Estimation of the Random Instantaneous Frequency with Time-Varying Amplitude Using the Maximum Likelihood and the Malliavin Calculus

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Abstract: The maximum likelihood method is used to estimate the frequency of the sinusoid when the frequency is described as a known stochastic process (for example Ornstein-Uhlenbeck process). We use Malliavin calculus and Ito calculus to derive expressions for the estimate of the slowly varying amplitude. The need for these estimates occurs in EEG analysis, inverse synthetic aperture radar (ISAR), amplitude modulation-frequency modulation (AM-FM) problem etc .... The observations are one sinusoid with slowly varying amplitude. The observations are described as a stochastic differential equation (SDE).

Key words: Ito Calculus, Malliavin calculus, Time-Varying Parameters, Girsanov Theory, Ornstein-Uhlenbeck process, AM-FM estimation.

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1. Introduction

In not so few applications one is confronted with the observation of a sinusoid that has a stochastic frequency that could be bouncing around a fixed unknown quantity. This behavior of the frequency could be modeled as an Ornstein-Uhlenbeck (OU) stochastic process. This situation occurs for example when measuring the EEG signal [1], the EKG signal [2] where the measured frequency/frequencies are changing from period to period or from cycle to cycle. The situation is also observed when measuring the echo of a moving target as in inverse synthetic aperture radar (ISAR) [3] where it is noticed that the Doppler shift frequency is a random quantity.

Sometimes the Radar echo is modeled as two closely separated targets that continuously appear and disappear. This happens when a small target is hiding behind a big target and it moves around the big target. This causes the echo to have a strong frequency component and a small component that is bouncing around the big component.

The literature is also concerned with the estimation of the frequency and the amplitude of an observed signal in a deterministic environment (the amplitude and the frequency are both deterministic) which is known as the AM-FM problem [4]. In this situation, the concern is the estimate of the time-varying frequency/amplitude [5].

In this report, we focus on modeling the stochastic frequency as an Ornstein-Uhlenbeck (OU) process. The frequency is bouncing around some unknown constant frequency. We use the maximum likelihood method to estimate the parameter of the OU process. The observation is a sine wave with slowly varying amplitude. This paper is divided as follows: Section 2 is the description of the estimation problem and the introduction of the OU process. We also describe the TEO approach to find the instantaneous amplitude and the instantaneous frequency. In Section 3 we introduce Girsanov theory and the maximum likelihood method to estimate the parameters of the OU process. We also present the statistical properties of the estimates. In Section 3 we also present the estimate of the amplitude using the Ito calculus and the Malliavin calculus [6]. In Section 4, we apply the proposed method to simulated data and compare the results with the TEO method. We present summary and future work. There is an Appendix which has most of the necessary derivations.

2. Problem Formulation:

The observed signal is modeled as a single sinusoid with time varying amplitude and stochastic frequency. For EEG, we use band pass filter to separate the different components such as the alpha, delta, beta and theta. For Doppler shifted signal, we directly measure the echo of a stochastic frequency.

Assume that the observed signal y(t) is given by the equation:

$$y(t) = a(t)\sin(2\pi\ f(t)t + \varphi) \qquad (1)$$

Thus, $y^2(t) = a^2(t)\sin^2(2\pi\ f(t)t + \varphi)$

$$= a^2(t)\left(1 - \cos^2(2\pi\ f(t)t + \varphi)\right)$$

i.e. $a^2(t)\cos^2(2\pi\ f(t)t + \varphi) = a^2(t) - y^2(t)$

a(t) is unknown slowly varying amplitude, and φ is unknown phase.

The frequency $f(t)$ is a stochastic process described by the Ornstein-Uhlenbeck SDE:

$$df(t) = \alpha(\beta - f(t))dt + \gamma dW(t) \qquad (2)$$

Where *W*(t) is a Wiener process, α , β , and γ are unknown parameters .

The Malliavin derivative of y(t), $D_s y(t)$, is derived as:

$$D_s y(t) = a(t)\cos(2\pi f(t)t + \varphi)2\pi t D_s f(t)$$

$$= 2\pi t a(t)\cos(2\pi f(t)t + \varphi)D_s f(t)$$

$$= 2\pi\, t\, a(t)\cos(2\pi f(t)t + \varphi)\gamma\, e^{-\alpha(t-s)} 1_{s\in[0,t]}(s) \qquad (3a)$$

where $1_{s\in[0,t]}(s) = \begin{cases} 1 & s \in [0,t] \\ 0 \text{ elsewhere} & \end{cases}$

$D_s f(t) = \gamma\, e^{-\alpha(t-s)} 1_{s\in[0,t]}(s)$ is the Malliavin derivative of f(t)

and
$$D_r D_s y(t) = -(2\pi t)^2 a(t)\sin(2\pi\ f(t)t + \varphi)\gamma^2\, e^{-(t-s)}\, e^{-(t-r)} 1_{r\in[0,s]}(r) 1_{s\in[0,t]}(s)$$

Also

$$(D_s y(t))^2 = (2\pi t)^2 a^2(t)\cos^2(2\pi f(t)t + \varphi)D_s f(t)D_s f(t)$$

i.e.

$$(D_s y(t))^2 = (2\pi t)^2 \left(a^2(t) - y^2(t)\right) D_s f(t) D_s f(t) \qquad (3b)$$

If we have an estimate for a(t) then we know the Malliavin derivative of y(t).

2.1Why do we need an OU model for the frequency:

It has been observed that a single sinusoid with relatively high SNR exhibits a strong peak at the unknown frequency when the Fourier transform is applied to segments of the observed data. This is evident in Fig.1 where we show the Fourier transform of a signal with constant amplitude and constant frequency at 10 Hz.

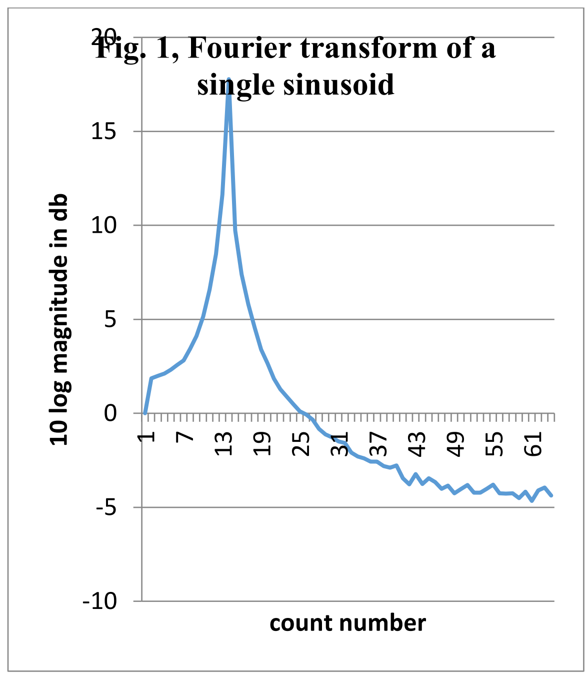


Fig. 1, Fourier transform of a single sinusoid

As we reduce the SNR, the peak is still clear and at the same location i.e. same value.

In Fig.2a, we present the observed sinusoidal in the time y(t) of eqn. (1). The frequency follows an OU process. Notice that the time-varying amplitude could be estimated through an envelope detector.

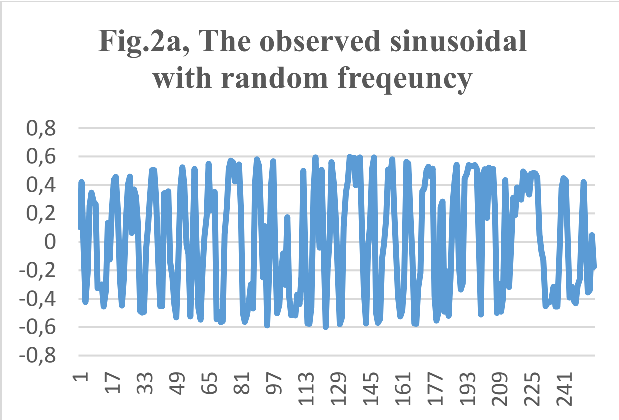


Fig.2a, The observed sinusoidal with random freqeuncy

When we model the frequency of the signal as a random quantity that follows an OU process, several things start to happen. The background noise level increases and several peaks start to emerge at different frequencies as shown in Fig. 2b.

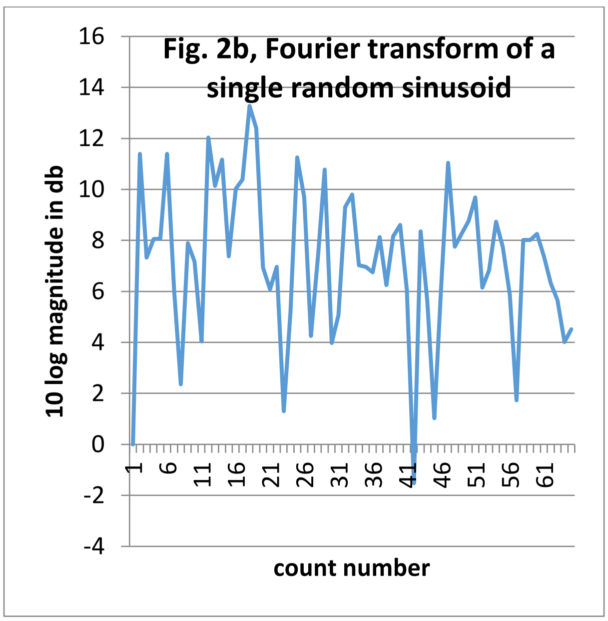


Fig. 2b, Fourier transform of a single random sinusoid

The dominant frequency, in the simulation (which is supposed to be 10 Hz), starts to move around and small peaks appear and disappear at different locations. This is exactly what we observe for the EEG signals, see Fig.3, and for the echo from Radar targets Fig. 4.

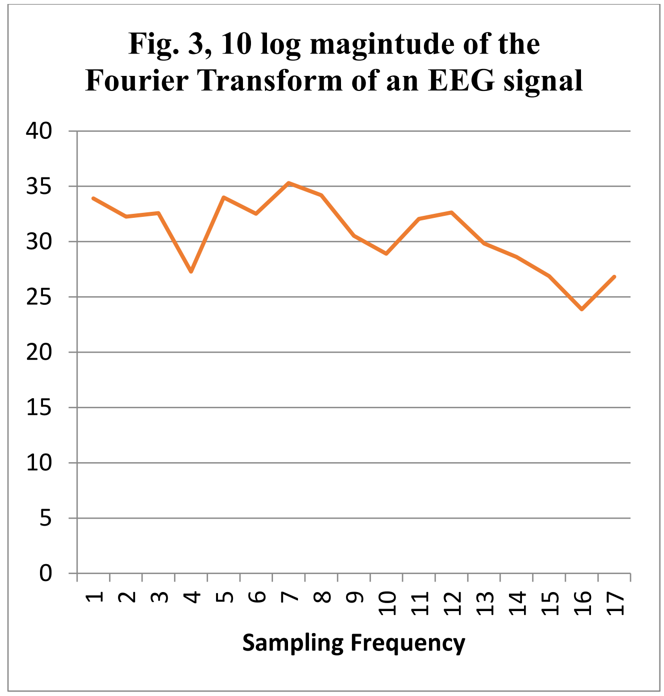


Fig. 3, 10 log magnitude of the Fourier Transform of an EEG signal

In Fig. 4, we show the Fourier transform of the echo of Radar targets. We took the Fourier transform using two segments of data. We have two targets. Notice the peaks are moving around and the background noise level is relatively high.

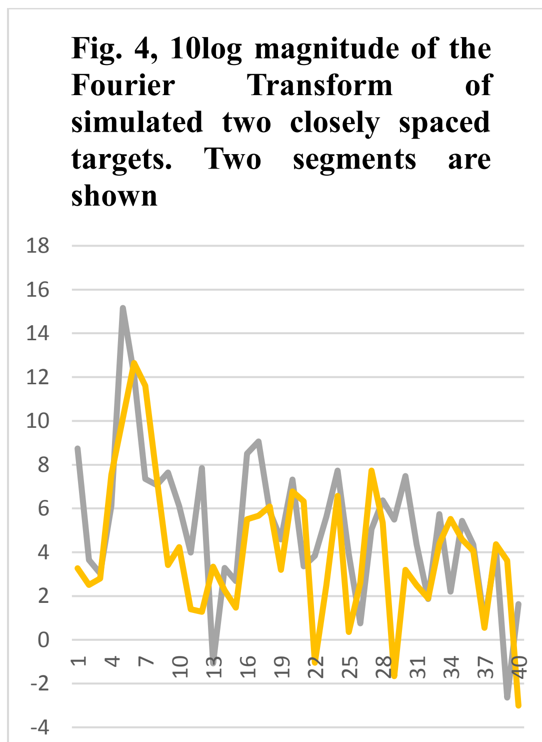


Fig. 4, 10log magnitude of the Fourier Transform of simulated two closely spaced targets. Two segments are shown

One could argue that using the Fourier transform would find the frequency estimate. As we see in Fig. 2b, the random frequency will flatten the Fourier transform which makes it difficult to estimate the frequency. Even small but random changes, less than 5%, in the dominant frequency will blur the spectrum or the Fourier transform. Thus, one has to resort to time domain estimation techniques albeit more complicated. The almost flat Fourier transform should be taken as an indication of the presence of random frequency.

2.2 TEO for Ampliltude and Frequency Estimation [1]:

In the TEO method, we assume slowly varying amplitude and slowly varying frequency. Taking the first derivative with respect to time of the signal y(t) we get:

$$\frac{dy(t)}{dt} = a(t)(2\pi f)\cos[\phi(t)] \quad (4)$$

where $\phi(t) = (2\pi f(t)t + \varphi)$

Taking the second derivative with respect to time, assuming constant amplitude, we get:

$$\frac{d^2 y(t)}{dt^2} = -a(t)(2\pi f)^2 \sin[\phi(t)]$$

$$= -(2\pi f)^2 y(t) \quad (5)$$

Define the energy tracking operator $\Psi(y(t))$ as:

$$\Psi(y(t)) = \left[\frac{dy(t)}{dt}\right]^2 - y(t)\frac{d^2 y(t)}{dt^2} \quad (6)$$

$$= \{a(t)(2\pi f)\cos[\phi(t)]\}^2 + a(t)(2\pi f)^2 \sin[\phi(t)]a(t)\sin[\phi(t)]$$

$$= a^2(t)(2\pi f)^2\{1 - \sin^2[\phi(t)]\} + a^2(t)(2\pi f)^2 \sin^2[\phi(t)]$$

$$= a^2(t)(2\pi f)^2$$

Which has a discrete version [7]:

$$\Psi(y(n)) = y^2(n) - y(n-1)y(n+1)$$

Or $\quad \Psi(y(n)) = \left[y^2(n) - y(n-1)y(n+1)\right]/\Delta^2$

Applying the TEO operator we get:

$$\Psi(y(t)) = a^2(t)(2\pi f)^2 \quad (7)$$

and $\quad \Psi\left[\frac{dy(t)}{dt}\right] = a^2(t)(2\pi f)^4 \quad (8)$

Hence the estimates of the instantaneous amplitude and the instantaneous frequency are obtained as:

$$(2\pi f) = \sqrt{\frac{\Psi\left[\frac{dy(t)}{dt}\right]}{\Psi[y(t)]}} \quad (9)$$

And $\quad a(t) = \pm\frac{\Psi[y(t)]}{\sqrt{\Psi\left[\frac{dy(t)}{dt}\right]}} \quad (10)$

In this report, it was observe that $\Psi(y(n))$ is a better estimate for a(t) i.e. less noisy. Unfortunately it has to be scaled .

3.Problem Solution; the Proposed Approach:

In this section we develop an SDE for the observations y(t). We use the Ito calculus rules to find an estimate for the slowly varying amplitude. Girsanov theory is used to find the maximum likelihood estimates of the parameters of the OU process describing the frequency. We also present the statistical properties of the estimates [see the Appendix]. The Malliavin calculus is used to find an estimate for some of the OU parameters.

3.1An SDE for the Observations:

We need to find an estimate for the unknown slowly varying amplitude a(t) and an estimate for the unknown parameters of the OU process α , β , and γ .

Using Ito lemma we get an SDE for the observations y(t) as:

$$dy(t) = \frac{\partial y(t)}{\partial t}dt + \frac{\partial y(t)}{\partial f}df(t) + \frac{1}{2}\frac{\partial^2 y(t)}{\partial f^2}\left(df(t)\right)^2 \qquad (11)$$

$$dy(t) = 2\pi\ f(t)a(t)\cos(2\pi\ f(t)t+\varphi)dt + \frac{\partial a(t)}{\partial t}\sin(2\pi\ f(t)t+\varphi)dt + (2\pi t)a(t)\cos(2\pi\ f(t)t+\varphi)df$$
$$-\frac{1}{2}(2\pi t)^2 a(t)\sin(2\pi\ f(t)t+\varphi)\left(df(t)\right)^2$$

$$dy(t) = \begin{bmatrix} \frac{\partial a(t)}{\partial t}\sin(2\pi\ f(t)t+\varphi) + 2\pi\ f(t)a(t)\cos(2\pi\ f(t)t+\varphi) \\ +(2\pi t)a(t)\cos(2\pi\ f(t)t+\varphi)\alpha(\beta-f(t)) - \frac{1}{2}(2\pi t)^2 a(t)\sin(2\pi\ f(t)t+\varphi)\gamma^2 \end{bmatrix}dt$$
$$+(2\pi t)a(t)\cos(2\pi\ f(t)t+\varphi)\gamma dW(t) \qquad (12)$$

For slowly varying amplitude $\frac{\partial a(t)}{\partial t} \approx 0$, and we get:

$$dy(t) \approx \left[2\pi\ f(t)a(t)\cos(2\pi\ f(t)t+\varphi) - \frac{1}{2}(2\pi t)^2 a(t)\sin(2\pi\ f(t)t+\varphi)\gamma^2 + (2\pi t)a(t)\cos(2\pi\ f(t)t+\varphi)\alpha(\beta-f(t))\right]dt$$
$$+(2\pi t)a(t)\cos(2\pi\ f(t)t+\varphi)\gamma dW(t) \qquad (13)$$

Or

$$dy(t) \approx a(t)\left[2\pi\ f(t)\cos(2\pi\ f(t)t+\varphi) - \frac{1}{2}(2\pi t)^2\sin(2\pi\ f(t)t+\varphi)\gamma^2 + (2\pi t)\cos(2\pi\ f(t)t+\varphi)\alpha(\beta-f(t))\right]dt$$
$$+(2\pi t)a(t)\cos(2\pi\ f(t)t+\varphi)\gamma dW(t)$$

which has the form:

$$dy(t) \approx \left[h_1(y,t) + h_2(y,t)\alpha\beta + \alpha h_3(y,t)\right]dt$$
$$+ h_2(y,t)\gamma dW(t) \qquad (14)$$

where
$$h_1(y,t) = 2\pi\ f(t)a(t)\cos(2\pi\ f(t)t+\varphi)$$
$$-\frac{1}{2}(2\pi t)^2 a(t)\sin(2\pi\ f(t)t+\varphi)\gamma^2$$

i.e.

$$h_1(y,t) = 2\pi\ f(t)\sqrt{a^2(t)-y^2(t)} - \frac{1}{2}(2\pi t)^2 y(t)\gamma^2 \qquad (15a)$$

$$h_2(y,t) = (2\pi t)a(t)\cos(2\pi\ f(t)t+\varphi)$$
$$= (2\pi t)\sqrt{a^2(t)-y^2(t)} \qquad (15b)$$

$$h_3(y,t) = h_2(y,t)\left(-\alpha f(t)\right) \qquad (15c)$$

3.2 Amplitude Estimation:

To find an estimate for the diffusion and thus the amplitude, we square dy(t) to get:

$$\left(dy(t)\right)^2 = (2\pi t)^2 a^2(t)\cos^2(2\pi\ f(t)t+\varphi)\gamma^2 dt$$
$$= (2\pi t)^2 a^2(t)\left(1-\sin^2(2\pi\ f(t)t+\varphi)\right)\gamma^2 dt$$

$$= (2\pi t)^2 \left(a^2(t) - y^2(t)\right)\gamma^2 dt \qquad (16)$$

In the above equation we used the Ito rules: dtdt=0, dtdW=0, dWdW=dt.

Rearrange eqn. (16) and after some manipulations we get an expression for a(t) as:

$$a^2(t) = y^2(t) + \frac{1}{(2\pi t)^2 \gamma^2} \frac{(dy(t))^2}{dt} \qquad (17)$$

This is an exact expression for the unknown amplitude a(t). We could improve this estimate by passing it through a low pass filter made of moving average. We could also use this equation to find an estimate for γ^2 provided that we have an estimate for a(t). Such an estimate for a(t) could be obtained through TEO .

3.3 The maximum likelihood estimate of the OU parameters:

Given the observation y(t), we need to find the unknown parameters of the OU model of the frequency. The frequency has the solution:

$$f(t) = \beta + e^{-\alpha t}\left(f(0) - \beta\right) + \gamma \int_0^t e^{-\alpha(t-u)} dW(u) \qquad (18)$$

i.e.

$$f(t) - \beta = e^{-\alpha t}\left(f(0) - \beta\right) + \gamma \int_0^t e^{-\alpha(t-u)} dW(u)$$

The case when $\alpha = 1$:

To simplify the analysis, we set the parameter $\alpha = 1$. This will facilitate the analysis and still we have a good model for the frequency. Thus, the SDE for the observations y(t) becomes:

$$dy(t) \approx \begin{bmatrix} 2\pi\ f(t)a(t)\cos(2\pi\ f(t)t + \varphi) \\ -\frac{1}{2}(2\pi t)^2 a(t)\sin(2\pi\ f(t)t + \varphi)\gamma^2 \\ +(2\pi t)a(t)\cos(2\pi\ f(t)t + \varphi)(\beta - f(t)) \end{bmatrix} dt + (2\pi t)a(t)\cos(2\pi\ f(t)t + \varphi)\gamma dW(t) \qquad (19)$$

For large values of "t", we have the approximation (verified through simulation)

$$dy(t) \approx -\frac{1}{2}(2\pi t)^2 y(t)\gamma^2 dt + (2\pi t)a(t)\cos(2\pi\ f(t)t + \varphi)\gamma dW(t), \quad t >> 1 \qquad (20)$$

Comparing eqn. (3b) and eqn. (17) we deduce that:

$$\left(D_s y(t)\right)^2 = \frac{(dy(t))^2}{dt} e^{-2(t-s)} 1_{s\in[0,t]}(s), \quad \alpha = 1 \qquad (21)$$

i.e. the Malliavin derivative of y(t) is completely known from the observed data y(t). Also the Malliavin derivative of the frequency $D_s f(t)$ is given as:

$$D_s f(t) = \gamma D_s \int_0^t e^{-(t-u)} dW(u) = \gamma\, e^{-(t-s)} 1_{s\in[0,t]}(s), \alpha = 1 \qquad (22)$$

and
$$dy(t) \approx \left[h_1(y,t) + h_2(y,t)\beta + h_3(y,t)\right] dt + h_2(y,t)\gamma dW(t), \quad \alpha = 1$$

For known diffusion term i.e. $h_2(y,t)\gamma$ is completely known, the maximum likelihood estimate of β is obtained by maximizing the likelihood function [8, Ch.1]:

$$\log L(\underline{\theta}, y) = \int_0^T \frac{b(\underline{\theta}, y(t))}{\sigma^2(y(t))} dy(t) - \frac{1}{2}\int_0^T \frac{b^2(\underline{\theta}, y(t))}{\sigma^2(y(t))} dt \qquad (23)$$

Where $b(\underline{\theta}, y(t)) = \left[h_1(y,t) + h_2(y,t)\beta + h_3(y,t)\right]$

$$\sigma^2(y(t)) = h_2^2(y,t)\gamma^2 = \gamma^2 (2\pi t)^2 \left(a^2(t) - y^2(t)\right)$$

$$\underline{\theta} = \beta$$

and $\partial b(\underline{\theta}, y(t)) / \partial \underline{\theta} = h_2(y,t)$

We could also use "$f(t)=\beta+e^{-\alpha t}\left(f(0)-\beta\right)+\gamma\int_0^t e^{-\alpha(t-u)}dW(u)$".

In this case

$$\partial b(\underline{\theta}, y(t))/\partial\underline{\theta}=\frac{\partial h_1(y,t)}{\partial\beta}+e^{-t}h_2(y,t)$$

$$\begin{aligned}&=2\pi\left(1-e^{-t}\right)a(t)\cos(2\pi\ f(t)t+\varphi)\\&+e^{-t}\left(2\pi t\right)a(t)\cos(2\pi\ f(t)t+\varphi)\\&=2\pi a(t)\cos(2\pi\ f(t)t+\varphi)\\&+2\pi e^{-t}\left(t-1\right)a(t)\cos(2\pi\ f(t)t+\varphi)\end{aligned}$$

$$\begin{aligned}&=2\pi\sqrt{a^2(t)-y^2(t)}+2\pi e^{-t}\left(t-1\right)\sqrt{a^2(t)-y^2(t)}\\&=2\pi\sqrt{a^2(t)-y^2(t)}\left[1+e^{-t}\left(t-1\right)\right]\end{aligned}$$

The maximization w.r.t. the unknown parameters results into the equation:

$$\int_0^T\frac{\partial b(\underline{\theta}, y(t))/\partial\underline{\theta}}{\sigma^2(y(t))}\left[dy(t)-b(\underline{\theta}, y(t))dt\right]=0 \quad (24)$$

Substitute into the above equation we get:

$$\int_0^T\frac{h_2(y,t)}{\sigma^2(y(t))}\left[dy(t)-\left(h_1(y,t)+h_2(y,t)\beta+h_3(y,t)\right)dt\right]=0$$

, for given f(t) (25)

i.e.

$$\hat{\beta}\int_0^T\frac{h_2^2(y,t)}{\sigma^2(y(t))}dt=\int_0^T\frac{h_2(y,t)}{\sigma^2(y(t))}\left[dy(t)-\left(h_1(y,t)+h_3(y,t)\right)dt\right]$$

which yields

$$\hat{\beta}(T)=\frac{\int_0^T\frac{h_2(y,t)}{\sigma^2(y(t))}\left[dy(t)-\left(h_1(y,t)+h_3(y,t)\right)dt\right]}{\int_0^T\frac{h_2^2(y,t)}{\sigma^2(y(t))}dt}$$

Substitute $\sigma^2(y(t))=h_2^2(y,t)\gamma^2=\gamma^2\left(2\pi t\right)^2\left(a^2(t)-y^2(t)\right)$, we get:

$$\hat{\beta}(T)=\gamma^2\frac{\int_0^T\left(\frac{1}{\gamma^2h_2(y,t)}\right)\left[dy(t)-\left(h_1(y,t)+h_3(y,t)\right)dt\right]}{\int_0^T dt}$$

$$=\frac{1}{T}\int_0^T\left(\frac{1}{h_2(y,t)}\right)\left[dy(t)-\left(h_1(y,t)+h_3(y,t)\right)dt\right] \quad (26)$$

This is the maximum likelihood estimate of the unknown β. This estimate is dependent on a(t) and f(t). An initial guess for the value of f(t) could be obtained from the Fourier transform. The estimate for a(t) is obtained through eqn. (17). Also the estimate is function of γ. The estimate of γ is given below in eqn. (29). It could also be obtained from eqn. (17) if we use TEO to find a(t).

3.4 Statistical properties of the estimate:

As shown in the appendix, the estimate of β is unbiased and its variance equals the Cramer Rao lower bound.

3.5 An equation for γ^2:

To find an estimate for γ, we obtain an equation for the Malliavin derivative of y(t). We obtain another equation from the SDE of y(t). Equating both equations will yield an expression for the value of γ^2 [see the Appendix]. We could also use eqn. (3b) if we have an estimate for a(t). This estimate could be found through TEO of eqn. (10).

Recall that:

$$D_s y(t)=2\pi t a(t)\cos(2\pi f(t)t+\varphi)D_s f(t) \quad (27)$$

Squaring we get:

$$\left(D_s y(t)\right)^2 = (2\pi t)^2 \left(a^2(t) - y^2(t)\right) D_s f(t) D_s f(t) \tag{28}$$

From Ito calculus,

$$a^2(t) - y^2(t) = \frac{1}{(2\pi t)^2 \gamma^2} \frac{\left(dy(t)\right)^2}{dt}$$

Substitute eqn. (3b) into eqn. (28) we get:

$$\left(D_s y(t)\right)^2 = \frac{1}{\gamma^2} \frac{\left(dy(t)\right)^2}{dt} D_s f(t) D_s f(t) \tag{29}$$

3.6 Estimation of *f*(t) or γ^2 through the Malliavin derivative:

We know that the Malliavin derivative of y(t) is function of a(t) which has been estimated. It is also function of the value of γ . The value of γ^2 could have been estimated if we use an estimate for a(t). This estimate could be obtained through TEO. In this case we use eqn. (3b) to find an estimate for γ^2. If we use the Malliavin calclulus alone, we need to find another equation for $\left(D_s y(t)\right)^2$. Equating both expressions we get an estimate for γ^2. On the other hand if we have an estimate for γ^2, we use the $\left(D_s y(t)\right)^2$ equation to find and estimate for f(t).

Through the transformation $z(t) = \frac{(2\pi t)}{\gamma} \arcsin y(t)$, we get an expression for the Malliavin derivative of z(t), $D_s z(t)$, which is completely known from the observed data y(t) except for f(t).

$$D_s z(t) = \frac{1}{\gamma(2\pi t)} \frac{d \arcsin y}{dy} D_s y(t) = \frac{1}{\gamma(2\pi t)} \frac{1}{\sqrt{1 - y^2(t)}} D_s y(t) \tag{30}$$

Since the diffusion of the z(t) SDE is unity, the Malliavin derivative of z(t), $D_s z(t)$, satisfies an ordinary differential equation which could be solved analytically or numerically. This is the second equation. Equating both equations we get an expression for f(t) as function of y(t).

$$dD_s z(t) = \frac{\partial g(t,z)}{\partial z} D_s z(t) dt, s \le t$$

Where $g(z,t) = \begin{pmatrix} \frac{f(t)\sqrt{a^2(t) - \sin^2\left(z(2\pi t)\gamma\right)}}{\gamma \, t \cos\left(z(2\pi t)\gamma\right)} \\ -\frac{1}{2}(2\pi t)\gamma \tan\left(z(2\pi t)\gamma\right) + \frac{\beta}{\gamma} \\ -\frac{f(t)}{\gamma} + \frac{1}{2}\gamma \end{pmatrix}$

[see the appendix eqn. A.7 for the derivation].

The Malliavin derivative $D_s z(t)$ has the explicit solution

$$D_s z(t) = \exp\left[\int_s^t \left(\frac{\partial g(u,z)}{\partial z}\right) du\right], s \le t$$

Equating both expressions for the Malliavin derivative of z(t), eqn. (30) and eqn. (A.10) we get:

$$\exp\left[\int_s^t \left(\frac{\partial g(u,z)}{\partial z}\right) du\right] = \frac{\sqrt{a^2(t) - y^2(t)}}{\sqrt{1 - y^2(t)}} e^{-2\alpha(t-s)} 1_{s \in [0,t]}(s)$$

This is a closed form equation in the value of f(t) where the right hand side is completely known. The solution of this equation is not easy and numerical methods should be employed.

4. Results and Conclusions:

In this Section we simulate a sinusoidal with slowly varying amplitude and random frequency. We then apply the proposed method to find an estimate for: (1) a(t), (2) The parameter of the OU β, (3) $f(t)$, and (4) γ^2.

Assume that the observed signal y(t) is given by the equation:

$$y(t) = a(t)\sin(2\pi\ f(t)t)$$

$$a(t) = 0.6\sin(2\pi\ f_a t)$$

Where $f_a = 1.0$

The frequency $f(t)$ is a stochastic process described by the Ornstein-Uhlenbeck SDE:

$$df(t) = (\beta - f(t))dt + \gamma dW(t)$$

where $\beta = 10$, $\gamma = 0.1$

The sampling interval $\Delta t = 0.01$

4.1 The estimate of the amplitude and of γ:

In Fig. 5, we show the true amplitude a(t), the estimates of the smoothed amplitude using Ito calculus (MA-Ito) and using TEO (psi(y)MA).

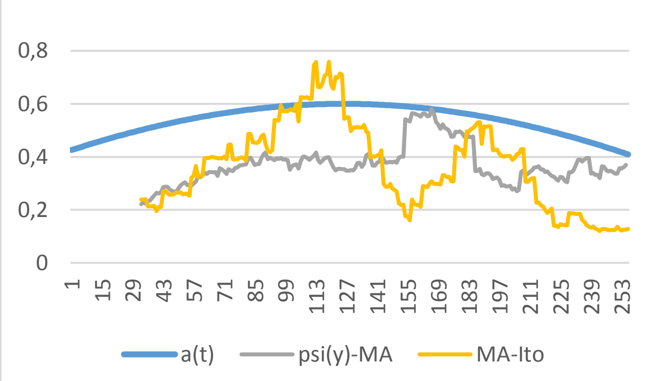


Fig.5, True a(t) and estimated smoothed amplitude using TEO (psi(y)-MA, and Ito calculus (MA-Ito)

The estimate of γ was obtained by minimizing the sum of squared error between the estimated smooth amplitude of TEO (eqn.10) and the estimated smooth amplitude of the Ito calculus (eqn. 17). The estimated amplitudes were biased downward. We must adjust the estimated amplitudes by multiplying by a scale factor. Since the amplitude is slowly varying, the scale factor could be the maximum y(t) divided by the maximum of the amplitude estimates. Following the above procedures, we found the estimate of γ =0.09 which is close to the true value.

4.2 The estimate of β:

Through simulation, it was found that the peak of the FFT changes from one segment to the other (we divide the data into segments of 128 points). This is due to the randomness of the frequency. Taking the average of the peaks yields good initial estimate for β that could be used as an estimate for f(t) in eqn. (26).

$$\hat{\beta}(T) = \frac{1}{T}\int_0^T \left(\frac{1}{h_2(y,t)}\right)\left[dy(t) - \left(h_1(y,t) + h_3(y,t)\right)dt\right] \tag{26}$$

Recall that:

$$h_1(y,t) = 2\pi\ f(t)\sqrt{a^2(t) - y^2(t)} - \frac{1}{2}(2\pi t)^2\, y(t)\gamma^2$$

$h_2(y,t)$

$= (2\pi t)\sqrt{}$

$h_3(y,t) =$

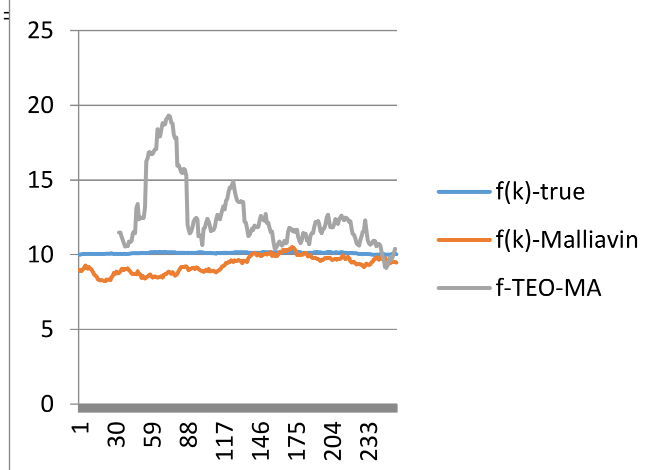
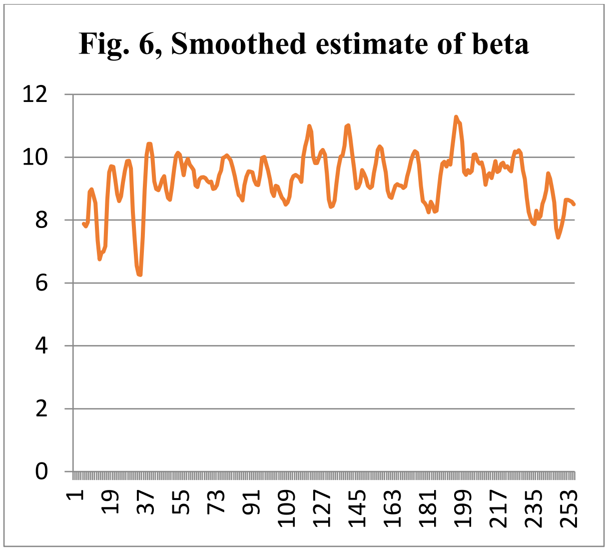


Fig.7, True frequency, estimated frequency using Malliavin calculus, and estimated frequenc using TEO followed by moving average

For numerical reasons, one should make sure that $(estimated\ a^2(t)) \geq y^2(t)$ even if we have to scale up the estimates of the amplitude.

The value of the smoothed $\hat{\beta}(T)$ is shown in Fig. 6.



4,3 The estimate of *f*(t):

It was found that [see the Appendix] the unknown frequency satisfy the equation:

$$\int_s^t [f(u)(.)+(.)]du$$

$$= \ln\left[\frac{1}{\gamma^2}\frac{(dy(t))^2}{dt}D_s f(t)D_s f(t)1_{s\in[0,t]}(s)\right]$$

This is not an easy to solve equation and one has to resort to numerical methods.

The true frequency, the smoothed estimated frequency using eqn. (A.12) and the smoothed estimated frequency using TEO of eqn. 9 are shown in Fig. 7.

4.4 Conclusions:

In this report we studied a signal that is made of one sinusoid with random frequency and slowly varying amplitude. The frequency was modeled as an OU process. The Ito calculus was used to find an SDE that describes the signal. The amplitude was estimated using Ito calculus rules. The Radon-Nikodym derivative, which is the likelihood function, was used to estimate the parameters of the OU process. The Malliavin calculus was used to find an estimate for the frequency.

In the future we need to find a closed formula for discrete Malliavin derivative. This will offer a different way to study the problem of sum of sinusoids.

Appendix A

In this appendix we present the derivations for the statistical properties of the maximum likelihood estimate of β. We also find the Malliavin calculus based estimates of f(t) and γ.

To study the statistical properties of the estimate, we substitute for:

$$dy(t) \approx [h_1(y,t)+h_2(y,t)\beta+h_3(y,t)]dt$$
$$+h_2(y,t)\gamma dW(t)$$

into the maximum likelihood equation to obtain:

$$\hat{\beta}(T)=\frac{1}{T}\int_0^T\left(\frac{1}{h_2(y,t)}\right)\left\{\begin{array}{l}[h_1(y,t)+h_2(y,t)\beta+h_3(y,t)]dt\\ +h_2(y,t)\gamma dW(t)\\ -(h_1(y,t)+h_3(y,t))dt\end{array}\right\}$$

$$=\frac{1}{T}\int_0^T\left(\frac{1}{h_2(y,t)}\right)\{h_2(y,t)\beta dt+h_2(y,t)\gamma dW(t)\}$$

$$=\frac{1}{T}\int_0^T\{\beta dt+\gamma dW(t)\}$$

$$=\beta+\frac{\gamma}{T}\int_0^T dW(t) \qquad \text{(A.1)}$$

Taking the expectation of both sides, we obtain:

$$E\{\hat{\beta}(T)\}=\beta+\frac{\gamma}{T}E\left\{\int_0^T dW(t)\right\}$$

$$= \beta \qquad \text{(A.2)}$$

i.e. the estimate is unbiased.

We now derive an expression for the variance of the estimate. From eqn. (A.1), we have:

$$E\left\{\left(\hat{\beta}(T)-\beta\right)^2\right\}=\left(\frac{\gamma}{T}\right)^2\left(E\left\{\int_0^T dW(s)\int_0^T dW(t)\right\}\right)$$

$$=\left(\frac{\gamma}{T}\right)^2\int_0^T dt$$

$$=\left(\frac{\gamma^2}{T}\right) \qquad \text{(A.3)}$$

Cramer-Rao lower bound:

In order to find the Cramer-Rao lower bound for the unbiased estimate, it is easier if we have the diffusion part unity. Thus, we use a transformation on the observed SDE as follows:

Find the transformation z=U(y) such that the SDE of z(t) has unity diffusion. Using Ito lemma we get:

$$dz(t)=\frac{\partial U}{\partial y}dy(t)+\frac{1}{2}\frac{\partial^2 U}{\partial y^2}\left(dy(t)\right)^2$$

Recall that $dy(t)\approx\left[h_1(y,t)+h_2(y,t)\alpha\beta+\alpha h_3(y,t)\right]dt + h_2(y,t)\gamma dW(t)$

where
$$h_1(y,t)=2\pi\ f(t)a(t)\cos(2\pi\ f(t)t+\varphi) -\frac{1}{2}(2\pi t)^2 a(t)\sin(2\pi\ f(t)t+\varphi)\gamma^2$$

i.e.

$$h_1(y,t)=2\pi\ f(t)\sqrt{a^2(t)-y^2}-\frac{1}{2}(2\pi t)^2 y(t)\gamma^2$$

$$h_2(y,t)=(2\pi t)a(t)\cos(2\pi\ f(t)t+\varphi)=(2\pi t)\sqrt{a^2(t)-y^2}$$

$$h_3(y,t)=h_2(y,t)\left(-f(t)\right)$$

We need $\frac{\partial U}{\partial y}=1/\left(h_2(y,t)\gamma\right)=\frac{(2\pi t)^{-1}}{\gamma}\left(1-y^2\right)^{-1/2}$ (A.4)

Integrating w.r.t. y , we get:

$$z=U(y)=\frac{1}{\gamma(2\pi t)}\arcsin y \quad \text{and}$$
$$y=\sin\left(z(2\pi t)\gamma\right) \qquad \text{(A.5)}$$

Thus, $\frac{\partial^2 U}{\partial y^2}=\frac{(2\pi t)^{-1}}{\gamma}y\left(1-y^2\right)^{-3/2}$ and the SDE for z(t) becomes:

$$\begin{aligned}dz(t)&=\left(\frac{(2\pi t)^{-1}}{\gamma}\left(a^2(t)-y^2\right)^{-1/2}\right)\\&\left[h_1(y,t)+h_2(y,t)\alpha\beta+\alpha h_3(y,t)\right]dt\\&+dW(t)\\&+\frac{1}{2}\left(\frac{(2\pi t)^{-1}}{\gamma}y\left(a^2(t)-y^2\right)^{-3/2}\right)\left(h_2(y,t)\gamma\right)^2dt\end{aligned}$$

$$=\left[\begin{array}{l}\left(\frac{(2\pi t)^{-1}}{\gamma}\left(a^2(t)-y^2\right)^{-1/2}\right)\left(\begin{array}{l}h_1(y,t)+h_2(y,t)\alpha\beta\\+\alpha h_3(y,t)\end{array}\right)\\+\left(\frac{1}{2}\left(\frac{(2\pi t)^{-1}}{\gamma}\left(a^2(t)-y^2\right)^{-1/2}\right)\left(h_2(y,t)\gamma\right)^2\right)\end{array}\right]dt$$
$$+dW(t)$$

$$=\left[\begin{array}{l}\frac{1}{\gamma h_2(y,t)}\left(h_1(y,t)+h_2(y,t)\alpha\beta+\alpha h_3(y,t)\right)\\+\left(\frac{1}{2}h_2(y,t)\gamma\right)\end{array}\right]dt$$
$$+dW(t)$$

Substitute

$$h_1(y,t)=2\pi\ f(t)\sqrt{a^2(t)-y^2}-\frac{1}{2}(2\pi t)^2 y(t)\gamma^2$$

$$= 2\pi\ f(t)\sqrt{a^2(t)-\sin^2\left(z(2\pi t)\gamma\right)} - \frac{1}{2}(2\pi t)^2\gamma^2\sin\left(z(2\pi t)\gamma\right)$$

$$h_2(y,t) = (2\pi t)a(t)\cos(2\pi\ f(t)t+\varphi) = (2\pi t)\cos\left(z(2\pi t)\gamma\right)$$

$$h_3(y,t) = h_2(y,t)\left(-f(t)\right)$$

We get:

$$dz = \begin{bmatrix} \dfrac{1}{\gamma h_2(y,t)}\begin{pmatrix} h_1(y,t)+h_2(y,t)\alpha\beta \\ +\alpha h_3(y,t)\end{pmatrix} \\ +\left(\dfrac{1}{2}h_2(y,t)\gamma\right)\end{bmatrix}dt + dW(t)$$

$$= \left[\left(\frac{h_1(y,t)}{\gamma h_2(y,t)} + \frac{\alpha\beta}{\gamma} - \alpha\frac{f(t)}{\gamma}\right) + \frac{1}{2}\gamma\right]dt + dW(t),$$

Substitute $y = \sin\left(z(2\pi t)\gamma\right)$ we get:

$$dz = \left[\left(\frac{2\pi\ f(t)\sqrt{a^2(t)-\sin^2\left(z(2\pi t)\gamma\right)} - \frac{1}{2}(2\pi t)^2\gamma^2\sin\left(z(2\pi t)\gamma\right)}{\gamma(2\pi t)\cos\left(z(2\pi t)\gamma\right)} + \frac{\alpha\beta}{\gamma} - \alpha\frac{f(t)}{\gamma}\right) + \frac{1}{2}\gamma\right]dt + dW(t)$$

Rearrange we get:

$$dz = \begin{pmatrix} \dfrac{f(t)\sqrt{a^2(t)-\sin^2\left(z(2\pi t)\gamma\right)}}{\gamma\ \mathrm{t}\cos\left(z(2\pi t)\gamma\right)} \\ -\dfrac{1}{2}(2\pi t)\gamma\tan\left(z(2\pi t)\gamma\right) + \dfrac{\alpha\beta}{\gamma} - \alpha\dfrac{f(t)}{\gamma} + \dfrac{1}{2}\gamma \end{pmatrix}dt$$

$$+ dW(t)$$

$$= g(z,t)dt + dW(t) \qquad \text{(A.6)}$$

Where

$$g(z,t) = \left(\frac{f(t)\sqrt{a^2(t)-\sin^2\left(z(2\pi t)\gamma\right)}}{\gamma\ \mathrm{t}\cos\left(z(2\pi t)\gamma\right)} - \frac{1}{2}(2\pi t)\gamma\tan\left(z(2\pi t)\gamma\right) + \frac{\alpha\beta}{\gamma} - \alpha\frac{f(t)}{\gamma} + \frac{1}{2}\gamma\right)$$

(A.7)

The Cramer-Rao lower bound is given as [9, Ch.7]:

$$E\left\{\left(\beta-\hat{\beta}(t)\right)^2\right\} \geq \frac{1}{E\left\{\int_0^t\left[\dfrac{\partial g(z,s)}{\partial\beta}\right]^2 ds\right\}}$$

$$= \frac{\gamma^2}{E\left\{\int_0^t \alpha^2 ds\right\}} = \frac{\gamma^2}{\alpha^2 t}$$

(A.8)

Thus, the variance of the estimates is equal to the Cramer-Rao lower bound.

Also

$$\frac{\partial g(t,z)}{\partial z}=\frac{(\gamma\, t)\cos(z(2\pi t)\gamma)f(t)\left(a^2(t)-\sin^2(z(2\pi t)\gamma)\right)^{-1/2}(-2(2\pi t)\gamma)\sin(z(2\pi t)\gamma)\cos(z(2\pi t)\gamma)}{(\gamma\, t)^2\cos^2(z(2\pi t)\gamma)}$$

$$+\frac{(\gamma\, t)(2\pi t)\gamma\sin(z(2\pi t)\gamma)f(t)\sqrt{a^2(t)-\sin^2(z(2\pi t)\gamma)}}{(\gamma\, t)^2\cos^2(z(2\pi t)\gamma)}-\frac{1}{2}(2\pi t)\gamma\frac{\partial\tan(z(2\pi t)\gamma)}{\partial z}$$

which is reduced to:

$$\frac{\partial g(t,z)}{\partial z}=\frac{f(t)\left(a^2(t)-\sin^2(z(2\pi t)\gamma)\right)^{-1/2}(-4\pi)\sin(z(2\pi t)\gamma)}{(\gamma\, t)}$$

$$+\frac{2\pi\sin(z(2\pi t)\gamma)f(t)\sqrt{a^2(t)-\sin^2(z(2\pi t)\gamma)}}{\cos^2(z(2\pi t)\gamma)}-(\pi t)\gamma Sec^2(z(2\pi t)\gamma)$$

(A.9)

All the variables are known except for f(t).

Define

$$\zeta(t,s)=D_s z(t)=\frac{1}{\gamma(2\pi t)}\frac{d\arcsin y}{dy}D_s y(t)$$

$$=\frac{1}{\gamma(2\pi t)}\frac{1}{\sqrt{1-y^2(t)}}D_s y(t)$$

Substitute for $D_s y(t)$, we get:

$$\zeta(t,s)=D_s z(t)=\frac{\sqrt{a^2(t)-y^2(t)}}{\sqrt{1-y^2(t)}}e^{-(t-s)}1_{s\in[0,t]}(s)$$

and $d\zeta(t,s)=\frac{\partial g(t,z)}{\partial z}\zeta(t,s)dt,\ s\le t$

The Malliavin derivative $D_s z(t)$ has the explicit solution

$$D_s z(t)=\exp\left[\int_s^t\left(\frac{\partial g(u,z)}{\partial z}\right)du\right],\ s\le t \quad \text{(A.10)}$$

Equating both expressions for the Malliavin derivative of z(t) (eqn.29 and eqn. A.10) we get:

$$\exp\left[\int_s^t\left(\frac{\partial g(u,z)}{\partial z}\right)du\right]$$

$$=\frac{1}{\gamma^2}\frac{(dy(t))^2}{dt}D_s f(t)D_s f(t)1_{s\in[0,t]}(s) \quad \text{,(A.11)}$$

$$z(t)=\frac{(2\pi t)}{\gamma}\arcsin y(t)$$

Notice that right hand side of eqn. (A.11) is completely known at any instant "t".

Equation (A.11) has the form:

$$\int_s^t[f(u)(.)+(.)]du$$

$$=\ln\left[\frac{1}{\gamma^2}\frac{(dy(t))^2}{dt}D_s f(t)D_s f(t)1_{s\in[0,t]}(s)\right]$$

(A.12)

This is a closed form equation in the value of f(t).

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