$+(n+a-1)\ln\alpha + (n+c-1)\ln\beta + (n+e-1)\ln\gamma$$

$$-\alpha b - \gamma f + (\alpha-1)\sum_{i=1}^{n}\ln x$$

$$-\beta\left(d - \sum_{i=1}^{N} x_i^\alpha\right) - \sum_{i=1}^{n} ln\left(1+e^{\beta x_i^\alpha}\right) -$$

$$\gamma[\textstyle\sum_{i=1}^{n}(1+R_i))ln\left(1+e^{\beta x_i^\alpha}\right) + \delta R_T^* \ln\left(1+e^{\beta T^\alpha}\right)].$$

$$\frac{\partial H}{\partial\alpha} = p_1\frac{\hat{g}_1^l(\alpha)}{\hat{g}_1(\alpha)} + \frac{n+a-1}{\alpha} - b + \sum_{i=1}^{n}\ln x_i$$

$$+\beta\sum_{i=1}^{n} x_i^\alpha \ln x_i - \beta[\sum_{i=1}^{n}\frac{e^{\beta x_i^\alpha} x_i^\alpha \ln x_i}{1+e^{\beta x_i^\alpha}}$$

$$-\beta\gamma[\textstyle\sum_{i=1}^{n}(1+R_i)\frac{e^{\beta x_i^\alpha} x_i^\alpha \ln x_i}{1+e^{\beta x_i^\alpha}} + \delta R_T^*\frac{e^{\beta T^\alpha} T^\alpha \ln T}{1+e^{\beta T^\alpha}}].$$

$$\frac{\partial^2 H}{\partial\alpha\partial x} = \sum_{i=1}^{n} 1/x_i + \beta\sum_{i=1}^{n} x_i^{\alpha-1}[\alpha \ln x_i + 1]$$

$$-\beta[\textstyle\sum_{i=1}^{n}\frac{\left(1+e^{\beta x_i^\alpha}\right)\left[\alpha e^{\beta x_i^\alpha} x_i^{\alpha-1}\ln x_i + e^{\beta x_i^\alpha} x_i^{\alpha-1})\right]}{(1+e^{\beta x_i^\alpha})^2}$$

$$-\beta\gamma[\sum_{i=1}^{n}(1+R_i)\frac{\left(1+e^{\beta x_i^\alpha}\right)\left[\alpha e^{\beta x_i^\alpha} x_i^{\alpha-1}\ln x_i + e^{\beta x_i^\alpha} x_i^{\alpha-1})\right]}{(1+e^{\beta x_i^\alpha})^2}$$

$$-\beta\textstyle\sum_{i=1}^{n}\frac{\alpha\beta ln x e^{\beta x_i^\alpha} x_i^{2\alpha-1}}{(1+e^{\beta x_i^\alpha})^2}$$

$$-\beta\gamma\textstyle\sum_{i=1}^{n}(1+R_i)\frac{+\alpha\beta ln x e^{\beta x_i^\alpha} x_i^{2\alpha-1}}{(1+e^{\beta x_i^\alpha})^2}$$

$$\frac{\partial H}{\partial\beta} = p_2\frac{\hat{g}_2^l(\beta)}{\hat{g}_2(\beta)} + \frac{n+c-1}{\beta} - d$$

$$+\sum_{i=1}^{n} x_i^\alpha - \sum_{i=1}^{n}\frac{e^{\beta x_i^\alpha} x_i^\alpha}{1+e^{\beta x_i^\alpha}}$$

$$-\gamma\sum_{i=1}^{n}[(1+R_i)\frac{e^{\beta x_i^\alpha} x_i^\alpha}{1+e^{\beta x_i^\alpha}} + \delta R_T^*\frac{e^{\beta T^\alpha} T^\alpha}{1+e^{\beta T^\alpha}}]$$

$$\frac{\partial^2 H}{\partial\beta\partial x} = \alpha\sum_{i=1}^{n} x_i^{\alpha-1}$$

$$-\alpha[\sum_{i=1}^{n}\frac{\left(1+e^{\beta x_i^\alpha}\right)\left[e^{\beta x_i^\alpha} x_i^{\alpha-1}\right] + \beta e^{\beta x_i^\alpha} x_i^{2\alpha-1}}{(1+e^{\beta x_i^\alpha})^2}$$

$$-\alpha\gamma\sum_{i=1}^{n}(1+R_i)\frac{\left(1+e^{\beta x_i^\alpha}\right)\left[e^{\beta x_i^\alpha} x_i^{\alpha-1}\right] + \beta e^{\beta x_i^\alpha} x_i^{2\alpha-1}}{(1+e^{\beta x_i^\alpha})^2}$$

$$\frac{\partial H}{\partial\gamma} = p_3\frac{\hat{g}_3^l(\gamma)}{\hat{g}_3(\gamma)} + \frac{n+e-1}{\gamma} - f$$

$$-\sum_{i=1}^{n}[(1+R_i)ln\left(1+e^{\beta x_i^\alpha}\right) + \delta R_T^* \ln\left(1+e^{\beta T^\alpha}\right)]$$

$$\frac{\partial^2 H}{\partial\gamma\partial x} = -\alpha\beta\sum_{i=1}^{n}(1+R_i)\frac{e^{\beta x_i^\alpha} x_i^{\alpha-1}}{1+e^{\beta x_i^\alpha}}$$

$$\frac{\partial^2 H}{\partial\alpha^2} = p_1\frac{\hat{g}_1(\alpha)\hat{g}_1^{ll}(\alpha) - \hat{g}_1^{l^2}(\alpha)}{\hat{g}_1^2(\alpha)}$$

$$-(n+a-1)/\alpha^2 + \beta\sum_{i=1}^{n} x_i^\alpha(ln x_i)^2$$

$$-\beta^2[\sum_{i=1}^{n}\frac{\left(1+e^{\beta x_i^\alpha}\right)\left[e^{\beta x_i^\alpha} x_i^\alpha(ln x_i)^2\right] + e^{\beta x_i^\alpha}(x_i^\alpha ln x_i)^2}{\left(1+e^{\beta x_i^\alpha}\right)^2}$$

$$-\gamma\beta^2[\sum_{i=1}^{n}(1+R_i)\frac{\left(1+e^{\beta x_i^\alpha}\right)\left[e^{\beta x_i^\alpha} x_i^\alpha(ln x_i)^2\right] + e^{\beta x_i^\alpha}(x_i^\alpha ln x_i)^2}{\left(1+e^{\beta x_i^\alpha}\right)^2}$$

$$+\delta R_T^*\frac{\left(1+e^{\beta T^\alpha}\right)\left[e^{\beta T^\alpha} T^\alpha(lnT)^2\right] + e^{\beta T^\alpha}(T^\alpha lnT)^2}{(1+e^{\beta T^\alpha})^2}]$$

$$\frac{\partial H}{\partial\beta} = p_2\frac{\hat{g}_2^l(\beta)}{\hat{g}_2(\beta)} + \frac{n+c-1}{\beta} - d$$

$$+\textstyle\sum_{i=1}^{n} x_i^\alpha - \sum_{i=1}^{n}\frac{e^{\beta x_i^\alpha} x_i^\alpha}{1+e^{\beta x_i^\alpha}}$$

$$-\gamma\sum_{i=1}^{n}[(1+R_i)\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}}{1+e^{\beta x_i^{\alpha}}}+\delta R_T^*\frac{e^{\beta T^{\alpha}}T^{\alpha}}{1+e^{\beta T^{\alpha}}}]$$

$$\frac{\partial^2 H}{\partial\beta^2}=p_2\frac{\hat{g}_2(\beta)\hat{g}_2^{ll}(\beta)-\hat{g}_2^{l^2}(\beta)}{\hat{g}_2^{\,2}(\beta)}$$

$$-\frac{n+c-1}{\beta^2}-\sum_{i=1}^{n}\frac{e^{\beta x_i^{\alpha}}x_i^{2\alpha}}{\left(1+e^{\beta x_i^{\alpha}}\right)^2}$$

$$-\gamma[\sum_{i=1}^{n}(1+R_i)\frac{e^{\beta x_i^{\alpha}}x_i^{2\alpha}}{\left(1+e^{\beta x_i^{\alpha}}\right)^2}+\delta R_T^*\frac{e^{\beta T^{\alpha}}T^{2\alpha}}{(1+e^{\beta T^{\alpha}})^2}]$$

$$\frac{\partial^2 H}{\partial\gamma^2}=p_3\frac{\hat{g}_3(\gamma)\hat{g}_3^{ll}(\gamma)-\hat{g}_3^{l\,2}(\gamma)}{\hat{g}_3^{\,2}(\gamma)}-(n+e-1)/\gamma^2$$

$$\frac{\partial H}{\partial\alpha\,\partial\beta}=\sum_{i=1}^{n}x_i^{\alpha}\ln x_i$$

$$-[\sum_{i=1}^{n}\frac{\left(1+e^{\beta x_i^{\alpha}}\right)\left[e^{\beta x_i^{\alpha}}x_i^{\alpha}lnx_i\right]+\beta e^{\beta x_i^{\alpha}}x_i^{2\alpha}lnx_i}{\left(1+e^{\beta x_i^{\alpha}}\right)^2}$$

$$-\gamma[\sum_{i=1}^{n}(1+R_i)\frac{\left(1+e^{\beta x_i^{\alpha}}\right)\left[e^{\beta x_i^{\alpha}}x_i^{\alpha}lnx_i\right]+\beta e^{\beta x_i^{\alpha}}x_i^{2\alpha}lnx_i}{\left(1+e^{\beta x_i^{\alpha}}\right)^2}$$

$$+\delta R_T^*\frac{\left(1+e^{\beta T^{\alpha}}\right)\left[e^{\beta T^{\alpha}}T^{\alpha}lnT\right]+\beta e^{\beta T^{\alpha}}T^{2\alpha}lnT}{(1+e^{\beta T^{\alpha}})^2}]$$

$$\frac{\partial H}{\partial\gamma\,\partial\alpha}=-\beta[\sum_{i=1}^{n}(1+R_i)\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}\ln x_i}{1+e^{\beta x_i^{\alpha}}}+\delta R_T^*\frac{e^{\beta T^{\alpha}}T^{\alpha}\ln T}{1+e^{\beta T^{\alpha}}}]$$

$$\frac{\partial H}{\partial\gamma\,\partial\beta}=-\left[\sum_{i=1}^{n}(1+R_i)\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}}{1+e^{\beta x_i^{\alpha}}}+\delta R_T^*\frac{e^{\beta T^{\alpha}}T^{\alpha}}{1+e^{\beta T^{\alpha}}}\right].$$

where the r[th] derivative of the kernel density estimation can be defined as

$$\frac{\partial H}{\partial\alpha\,\partial\beta}=\sum_{i=1}^{n}x_i^{\alpha}\ln x_i$$

$$-[\sum_{i=1}^{n}\frac{\left(1+e^{\beta x_i^{\alpha}}\right)\left[e^{\beta x_i^{\alpha}}x_i^{\alpha}lnx_i\right]+\beta e^{\beta x_i^{\alpha}}x_i^{2\alpha}lnx_i}{\left(1+e^{\beta x_i^{\alpha}}\right)^2}$$

$$-\gamma[\sum_{i=1}^{n}(1+R_i)\frac{\left(1+e^{\beta x_i^{\alpha}}\right)\left[e^{\beta x_i^{\alpha}}x_i^{\alpha}lnx_i\right]+\beta e^{\beta x_i^{\alpha}}x_i^{2\alpha}lnx_i}{\left(1+e^{\beta x_i^{\alpha}}\right)^2}$$

$$+\delta R_T^*\frac{\left(1+e^{\beta T^{\alpha}}\right)\left[e^{\beta T^{\alpha}}T^{\alpha}lnT\right]+\beta e^{\beta T^{\alpha}}T^{2\alpha}lnT}{(1+e^{\beta T^{\alpha}})^2}]$$

$$\frac{\partial H}{\partial\gamma\,\partial\alpha}=-\beta[\sum_{i=1}^{n}(1+R_i)\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}\ln x_i}{1+e^{\beta x_i^{\alpha}}}+\delta R_T^*\frac{e^{\beta T^{\alpha}}T^{\alpha}\ln T}{1+e^{\beta T^{\alpha}}}]$$

$$\frac{\partial H}{\partial\gamma\,\partial\beta}=-\left[\sum_{i=1}^{n}(1+R_i)\frac{e^{\beta x_i^{\alpha}}x_i^{\alpha}}{1+e^{\beta x_i^{\alpha}}}+\delta R_T^*\frac{e^{\beta T^{\alpha}}T^{\alpha}}{1+e^{\beta T^{\alpha}}}\right].$$

where the r[th] derivative of the kernel density estimation can be defined as follows:

$$\frac{d^r\hat{g}_1(\alpha)}{d\alpha^r}=\hat{g}_1^{\,r}(\alpha)=\frac{1}{nh_1^{r+1}}\sum_{i=1}^{n}K^r\left(\frac{\alpha-\hat{\alpha}_i}{h_1}\right),$$ where r=0,1,2,3,-----. (14)

Using the Gaussian kernel and (14), we have

$$\hat{g}_1(\alpha)=\frac{1}{n\square_1\sqrt{2\pi}}\sum_{i=1}^{n}e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{\square_1})^2},$$

$$\hat{g}_1^{'}(\alpha)=-\frac{1}{n\square_1^2\sqrt{2\pi}}\sum_{i=1}^{n}\left(\frac{\alpha-\hat{\alpha}_i}{\square_1}\right)e^{-0.5(\frac{\alpha-\hat{\alpha}_i}{\square_1})^2},$$

and

$$\hat{g}_1^{''}(\alpha)=\frac{1}{n\square_1^3\sqrt{2\pi}}\sum_{i=1}^{n}[(\frac{\alpha-\alpha_i}{\square_1})^2-1]\,e^{-0.5(\frac{\alpha-\alpha_i}{\square_1})^2}.$$

Similarly for the kernel priors $\hat{g}_2(\beta)$ and $\hat{g}_3(\gamma)$.

Table 2: The critical and calculated values for the K-S, A-D and CH2 tests and their powers (p-values) for the GGD.

Data	The Tests	Calculated value	Critical value	p-values	MLES		
					α	β	γ
Reactor pumps Data N=23	K-S	0.4719	0.9001	0.8163	0.7399	3.8584	0.2085
	A-D	0.2466	0.7714	0.7563			
	CH2	9.5741	12.234	0.1355			
Rainfall data N=50	K-S	0.6064	0.9074	0.4872	1.5604	0.3892	1.3286
	A-D	0.3010	0.7711	0.6134			
	CH2	5.6472	15.086	0.3854			

Table 3: The estimate and the mean squared errors (MSEs) for the parameter α, β and γ based on the numerical and Bayes methods under the squared error loss function based on the GPHCS: for $m = n/2,\ k = m/2$.

Data	T	Par	Picard Estamite		Runge-Kutta		Adams		Gamma Prior		Kernel prior	
			Estimate	MSE	Est.	MSE	Est.	MSE	Estimate	MSE	Estimate	MSE
The reactor Pumps n=23	0.5	α	0.7023	0.0014	0.6927	0.0022	9,7383	0.0012	0.6853	0.0029	0.7251	0.0022
		β	3.3225	0.2849	3.3267	0.2827	3.6991	0.0254	3.2331	0.3909	3.3188	0.2911
		γ	0.1739	9E-04	0.2332	8E-04	0.1946	1E-04	0.2249	4E-04	0.2554	0.0026
	4.5	α	0.7030	0.0014	0.6928	0.0022	0.7390	8E-07	0.6855	0.0029	0.7254	2E-04
		β	3.3333	0.2757	3.3284	0.2809	3.7089	0.0223	3.2545	0.3646	3.3323	0.2767
		γ	0.1750	9E-04	0.2333	8E-04	0.1957	8E-05	0.2269	5E-04	0.2610	0.0032
The Rainfall n=50	5	α	1.4826	0.00605	1.3364	0.0502	1.5611	0.0049	1.387	0.0299	1.38735	0.0299
		β	0.34066	0.0024	0.3407	0.0024	0.4057	0.0027	0.3979	0.0075	0.3949	0.0032
		γ	0.0289	1.6E-05	0.0379	3E-05	0.0615	8E-04	0.0897	0.0032	0.0889	0.0031
	25	α	1.4825	0.0061	1.3364	0.0502	1.5597	5E-05	1.3569	0.0414	1.3550	0.0422
		β	0.3414	0.0023	0.3408	0.0024	0.4079	0.0004	0.3473	0.0018	0.2519	0.0189
		γ	0.0289	2E-05	0.0379	3E-05	0.0618	8E-04	0.0557	5E-04	0.0379	2E-05

From the results in Table 2, the GGD is a very good fit for these datasets since the calculated values for the goodness of fit tests are less than the critical values. Moreover, the power of the tests is greater than the significance level 0.05 of the tests. From the results in Table 3, the estimated values of MSEs based on the numerical methods are approximately smaller than those for the Bayes methods for these datasets, with considering the MLEs are the true values of the parameters.

Table 4: The Average Bias (AVB) and Mean Squared Errors (MSEs) in prarenthses for GGD parameter α Using the numerical and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75.

n	m	k	α	β	γ	Numerical methods			Bayes' method	
						Picard	Runge-Kutta	Adams	Gamma Prior	Kernel Prior
20	10	5	1	0.75	1.5	0.0891(0.0081)	0.0984(0.0098)	0.1037(0.0153)	0.2271(0.1181)	0.2110(0.0630)
				1.5	2.5	0.1844(0.0350)	0.1970(0.0393)	0.5204(0.4881)	0.4333(0.1926)	0.3731(0.1406)
			2	0.75	1.5	0.0991(0.0100)	0.0955(0.0093)	0.0976(0.0193)	0.1861(0.0350)	0.1805(0.0329)
				1.5	2.5	0.2001(0.0402)	0.1883(0.0356)	0.1813(0.0805)	0.3838(0.1491)	0.3424(0.1197)
		8	1	0.75	1.5	0.1004(0.0102)	0.0916(0.0086)	0.0557(0.0051)	0.1898(0.0410)	0.1808(0.0334)
				1.5	2.5	0.1853(0.0344)	0.2059(0.0427)	0.5850(0.4078)	0.4074(0.1832)	0.3580(0.1285)
			2	0.75	1.5	0.1145(0.0132)	0.0882(0.0079)	0.0626(0.0060)	0.1662(0.0278)	0.1641(0.0271)
				1.5	2.5	0.1939(0.0377)	0.1937(0.0377)	0.3095(0.1922)	0.3619(0.1312)	0.3398(0.1168)
		8	1	0.75	1.5	0.1002(0.0101)	0.0918(0.0086)	0.0559(0.0052)	0.1896(0.0399)	0.1836(0.0385)
				1.5	2.5	0.1853(0.0345)	0.2053(0.0424)	0.5758(0.3949)	0.3988(0.1632)	0.3570(0.1277)
			2	0.75	1.5	0.1146(0.0132)	0.0873(0.0078)	0.0631(0.0060)	0.1654(0.0276)	0.1634(0.0270)
				1.5	2.5	0.1942(0.0378)	0.1930(0.0374)	0.3019(0.1871)	0.3610(0.1306)	0.3368(0.1138)

	15	11	1	0.75	1.5	0.0991(0.0099)	0.0929(0.0088)	0.0462(0.0032)	0.1802(0.0332)	0.1768(0.0325)
				1.5	2.5	0.1905(0.0363)	0.2088(0.0437)	0.4720(0.2347)	0.4011(0.1965)	0.3582(0.1931)
			2	0.75	1.5	0.1212(0.0148)	0.0869(0.0077)	0.0590(0.0052)	0.1619(0.0264)	0.1603(0.0258)
				1.5	2.5	0.1900(0.0362)	0.1992(0.0398)	0.3887(0.2028)	0.3453(0.1193)	0.3320(0.1103)
40	20	10	1	0.75	1.5	0.0937(0.0089)	0.0977(0.0096)	0.0660(0.0063)	0.1830(0.0388)	0.1784(0.0371)
				1.5	2.5	0.1891(0.0360)	0.1994(0.0400)	0.4747(0.3624)	0.3602(0.1303)	0.3394(0.1157)
			2	0.75	1.5	0.1012(0.0103)	0.0959(0.0093)	0.0644(0.0073)	0.1636(0.0269)	0.1624(0.0265)
				1.5	2.5	0.2006(0.0402)	0.1891(0.0358)	0.1177(0.0258)	0.3379(0.1146)	0.3200(0.1026)
		15	1	0.75	1.5	0.0989(0.0098)	0.0933(0.0088)	0.0401(0.0024)	0.1702(0.0354)	0.1659(0.0282)
				1.5	2.5	0.1865(0.0348)	0.2078(0.0433)	0.6062(0.4041)	0.3547(0.1273)	0.3365(0.1133)
			2	0.75	1.5	0.1130(0.0128)	0.0882(0.0079)	0.0448(0.0031)	0.1522(0.0233)	0.1520(0.0232)
				1.5	2.5	0.1947(0.0379)	0.1933(0.0375)	0.2522(0.1146)	0.3294(0.1086)	0.3200(0.1025)
	30	15	1	0.75	1.5	0.0994(0.0099)	0.0929(0.0087)	0.0397(0.0023)	0.1669(0.0287)	0.1638(0.0271)
				1.5	2.5	0.1866(0.0349)	0.2078(0.0434)	0.6135(0.4112)	0.3538(0.1257)	0.3364(0.1133)
			2	0.75	1.5	0.1127(0.0127)	0.0888(0.0080)	0.0421(0.0028)	0.1525(0.0234)	0.1524(0.0233)
				1.5	2.5	0.1942(0.0378)	0.1942(0.0378)	0.2710(0.1318)	0.3299(0.1089)	0.3205(0.1028)
		23	1	0.75	1.5	0.0978(0.0096)	0.0952(0.0091)	0.0387(0.0019)	0.1694(0.0375)	0.1628(0.0270)
				1.5	2.5	0.1936(0.0375)	0.2095(0.0439)	0.4024(0.1660)	0.3547(0.1324)	0.3330(0.1113)
			2	0.75	1.5	0.1180(0.0140)	0.0897(0.0081)	0.0357(0.0020)	0.1514(0.0230)	0.1512(0.0229)
				1.5	2.5	0.1909(0.0365)	0.2010(0.0405)	0.3488(0.1383)	0.3196(0.1022)	0.3160(0.0999)
60	30	15	1	0.75	1.5	0.0925(0.0086)	0.0992(0.0099)	0.0716(0.0068)	0.1724(0.0311)	0.1700(0.0300)
				1.5	2.5	0.1899(0.0362)	0.1984(0.0395)	0.4397(0.3104)	0.3391(0.1152)	0.3270(0.1072)
			2	0.75	1.5	0.1028(0.0106)	0.0953(0.0092)	0.0473(0.0037)	0.1569(0.0247)	0.1563(0.0245)
				1.5	2.5	0.2008(0.0403)	0.1890(0.0357)	0.0986(0.0176)	0.3220(0.1038)	0.3123(0.0976)
		23	1	0.75	1.5	0.0994(0.0099)	0.0936(0.0088)	0.0343(0.0016)	0.1586(0.0253)	0.1579(0.0255)
				1.5	2.5	0.1864(0.0348)	0.2092(0.0439)	0.6307(0.4179)	0.3374(0.1140)	0.3279(0.1075)
			2	0.75	1.5	0.1135(0.0129)	0.0883(0.0079)	0.0375(0.0022)	0.1474(0.0218)	0.1476(0.0219)
				1.5	2.5	0.1941(0.0377)	0.1938(0.0376)	0.2581(0.1059)	0.3179(0.1011)	0.3131(0.0980)
	45	23	1	0.75	1.5	0.0992(0.0099)	0.0937(0.0088)	0.0349(0.0017)	0.1584(0.0252)	0.1580(0.0252)
				1.5	2.5	0.1865(0.0348)	0.2096(0.0440)	0.6376(0.4237)	0.3378(0.1142)	0.3282(0.1077)
			2	0.75	1.5	0.1137(0.0129)	0.0887(0.0079)	0.0384(0.0022)	0.1478(0.0219)	0.1478(0.0219)
				1.5	2.5	0.1942(0.0377)	0.1938(0.0376)	0.2537(0.1028)	0.3181(0.1012)	0.3132(0.0981)
		34	1	0.75	1.5	0.0978(0.0096)	0.0949(0.0090)	0.0390(0.0018)	0.1574(0.0248)	0.1564(0.0246)
				1.5	2.5	0.1934(0.0374)	0.2099(0.0441)	0.4139(0.1741)	0.3379(0.1229)	0.3252(0.1058)
			2	0.75	1.5	0.1179(0.0139)	0.0894(0.0080)	0.0288(0.0013)	0.1472(0.0217)	0.1472(0.0217)
				1.5	2.5	0.1905(0.0363)	0.2010(0.0404)	0.3639(0.1438)	0.3138(0.0985)	0.3116(0.0971)

Table 5: **The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for GGD parameter β Using the numerical and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75.**

n	m	k	α	β	γ	Numerical methods			Bayes' method	
						Picard	Runge-Kutta	Adams	Gamma Prior	Kernel Prior
20	10	5	1	0.75	1.5	0.0625(0.0039)	0.1017(0.0103)	0.1684(0.0287)	0.1674(0.2086)	0.1895(0.1215)
				1.5	2.5	0.0612(0.0037)	0.1082(0.0117)	0.4063(0.1677)	0.1869(0.4230)	0.2278(0.2873)
			2	0.75	1.5	0.1290(0.0167)	0.2133(0.0455)	0.5912(0.3526)	0.3496(0.4634)	0.3132(0.1996)
				1.5	2.5	0.1237(0.0153)	0.2266(0.0514)	0.4853(0.1534)	0.3716(0.4352)	0.4650(0.4216)
		8	1	0.75	1.5	0.0628(0.0039)	0.1005(0.0101)	0.1294(0.0168)	0.2164(0.2497)	0.2065(0.1211)
				1.5	2.5	0.0613(0.0038)	0.1062(0.0113)	0.3423(0.1176)	0.1384(0.1342)	0.1951(0.4433)
			2	0.75	1.5	0.1268(0.0161)	0.2078(0.0432)	0.4435(0.1970)	0.3452(0.4609)	0.4311(0.4528)
				1.5	2.5	0.1246(0.0155)	0.2210(0.0488)	0.9033(0.8182)	0.2592(0.1170)	0.3986(0.1342)
	15	8	1	0.75	1.5	0.0628(0.0039)	0.1005(0.0101)	0.1292(0.0167)	0.2063(0.2129)	0.2062(0.0942)
				1.5	2.5	0.0613(0.0038)	0.1062(0.0113)	0.3424(0.1176)	0.1215(0.0230)	0.1658(0.0685)
			2	0.75	1.5	0.1267(0.0161)	0.2078(0.0432)	0.4446(0.1980)	0.4665(0.4125)	0.3704(0.3154)
				1.5	2.5	0.1244(0.0155)	0.2209(0.0488)	0.9011(0.8142)	0.2532(0.1212)	0.3495(0.3236)
		11	1	0.75	1.5	0.0632(0.0040)	0.0997(0.0099)	0.1051(0.0110)	0.1828(0.1005)	0.2039(0.2149)
				1.5	2.5	0.0614(0.0038)	0.1047(0.0110)	0.3047(0.0932)	0.1140(0.0201)	0.1541(0.0671)
			2	0.75	1.5	0.1268(0.0161)	0.2045(0.0418)	0.3558(0.1266)	0.3810(0.8403)	0.3491(0.3659)

				1.5	2.5	0.1252(0.0157)	0.2169(0.0470)	0.8018(0.6453)	0.2578(0.9183)	0.2862(0.2181)
40	20	10	1	0.75	1.5	0.0627(0.0039)	0.1016(0.0103)	0.1641(0.0271)	0.1262(0.0439)	0.1382(0.0473)
				1.5	2.5	0.0610(0.0037)	0.1085(0.0118)	0.4152(0.1737)	0.1242(0.0245)	0.1721(0.1587)
			2	0.75	1.5	0.1296(0.0168)	0.2129(0.0453)	0.5824(0.3405)	0.2379(0.0789)	0.2562(0.1588)
				1.5	2.5	0.1240(0.0154)	0.2279(0.0519)	0.4532(0.4134)	0.2509(0.0708)	0.3499(0.5607)
		15	1	0.75	1.5	0.0628(0.0039)	0.1006(0.0101)	0.1340(0.0180)	0.1536(0.1473)	0.1644(0.0730)
				1.5	2.5	0.0612(0.0037)	0.1066(0.0114)	0.3550(0.1262)	0.1200(0.0393)	0.1351(0.2143)
			2	0.75	1.5	0.1275(0.0163)	0.2083(0.0434)	0.4609(0.2127)	0.2746(0.3722)	0.3175(0.3318)
				1.5	2.5	0.1249(0.0156)	0.2223(0.0494)	0.9368(0.8782)	0.2427(0.0594)	0.2557(0.3024)
	30	15	1	0.75	1.5	0.0628(0.0039)	0.1006(0.0101)	0.1343(0.0181)	0.1382(0.0492)	0.1742(0.1031)
				1.5	2.5	0.0611(0.0037)	0.1066(0.0114)	0.3550(0.1261)	0.1145(0.0136)	0.1197(0.0293)
			2	0.75	1.5	0.1277(0.0163)	0.2082(0.0434)	0.4603(0.2121)	0.2365(0.1131)	0.3189(0.9134)
				1.5	2.5	0.1250(0.0156)	0.2224(0.0495)	0.9387(0.8817)	0.2429(0.0592)	0.2473(0.2118)
		23	1	0.75	1.5	0.0632(0.0040)	0.0995(0.0099)	0.1015(0.0103)	0.1569(0.2086)	0.1913(0.4836)
				1.5	2.5	0.0614(0.0038)	0.1045(0.0109)	0.3046(0.0931)	0.1173(0.0148)	0.1133(0.0315)
			2	0.75	1.5	0.1274(0.0162)	0.2039(0.0416)	0.3421(0.1170)	0.2352(0.1026)	0.3024(0.2281)
				1.5	2.5	0.1255(0.0157)	0.2162(0.0468)	0.8002(0.6420)	0.2376(0.0565)	0.2390(0.0981)
60	30	15	1	0.75	1.5	0.0626(0.0039)	0.1018(0.0104)	0.1693(0.0287)	0.1129(0.0150)	0.1190(0.0226)
				1.5	2.5	0.0609(0.0037)	0.1086(0.0118)	0.4200(0.1773)	0.1154(0.0154)	0.1473(0.1607)
			2	0.75	1.5	0.1293(0.0167)	0.2132(0.0454)	0.5870(0.3453)	0.2379(0.0720)	0.2385(0.0901)
				1.5	2.5	0.1240(0.0154)	0.2282(0.0521)	0.53620.3251)	0.2527(0.0678)	0.2863(0.3176)
		23	1	0.75	1.5	0.0628(0.0039)	0.1005(0.0101)	0.1329(0.0177)	0.1354(0.0983)	0.1447(0.0395)
				1.5	2.5	0.0611(0.0037)	0.1066(0.0114)	0.3524(0.1242)	0.1165(0.0137)	0.1114(0.0140)
			2	0.75	1.5	0.1276(0.0163)	0.2080(0.0433)	0.4547(0.2070)	0.2381(0.1327)	0.2850(0.3649)
				1.5	2.5	0.1250(0.0156)	0.2221(0.0494)	0.9305(0.8661)	0.2463(0.0607)	0.2360(0.0599)
	45	23	1	0.75	1.5	0.0628(0.0039)	0.1005(0.0101)	0.1327(0.0176)	0.1225(0.0247)	0.1644(0.3442)
				1.5	2.5	0.0611(0.0037)	0.1066(0.0114)	0.3527(0.1244)	0.1166(0.0136)	0.1146(0.0364)
			2	0.75	1.5	0.1274(0.0162)	0.2080(0.0433)	0.4556(0.2078)	0.3203(0.1321	0.2574(0.1148)
				1.5	2.5	0.1250(0.0156)	0.2222(0.0494)	0.9304(0.8660)	0.2462(0.0607)	0.2361(0.0592)
		34	1	0.75	1.5	0.0632(0.0040)	0.0996(0.0099)	0.1024(0.0105)	0.1372(0.0780)	0.1559(0.0613)
				1.5	2.5	0.0613(0.0038)	0.1046(0.0109)	0.3051(0.0933)	0.1165(0.0136)	0.1125(0.0157)
			2	0.75	1.5	0.1276(0.0163)	0.2040(0.0416)	0.3452(0.1192)	0.2377(0.1225)	0.2601(0.1285)
				1.5	2.5	0.1255(0.0158)	0.2165(0.0469)	0.8040(0.6474)	0.2410(0.0581)	0.2331(0.0545)

Table 6: The Average Bias (AVB) and Mean Squared Errors (MSEs) in parantheses for GGD parameter γ using the numerical and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75.

n	m	k	α	β	γ	Numerical Methods			Bayes' method	
						Picard	Runge-Kutta	Adams	Gamma Prior	Kernel Prior
20	10	5	1	0.75	1.5	0.0816(0.0069)	0.2074(0.0430)	0.2324(0.0540)	0.3435(0.1435)	0.4339(0.4523)
				1.5	2.5	0.5478(0.8458)	0.3737(0.1398)	0.3931(0.1546)	0.5344(0.1362)	0.5726(0.8626)
			2	0.75	1.5	0.0956(0.0092)	0.2229(0.0497)	0.2420(0.0586)	0.2999(0.2311)	0.4419(0.4231)
				1.5	2.5	0.1097(0.0506)	0.3983(0.1590)	0.4032(0.1626)	0.5607(0.4363)	0.5919(0.9106)
		8	1	0.75	1.5	0.0954(0.0091)	0.2030(0.0412)	0.2303(0.0530)	0.4679(0.3425)	0.4076(0.4796)
				1.5	2.5	0.0495(0.0052)	0.3657(0.1338)	0.3925(0.1540)	0.5216(0.4623)	0.5129(0.6792)
			2	0.75	1.5	0.1022(0.0105)	0.2127(0.0452)	0.2368(0.0561)	0.3797(0.5425)	0.4094(0.3993)
				1.5	2.5	0.0505(0.0041)	0.3881(0.1508)	0.4029(0.1623)	0.4579(0.7800)	0.5468(0.8199)
	15	8	1	0.75	1.5	0.0951(0.0091)	0.2030(0.0412)	0.2303(0.0530)	0.4702(0.4362)	0.4233(0.5519)
				1.5	2.5	0.0483(0.0050)	0.3654(0.1336)	0.3924(0.1540)	0.4647(0.3853)	0.5369(0.4635)
			2	0.75	1.5	0.1023(0.0105)	0.2127(0.0452)	0.2368(0.0561)	0.5132(0.4536)	0.4059(0.3570)
				1.5	2.5	0.0529(0.0045)	0.3875(0.1503)	0.4026(0.1621)	0.4114(0.2375)	0.5755(0.4632)
		11	1	0.75	1.5	0.0952(0.0091)	0.2009(0.0403)	0.2290(0.0525)	0.4265(0.7536)	0.3873(0.2582)
				1.5	2.5	0.0358(0.0016)	0.3595(0.1293)	0.3912(0.1530)	0.4100(0.5856)	0.5005(0.6506)
			2	0.75	1.5	0.1021(0.0104)	0.2080(0.0433)	0.2340(0.0548)	0.4814(0.4231	0.3866(0.7553)
				1.5	2.5	0.0517(0.0032)	0.3793(0.1439)	0.4011(0.1609)	0.3791(0.1758)	0.4720(0.7113)
		10	1	0.75	1.5	0.0847(0.0073)	0.2072(0.0429)	0.2334(0.0545)	0.3043(0.4352)	0.3584(0.3521)
				1.5	2.5	0.4735(0.6534)	0.3755(0.1411)	0.3986(0.1589)	0.4265(0.4311)	0.5232(0.4352)
			2	0.75	1.5	0.0948(0.0090)	0.2230(0.0497)	0.2444(0.0597)	0.2458(0.1236)	0.2765(0.2325)

40	20			1.5	2.5	0.0612(0.0054)	0.4024(0.1621)	0.4122(0.1700)	0.4236(0.2019)	0.5175(0.4537)
		15	1	0.75	1.5	0.0928(0.0086)	0.2036(0.0414)	0.2312(0.0534)	0.3073(0.5786)	0.4286(04164)
				1.5	2.5	0.0537(0.0060)	0.3680(0.1355)	0.3972(0.1578)	0.3808(0.1569)	0.3908(0.2226)
			2	0.75	1.5	0.1006(0.0101)	0.2139(0.0458)	0.2388(0.0570)	0.2643(0.1917)	0.5611(0.4535)
				1.5	2.5	0.0382(0.0022)	0.3923(0.1540)	0.4106(0.1686)	0.4094(0.1700)	0.4081(0.4661)
	30	15	1	0.75	1.5	0.0931(0.0087)	0.2035(0.0414)	0.2312(0.0534)	0.4256(0.4352)	0.3644(0.4430)
				1.5	2.5	0.0517(0.0057)	0.3683(0.1356)	0.3973(0.1579)	0.3877(0.2002)	0.3901(0.2756)
			2	0.75	1.5	0.1003(0.0101)	0.2139(0.0457)	0.2388(0.0570)	0.2946(0.8420)	0.3226(0.3889)
				1.5	2.5	0.0373(0.0021)	0.3928(0.1544)	0.4109(0.1689)	0.4116(0.1715)	0.4347(0.4323)
		23	1	0.75	1.5	0.0947(0.0090)	0.2007(0.0403)	0.2293(0.0526)	0.3206(0.6713)	0.3376(0.4673)
				1.5	2.5	0.0481(0.0025)	0.3589(0.1289)	0.3942(0.1554)	0.4034(0.3151)	0.3812(0.2120)
			2	0.75	1.5	0.1010(0.0102)	0.2076(0.0431)	0.2346(0.0550)	0.2351(0.1121)	0.4200(0.4523)
				1.5	2.5	0.0573(0.0034)	0.3782(0.1430)	0.4057(0.1646)	0.4022(0.1621)	0.4011(0.4492)
60	30	15	1	0.75	1.5	0.0833(0.0070)	0.2078(0.0432)	0.2342(0.0548)	0.2324(0.0832)	0.2546(0.2950)
				1.5	2.5	0.4326(0.5763)	0.3754(0.1410)	0.4013(0.1611)	0.3816(0.1673)	0.4217(0.3226)
			2	0.75	1.5	0.0962(0.0093)	0.2229(0.0497)	0.2455(0.0603)	0.2627(0.5497)	0.2357(0.0781)
				1.5	2.5	0.0552(0.0041)	0.4028(0.1624)	0.4169(0.1738)	0.4285(0.1982)	0.4376(0.3211)
		23	1	0.75	1.5	0.0933(0.0087)	0.2034(0.0414)	0.2314(0.0535)	0.2842(0.0342)	0.3210(0.4054)
				1.5	2.5	0.0413(0.0037)	0.3678(0.1353)	0.3995(0.1596)	0.3913(0.1546)	0.3673(0.1453)
			2	0.75	1.5	0.1006(0.0101)	0.2135(0.0456)	0.2392(0.0572)	0.2333(0.1059)	0.2850(0.1757)
				1.5	2.5	0.0338(0.0016)	0.3922(0.1539)	0.4144(0.1717)	0.4216(0.1780)	0.3936(0.1804)
	45	23	1	0.75	1.5	0.0933(0.0087)	0.2033(0.0413)	0.2313(0.0535)	0.2931(0.5871)	0.3128(0.3493)
				1.5	2.5	0.0409(0.0036)	0.3681(0.1355)	0.3996(0.1597)	0.3922(0.1542)	0.3713(0.1910)
			2	0.75	1.5	0.1009(0.0102)	0.2134(0.0456)	0.2392(0.0572)	0.2293(0.1472)	0.3210(0.9012)
				1.5	2.5	0.0341(0.0016)	0.3920(0.1537)	0.4143(0.1716)	0.4211(0.1776)	0.3916(0.1623)
		34	1	0.75	1.5	0.0941(0.0089)	0.2007(0.0403)	0.2295(0.0527)	0.3093(0.8067)	0.3483(0.1452)
				1.5	2.5	0.0453(0.0022)	0.3592(0.1290)	0.3960(0.1568)	0.3927(0.1547)	0.3851(0.2957)
			2	0.75	1.5	0.1007(0.0101)	0.2078(0.0432)	0.2351(0.0553)	0.2276(0.0862)	0.3330(0.1342)
				1.5	2.5	0.0547(0.0031)	0.3789(0.1436)	0.4088(0.1671)	0.4115(0.1695)	0.3868(0.1500)

Table 7: **The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for the GGD parameter α uUsing the numerical and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5.**

n	m	k	α	β	γ	Numerical Methods			Bayes' method	
						Picard	Runge-Kutta	Adams	Gama Prior	Kernel Prior
20	10	5	1	0.75	1.5	0.0943(0.0090)	0.0953(0.0093)	0.1455(0.0213)	0.2125(0.0676)	0.1961(0.0412)
				1.5	2.5	0.1871(0.0365)	0.1978(0.0396)	0.2975(0.0886)	0.4290(0.1863)	0.3723(0.1449)
			2	0.75	1.5	0.0989(0.0100)	0.0961(0.0094)	0.1462(0.0215)	0.1872(0.0354)	0.1814(0.0332)
				1.5	2.5	0.1993(0.0398)	0.1889(0.0358)	0.2912(0.0849)	0.3866(0.1514)	0.3437(0.1203)
		8	1	0.75	1.5	0.1005(0.0102)	0.0915(0.0085)	0.1424(0.0204)	0.1921(0.0416)	0.1823(0.0340)
				1.5	2.5	0.1864(0.0348)	0.2060(0.0427)	0.3020(0.0913)	0.4034(0.1711)	0.3570(0.1278)
			2	0.75	1.5	0.1151(0.0133)	0.0873(0.0079)	0.1382(0.0193)	0.1656(0.0277)	0.1638(0.0271)
				1.5	2.5	0.1936(0.0376)	0.1947(0.0381)	0.2953(0.0873)	0.3632(0.1322)	0.3386(0.1149)
	15	8	1	0.75	1.5	0.1002(0.0101)	0.0918(0.0086)	0.1428(0.0205)	0.1994(0.0840)	0.1840(0.0354)
				1.5	2.5	0.1858(0.0346)	0.2054(0.0425)	0.3017(0.0911)	0.4048(0.1732)	0.3581(0.1288)
			2	0.75	1.5	0.1146(0.0132)	0.0876(0.0079)	0.1383(0.0193)	0.1655(0.0277)	0.1634(0.0269)
				1.5	2.5	0.1939(0.0377)	0.1939(0.0377)	0.2947(0.0869)	0.3619(0.1312)	0.3402(0.1169)
		11	1	0.75	1.5	0.0996(0.0100)	0.0924(0.0087)	0.1429(0.0206)	0.1803(0.0331)	0.1750(0.0310)
				1.5	2.5	0.1924(0.0370)	0.2090(0.0437)	0.3043(0.0927)	0.3989(0.1744)	0.3531(0.1453)
			2	0.75	1.5	0.1208(0.0147)	0.0870(0.0077)	0.1382(0.0192)	0.1622(0.0266)	0.1610(0.0261)
				1.5	2.5	0.1912(0.0366)	0.1990(0.0397)	0.2980(0.0889)	0.3449(0.1191)	0.3319(0.1102)
	20	10	1	0.75	1.5	0.0968(0.0094)	0.0958(0.0093)	0.1458(0.0213)	0.1752(0.0320)	0.1743(0.0324)
				1.5	2.5	0.1866(0.0350)	0.2036(0.0417)	0.3006(0.0904)	0.3571(0.1278)	0.3386(0.1148)
			2	0.75	1.5	0.1079(0.0117)	0.0909(0.0084)	0.1413(0.0201)	0.1563(0.0245)	0.1557(0.0243)
				1.5	2.5	0.1985(0.0394)	0.1910(0.0365)	0.2923(0.0855)	0.3355(0.1128)	0.3213(0.1033)
			1	0.75	1.5	0.0990(0.0098)	0.0931(0.0088)	0.1433(0.0206)	0.1671(0.0282)	0.1636(0.0270)
				1.5	2.5	0.1874(0.0352)	0.2088(0.0437)	0.3039(0.0924)	0.3549(0.1264)	0.3374(0.1140)

40		15	2	0.75	1.5	0.1129(0.0128)	0.0882(0.0079)	0.1387(0.0194)	0.1523(0.0233)	0.1519(0.0232)
				1.5	2.5	0.1956(0.0383)	0.1938(0.0376)	0.2944(0.0867)	0.3300(0.1089)	0.3210(0.1034)
	30	15	1	0.75	1.5	0.0990(0.0098)	0.0933(0.0088)	0.1435(0.0207)	0.1707(0.0318)	0.1659(0.0279)
				1.5	2.5	0.1872(0.0351)	0.2081(0.0435)	0.3034(0.0921)	0.3538(0.1255)	0.3368(0.1135)
			2	0.75	1.5	0.1130(0.0128)	0.0879(0.0078)	0.1384(0.0193)	0.1519(0.0232)	0.1517(0.0231)
				1.5	2.5	0.1955(0.0383)	0.1941(0.0377)	0.2946(0.0868)	0.3297(0.1088)	0.3204(0.1027)
		23	1	0.75	1.5	0.0975(0.0095)	0.0950(0.0091)	0.1448(0.0210)	0.1652(0.0277)	0.1635(0.0276)
				1.5	2.5	0.1962(0.0385)	0.2094(0.0439)	0.3054(0.0933)	0.4031(0.3524)	0.3315(0.1102)
			2	0.75	1.5	0.1179(0.0140)	0.0890(0.0080)	0.1395(0.0195)	0.1505(0.0227)	0.1506(0.0227)
				1.5	2.5	0.1927(0.0372)	0.2011(0.0405)	0.2994(0.0897)	0.3198(0.1023)	0.3162(0.1000)
60	30	15	1	0.75	1.5	0.0995(0.0099)	0.0941(0.0089)	0.1440(0.0208)	0.1608(0.0261)	0.1611(0.0284)
				1.5	2.5	0.1858(0.0346)	0.2064(0.0428)	0.3023(0.0915)	0.3385(0.1149)	0.3283(0.1078)
			2	0.75	1.5	0.1113(0.0124)	0.0887(0.0080)	0.1391(0.0194)	0.1482(0.0220)	0.1482(0.0221)
				1.5	2.5	0.1974(0.0390)	0.1920(0.0369)	0.2929(0.0858)	0.3206(0.1028)	0.3137(0.0985)
		23	1	0.75	1.5	0.0994(0.0099)	0.0931(0.0087)	0.1431(0.0205)	0.1608(0.0356)	0.1575(0.0251)
				1.5	2.5	0.1872(0.0351)	0.2098(0.0441)	0.3046(0.0928)	0.3381(0.1144)	0.3283(0.1078)
			2	0.75	1.5	0.1137(0.0129)	0.0879(0.0078)	0.1383(0.0192)	0.1471(0.0217)	0.1471(0.0217)
				1.5	2.5	0.1955(0.0383)	0.1937(0.0376)	0.2942(0.0866)	0.3179(0.1011)	0.3130(0.0980)
	45	23	1	0.75	1.5	0.0994(0.0099)	0.0932(0.0087)	0.1431(0.0205)	0.1578(0.0250)	0.1579(0.0253)
				1.5	2.5	0.1872(0.0351)	0.2095(0.0440)	0.3044(0.0927)	0.3377(0.1142)	0.3281(0.1077)
			2	0.75	1.5	0.1135(0.0129)	0.0886(0.0079)	0.1389(0.0194)	0.1477(0.0219)	0.1478(0.0219)
				1.5	2.5	0.1958(0.0384)	0.1936(0.0375)	0.2941(0.0865)	0.3179(0.1011)	0.3129(0.0980)
		34	1	0.75	1.5	0.0978(0.0096)	0.0950(0.0091)	0.1447(0.0210)	0.1603(0.0307)	0.1568(0.0247)
				1.5	2.5	0.1964(0.0386)	0.2100(0.0441)	0.3060(0.0936)	0.3368(0.1148)	0.3252(0.1059)
			2	0.75	1.5	0.1177(0.0139)	0.0896(0.0081)	0.1400(0.0196)	0.1476(0.0218)	0.1476(0.0218)
				1.5	2.5	0.1929(0.0372)	0.2006(0.0403)	0.2990(0.0894)	0.3135(0.0983)	0.3114(0.0970)

Table 8: **The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for the GWD parameter β using the numerical and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5.**

n	m	k	α	β	γ	Numerical Methods			Bayes' method	
						Picard	Runge-Kutta	Adams	Gama Prior	Kernel Prior
20	10	5	1	0.75	1.5	0.0773(0.0060)	0.0544(0.0030)	0.0910(0.0760)	0.1795(0.1014)	0.2180(0.2759)
				1.5	2.5	0.0763(0.0058)	0.0622(0.0039)	0.0645(0.0273)	0.1443(0.0492)	0.2416(0.5904)
			2	0.75	1.5	0.1599(0.0256)	0.1811(0.0329)	0.2084(0.2364)	0.2788(0.1714)	0.3322(0.4476)
				1.5	2.5	0.1559(0.0243)	0.1898(0.0362)	0.2289(0.2937)	0.3364(0.4275)	0.4063(0.4352)
		8	1	0.75	1.5	0.0774(0.0060)	0.0543(0.0031)	0.1171(0.1050)	0.2347(0.7698)	0.1951(0.0786)
				1.5	2.5	0.0764(0.0058)	0.0605(0.0037)	0.0507(0.3212)	0.1205(0.0425)	0.1916(0.1540)
			2	0.75	1.5	0.1566(0.0245)	0.1729(0.0301)	0.2398(0.2646)	0.3729(0.7943)	0.3968(0.4404)
				1.5	2.5	0.1566(0.0245)	0.1895(0.0359)	0.1590(0.0842)	0.2539(0.1334)	0.3372(0.3080)
	15	8	1	0.75	1.5	0.0774(0.0060)	0.0536(0.0029)	0.1342(0.4919)	0.1869(0.0780)	0.2112(0.1378)
				1.5	2.5	0.0764(0.0058)	0.0605(0.0037)	0.0554(0.1751)	0.1261(0.0450)	0.1865(0.1326)
			2	0.75	1.5	0.1566(0.0245)	0.1727(0.0301)	0.2494(0.4199)	0.3636(0.5864)	0.4050(0.9428)
				1.5	2.5	0.1565(0.0245)	0.1893(0.0358)	0.1476(0.0339)	0.2702(0.7630)	0.3440(0.3877)
		11	1	0.75	1.5	0.0778(0.0061)	0.0698(0.0052)	0.1222(0.1138)	0.1973(0.1176)	0.1998(0.1627)
				1.5	2.5	0.0765(0.0059)	0.0592(0.0035)	0.0337(0.0385)	0.1187(0.0291)	0.1482(0.0638)
			2	0.75	1.5	0.1565(0.0245)	0.1696(0.0289)	0.2043(0.1814)	0.3353(0.4444)	0.3758(0.5414)
				1.5	2.5	0.1565(0.0245)	0.1854(0.0344)	0.1308(0.0179)	0.2262(0.0537)	0.4070(0.3523)
	20	10	1	0.75	1.5	0.0774(0.0060)	0.0550(0.0031)	0.0347(0.0312)	0.1314(0.0505)	0.1552(0.0681)
				1.5	2.5	0.0762(0.0058)	0.0621(0.0039)	0.0184(0.0008)	0.1162(0.0191)	0.1370(0.0581)
			2	0.75	1.5	0.1585(0.0251)	0.1775(0.0316)	0.1379(0.0666)	0.2464(0.1178)	0.2981(0.2727)
				1.5	2.5	0.1566(0.0245)	0.1933(0.0374)	0.1493(0.0264)	0.2558(0.0919)	0.2725(0.1970)
		15	1	0.75	1.5	0.0775(0.0060)	0.0538(0.0030)	0.0399(0.0334)	0.1631(0.4915)	0.2057(0.1253)
				1.5	2.5	0.0764(0.0058)	0.0609(0.0037)	0.0216(0.0082)	0.1147(0.0145)	0.1276(0.0495)
			2	0.75	1.5	0.1576(0.0249)	0.1738(0.0304)	0.1890(0.0145)	0.2483(0.1185)	0.3084(0.4084)
				1.5	2.5	0.1569(0.0246)	0.1907(0.0364)	0.1472(0.0229)	0.2439(0.0599)	0.2430(0.1536)

40	30	15	1	0.75	1.5	0.0775(0.0060)	0.0540(0.0030)	0.0332(0.0101)	0.1387(0.0656)	0.1735(0.0868)
				1.5	2.5	0.0763(0.0058)	0.0609(0.0037)	0.0179(0.0005)	0.1136(0.0145)	0.1343(0.0681)
			2	0.75	1.5	0.1576(0.0248)	0.1730(0.0301)	0.1422(0.1411)	0.2698(0.3538)	0.3211(0.3696)
				1.5	2.5	0.1570(0.0246)	0.1907(0.0364)	0.1454(0.0212)	0.2427(0.0590)	0.2482(0.1073)
		23	1	0.75	1.5	0.0778(0.0061)	0.0713(0.0054)	0.0530(0.0226)	0.1524(0.1223)	0.2285(0.1432)
				1.5	2.5	0.0764(0.0058)	0.0589(0.0035)	0.0234(0.0016)	0.1162(0.0141)	0.1145(0.0192)
			2	0.75	1.5	0.1572(0.0247)	0.1693(0.0289)	0.1264(0.0355)	0.2534(0.2153)	0.2843(0.2459)
				1.5	2.5	0.1567(0.0246)	0.1848(0.0341)	0.1398(0.0196)	0.2372(0.0563)	0.2464(0.2314)
60	30	15	1	0.75	1.5	0.0774(0.0060)	0.0545(0.0030)	0.0199(0.0035)	0.1191(0.0267)	0.1641(0.3710)
				1.5	2.5	0.0762(0.0058)	0.0616(0.0038)	0.0190(0.0004)	0.1161(0.0135)	0.1186(0.0235)
			2	0.75	1.5	0.1579(0.0249)	0.1755(0.0310)	0.1256(0.0270)	0.2581(0.2666)	0.2787(0.2427)
				1.5	2.5	0.1569(0.0246)	0.1926(0.0371)	0.1506(0.0227)	0.2492(0.0624)	0.2531(0.1279)
		23	1	0.75	1.5	0.0775(0.0060)	0.0541(0.0030)	0.0171(0.0025)	0.1398(0.1127)	0.1654(0.2713)
				1.5	2.5	0.0763(0.0058)	0.0609(0.0037)	0.0191(0.0004)	0.1164(0.0136)	0.1157(0.0183)
			2	0.75	1.5	0.1575(0.0248)	0.1731(0.0301)	0.1244(0.0303)	0.2277(0.1287)	0.2944(0.4335)
				1.5	2.5	0.1570(0.0246)	0.1904(0.0363)	0.1484(0.0220)	0.2465(0.0608)	0.2424(0.0829)
	45	23	1	0.75	1.5	0.0775(0.0060)	0.0535(0.0029)	0.0184(0.0027)	0.1413(0.0792)	0.1533(0.0809)
				1.5	2.5	0.0763(0.0058)	0.0609(0.0037)	0.0198(0.0007)	0.1166(0.0136)	0.1217(0.0676)
			2	0.75	1.5	0.1575(0.0248)	0.1742(0.0305)	0.1225(0.0204)	0.2311(0.0889)	0.2889(0.2480)
				1.5	2.5	0.1569(0.0246)	0.1904(0.0363)	0.1483(0.0220)	0.2463(0.0607)	0.2326(0.0565)
		34	1	0.75	1.5	0.0778(0.0060)	0.0711(0.0054)	0.0390(0.0032)	0.1403(0.1535)	0.1541(0.0740)
				1.5	2.5	0.0764(0.0058)	0.0590(0.0035)	0.0219(0.0010)	0.1168(0.0137)	0.1107(0.0125)
			2	0.75	1.5	0.1572(0.0247)	0.1706(0.0292)	0.1204(0.0257)	0.2296(0.0914)	0.2647(0.1837)
				1.5	2.5	0.1567(0.0246)	0.1849(0.0342)	0.1425(0.0203)	0.2407(0.0580)	0.2331(0.0557)

Table 9: **The Average Bias (AVB) and Mean Squared Errors (MSEs) in parentheses for the GWD parameter γ using the numerical and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5.**

N	m	k	α	β	γ	Numerical Methods			Bayes' method	
						Picard	Runge-Kutta	Adams	Gama Prior	Kernel Prior
20	10	5	1	0.75	1.5	0.0443(0.0020)	0.1731(0.0300)	0.1301(0.0170)	0.4340(0.9654)	0.4111(0.8706)
				1.5	2.5	0.2347(0.0814)	0.3595(0.1294)	0.2931(0.0859)	0.6279(0.5362)	0.6243(0.5732)
			2	0.75	1.5	0.0473(0.0023)	0.1898(0.0361)	0.1408(0.0199)	0.3054(0.3280)	0.3311(0.2531)
				1.5	2.5	0.1020(0.0164)	0.3797(0.1444)	0.3037(0.0923)	0.5795(0.8989)	0.5760(0.5431)
		8	1	0.75	1.5	0.0474(0.0023)	0.1711(0.0294)	0.1513(0.0235)	0.4646(0.9489)	0.3927(0.3856)
				1.5	2.5	0.1474(0.0239)	0.3517(0.1237)	0.2924(0.0855)	0.4110(0.6624)	0.6088(0.5352)
			2	0.75	1.5	0.0511(0.0026)	0.1773(0.0316)	0.1323(0.0177)	0.4209(0.5327)	0.4911(0.4325)
				1.5	2.5	0.0972(0.0109)	0.3743(0.1402)	0.3034(0.0921)	0.4407(0.6875)	0.5300(0.9913)
	15	8	1	0.75	1.5	0.0474(0.0023)	0.1707(0.0292)	0.1520(0.0237)	0.5564(0.4537)	0.4259(0.8801)
				1.5	2.5	0.1478(0.0240)	0.3515(0.1236)	0.2923(0.0855)	0.4143(0.3770)	0.5029(0.4194)
			2	0.75	1.5	0.0510(0.0026)	0.1771(0.0316)	0.1323(0.0178)	0.5273(0.5325)	0.4086(0.3602)
				1.5	2.5	0.0953(0.0106)	0.3737(0.1397)	0.3031(0.0919)	0.4211(0.4654)	0.5946(0.4563)
		11	1	0.75	1.5	0.0475(0.0023)	0.1852(0.0347)	0.1517(0.0237)	0.4535(0.8659)	0.4055(0.7446)
				1.5	2.5	0.0878(0.0081)	0.3463(0.1199)	0.2913(0.0849)	0.4000(0.4563)	0.4714(0.4765)
			2	0.75	1.5	0.0508(0.0026)	0.1729(0.0301)	0.1296(0.0170)	0.4282(0.4352)	0.4077(0.5881)
				1.5	2.5	0.0792(0.0067)	0.3647(0.1331)	0.3011(0.0907)	0.3822(0.1955)	0.4747(0.5625)
40	20	10	1	0.75	1.5	0.0448(0.0020)	0.1738(0.0303)	0.1315(0.0173)	0.2835(0.4758)	0.3258(0.4514)
				1.5	2.5	0.2474(0.0719)	0.3586(0.1286)	0.2984(0.0890)	0.3836(0.1926)	0.4130(0.3421)
			2	0.75	1.5	0.0492(0.0024)	0.1843(0.0341)	0.1388(0.0194)	0.2607(0.3030)	0.3340(0.5371)
				1.5	2.5	0.0976(0.0116)	0.3833(0.1470)	0.3123(0.0976)	0.5817(0.4536)	0.4370(0.4215)
		15	1	0.75	1.5	0.0463(0.0021)	0.1710(0.0293)	0.1487(0.0227)	0.3080(0.4014)	0.3277(0.3787)
				1.5	2.5	0.1608(0.0272)	0.3546(0.1257)	0.2976(0.0886)	0.3801(0.1500)	0.4241(0.7349)
			2	0.75	1.5	0.0500(0.0025)	0.1790(0.0322)	0.1348(0.0184)	0.2704(0.3026)	0.3252(0.3128)
				1.5	2.5	0.1000(0.0109)	0.3777(0.1427)	0.3109(0.0967)	0.4130(0.1730)	0.3842(0.1807)
			1	0.75	1.5	0.0463(0.0022)	0.1713(0.0294)	0.1484(0.0226)	0.4377(0.3524)	0.3853(0.7954)
				1.5	2.5	0.1607(0.0273)	0.3543(0.1255)	0.2973(0.0884)	0.3781(0.1687)	0.4406(0.4535)

	30	15	2	0.75	1.5	0.0500(0.0025)	0.1782(0.0320)	0.1340(0.0182)	0.2811(0.3691)	0.3655(0.1354)
				1.5	2.5	0.0993(0.0108)	0.3777(0.1427)	0.3110(0.0967)	0.4108(0.1700)	0.3999(0.2597)
		23	1	0.75	1.5	0.0471(0.0022)	0.1871(0.0353)	0.1547(0.0245)	0.3140(0.4340)	0.3682(0.4413)
				1.5	2.5	0.0764(0.0060)	0.3455(0.1193)	0.2942(0.0865)	0.3986(0.2266)	0.3851(0.1884)
			2	0.75	1.5	0.0502(0.0025)	0.1728(0.0300)	0.1304(0.0172)	0.2488(0.2836)	0.3538(0.2342)
				1.5	2.5	0.0720(0.0054)	0.3637(0.1323)	0.3058(0.0935)	0.4011(0.1612)	0.3915(0.2607)
60	30	15	1	0.75	1.5	0.0462(0.0021)	0.1726(0.0299)	0.1309(0.0172)	0.2758(0.4621)	0.3109(0.3718)
				1.5	2.5	0.2202(0.0540)	0.3569(0.1274)	0.3006(0.0904)	0.3886(0.1525)	0.3840(0.2020)
			2	0.75	1.5	0.0498(0.0025)	0.1815(0.0331)	0.1376(0.0191)	0.2378(0.1071)	0.3056(0.3741)
				1.5	2.5	0.1010(0.0113)	0.3817(0.1458)	0.3162(0.1000)	0.4283(0.1948)	0.4115(0.2713)
		23	1	0.75	1.5	0.0465(0.0022)	0.1713(0.0294)	0.1322(0.0177)	0.2631(0.2061)	0.2953(0.1941)
				1.5	2.5	0.1565(0.0250)	0.3542(0.1255)	0.2997(0.0899)	0.3914(0.1537)	0.3786(0.1686)
			2	0.75	1.5	0.0501(0.0025)	0.1781(0.0319)	0.1347(0.0184)	0.2170(0.0823)	0.3184(0.4395)
				1.5	2.5	0.0997(0.0106)	0.3770(0.1421)	0.3143(0.0988)	0.4217(0.1781)	0.3965(0.1760)
	45	23	1	0.75	1.5	0.0465(0.0022)	0.1705(0.0292)	0.1340(0.0183)	0.2776(0.3707)	0.3041(0.2612)
				1.5	2.5	0.1558(0.0246)	0.3541(0.1254)	0.2997(0.0898)	0.3917(0.1539)	0.3852(0.2516)
			2	0.75	1.5	0.0502(0.0025)	0.1792(0.0322)	0.1361(0.0187)	0.2569(0.8435)	0.2928(0.2567)
				1.5	2.5	0.0992(0.0105)	0.3768(0.1420)	0.3142(0.0988)	0.4211(0.1776)	0.3864(0.1555)
		34	1	0.75	1.5	0.0470(0.0022)	0.1868(0.0352)	0.1546(0.0244)	0.2696(0.1694)	0.3147(0.2838)
				1.5	2.5	0.0782(0.0062)	0.3460(0.1197)	0.2962(0.0877)	0.3938(0.1556)	0.3725(0.1415)
			2	0.75	1.5	0.0502(0.0025)	0.1740(0.0304)	0.1321(0.0176)	0.2251(0.1076)	0.2937(0.3655)
				1.5	2.5	0.0736(0.0055)	0.3641(0.1326)	0.3087(0.0953)	0.4107(0.1688)	0.3856(0.1502)