

On the Solution of the Boundary Value Problem of First Order Lyapunov System of Differential Equations

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Abstract: - In this paper, we present results on the existence and uniqueness of a solution to the first order non-homogeneous Lyapunov system of differential equations satisfying general two-point boundary conditions. A variation of parameters type of formula is derived to construct an integral form of the solution. The uniqueness of the solution is established through the least squares solution of a double-sided matrix equation associated with the boundary value problem.

Key-Words: - Fundamental matrix solution; Lyapunov equation; Variation of Parameters; Green's matrix; Best least squares solution; Kronecker product

Received: July 9, 2025. Revised: August 13, 2025. Accepted: September 11, 2025. Published: October 22, 2025.

1 Introduction

Boundary value problems arise in almost all branches of physical sciences and engineering. The methods commonly used in solving two-point boundary value problems are based on idea of constructing a Green's function, in the form of an integral equation, and such a function give an analytical form for the solution of the two-point boundary value problems associated with linear system of differential equations. In this paper, our focus is to study the first order linear Lyapunov system of equations

$$T' = A(t)T + TB(t) + F(t), \quad a \leq t \leq b \quad (1)$$

where A , B and T are square matrices of order n and $F(t)$ is an $n \times n$ square matrix and is usually regarded as a forcing matrix function. We now adjoin the boundary condition in the matrix form

$$MT(a) + NT(b) = \alpha \quad (2)$$

where M and N are given square matrices of order n and α is also a given square matrix. The general solution of (1) is discussed in [1, 2], where the state matrices are non-invertible, and the analysis is based on the construction of ψ -bounded solutions on the

time scales. An alternative form of linear Sylvester system of equations is studied in [3-5], where the Kronecker product of matrices is used to derive a formula for the general solution of the equations. The importance of the Kronecker product boundary value problems for linear system of differential equations gained momentum in recent years due to its applications in system analysis [6-8], matrix calculus and stochastic equations [9-11]. The results on Moore-Penrose inverse are detailed in [12]. This paper is organized as follows: section 2 presents a criterion to establish the general solution of (1) in terms of two fundamental matrix solutions of $T' = AT$ and $T' = B^*T$, where $*$ refers to the transpose of the complex conjugate matrix, and then we develop a variation of parameters type of formula to obtain the general solution of the two-point boundary value problem (1) and (2). Section 3 is concerned with the least squares solution of the double-sided system of equations that closely related to the solution of the two-point boundary value problem in this paper, followed by conclusive remarks in section 4.

2 Integral form of the solution

In this section, we shall be concerned with the general solution of the two-point boundary value problem. We first present the variation of parameters formula, where A , B , and T stand for square matrices of order n and we assume that they are all functions of t on $[a, b]$, Y is a fundamental matrix solution of $T' = A(t)T$ and Z stands for a fundamental matrix solution of $T' = B^*(t)T$.

Theorem 2.1 T is a solution of $T' = A(t)T + TB(t)$ if and only if $T = YCZ^*$, where C is a constant n by n matrix.

Proof: It is easily verified that T defined by $T = YCZ^*$ is a solution of $T' = A(t)T + TB(t)$, since

$$\begin{aligned} T' &= Y' CZ^* + YCZ^{*'} = A(t)YCZ^* + YCZ^* B(t) \\ &= A(t)T(t) + T(t)B(t) \end{aligned}$$

Hence, $T = YCZ^*$ is a solution of (1) with $F(t) \equiv 0$. In order to prove that every solution of this equation is of the form $T = YCZ^*$, let T be a solution and K be a matrix given by the equation $K = Y^{-1}T$. Then, $Y'K + YK' = AYK + YKB$, which leads to $K' = KB^*$ or $K^* = BK^*$. Since Z is a fundamental matrix solution of $T' = B^*(t)T$, it follows that there exists a constant matrix C^* such that $K^* = ZC^*$, and it is readily seen that $T = YK = YCZ^*$. $\square$

We now turn to the non-homogeneous Lyapunov system (1). Any solution of the homogeneous linear Lyapunov system is of the form YCZ^* , where Y is a fundamental matrix solution of $T' = A(t)T$ and Z stands for a fundamental matrix solution of $T' = B^*(t)T$. Such a form cannot be a solution of (1) where $F(t) \neq 0$. Let $T(t)$ be any solution of (1) and $\bar{T}(t)$ be a particular solution of (1). Then, $T(t) - \bar{T}(t)$ is a solution of

$$T' = A(t)T + TB(t). \quad (3)$$

Therefore, $T(t) - \bar{T}(t) = Y(t)CZ^*(t)$ or

$$T(t) = \bar{T}(t) + Y(t)CZ^*(t). \quad (4)$$

We seek a particular solution of (1) in the form

$$\bar{T}(t) = Y(t)C(t)Z^*(t).$$

Then, from (1), we have

$$\bar{T}'(t) = A(t)\bar{T}(t) + \bar{T}(t)B(t) + F(t).$$

Hence, as a result of (2) and (3), we have

$$\begin{aligned} Y'(t)C(t)Z^*(t) + Y(t)C'(t)Z^*(t) + Y(t)C(t)Z^{*'}(t) \\ = A(t)Y(t)C(t)Z^*(t) + Y(t)C(t)Z^*(t)B(t) + F(t). \end{aligned}$$

Since Y and Z are fundamental matrix solutions of $T' = A(t)T$ and $T' = B^*(t)T$, respectively, it follows that

$$\begin{aligned} A(t)Y(t)C(t)Z^*(t) + Y(t)C'(t)Z^*(t) + Y(t)C(t)Z^{*'}(t)B(t) \\ = A(t)Y(t)C(t)Z^*(t) + Y(t)C(t)Z^*(t)B(t) + F(t) \end{aligned}$$

Hence, it follows that $Y(t)C'(t)Z^*(t) = F(t)$, from which, $C'(t) = Y^{-1}(t)F(t)Z^{*-1}(t)$, then,

$$C(t) = \int_a^t Y^{-1}(s)F(s)Z^{*-1}(s)ds$$

Hence, a particular solution $\bar{T}(t)$ of (1) is given by

$$\bar{T}(t) = Y(t) \left[\int_a^t Y^{-1}(s)F(s)Z^{*-1}(s)ds \right] Z^*(t)$$

and then, any solution $T(t)$ of (1) can be written as

$$T(t) = Y(t) \left[\int_a^t Y^{-1}(s)F(s)Z^{*-1}(s)ds \right] Z^*(t) + Y(t)CZ^*(t) \quad (5)$$

Note that in the equation (1), if we take $B = A^*$, then

$$T'(t) = A(t)T + TA^*(t) + F(t) \quad (6)$$

which is known as the non-homogeneous Lyapunov system.

Theorem 2.2 If Y is a fundamental matrix solution of $T' = A(t)T$, then the conjugate transpose Y^* is a fundamental matrix solution of $T' = TA^*(t)$

Proof: Differentiate both sides of the identity $Y(t)Y^{-1}(t) = I$ with respect to t , we get

$$Y'(t)Y^{-1}(t) + Y(t)(Y^{-1}(t))' = 0$$

$$\begin{aligned} Y(t)(Y^{-1}(t))' &= -A(t) \\ (Y^{-1})' &= -Y^{-1}A. \end{aligned}$$

Hence, $Y^{-1}(t)$ is a fundamental matrix solution of $T'(t) = -T(t)A(t)$. Similarly, from $Y^*(t)Y^{*-1}(t) = I$, we have, after differentiating both sides with respect to t ,

$$\begin{aligned} (Y^*)'Y^{*-1} + Y^*(Y^{*-1})' &= 0 \\ Y^*A^*Y^{*-1} + Y^*(Y^{*-1})' &= 0 \end{aligned}$$

from which, we have $(Y^{*-1})' = -A^*Y^{*-1}$. Hence, Y^{*-1} is a fundamental matrix solution of the system $T'(t) = -A^*(t)T(t)$. Now, since Y is a fundamental matrix solution of $T' = A(t)T$, we have $Y' = AY$, as such, $(Y^*)' = Y^*A^*$, and hence Y^* is a fundamental matrix solution of $T' = T(t)A^*(t)$. $\square$

Theorem 2.3 Any solution of the non-homogeneous Lyapunov system (6) is given by

$$T(t) = Y(t)CY^*(t) + Y(t)\left[\int_a^t Y^{-1}(s)F(s)Y^{*-1}(s)ds\right]Y^*(t)$$

The proof is immediate if we simply realize that $Z^* = Y^*$ in Theorem 2.1.

We are ready to move on to the two-point boundary value problem (1) that satisfies the boundary conditions (2). Substituting the general solution of (1) given by (5) in the boundary condition matrix equation (2), we get

$$\begin{aligned} MY(a)CZ^*(a) + NY(b)CZ^*(b) + \\ Y(b)\left[\int_a^b Y^{-1}(s)F(s)Z^{*-1}(s)ds\right]Z^*(b) = \alpha \end{aligned}$$

and then,

$$MY(a)CZ^*(a) + NY(b)CZ^*(b) = \beta,$$

where $\beta = \alpha - Y(b)\left[\int_a^b Y^{-1}(s)F(s)Z^{*-1}(s)ds\right]Z^*(b)$, or the equation can be written into a matrix product form

$$MY(a) \quad NY(b) \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} Z^*(a) \\ Z^*(b) \end{bmatrix} = \beta$$

which is written symbolically as $RXS = T$, where

$$R = MY(a) \quad NY(b), \quad X = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} Z^*(a) \\ Z^*(b) \end{bmatrix}.$$

Since R and S are rectangular matrices of orders n by $2n$ and $2n$ by n matrices, respectively, and non-invertible in the normal sense, we analyze the general form of X in the best least squares sense.

3 Least squares solution to a double-sided matrix equation

We will investigate the solution of the following double-sided matrix equation with the previously developed results,

$$RXS = T \quad (7)$$

where R , S , and T are given $n \times 2n$, $2n \times n$ and $n \times n$ matrices. Such a double-sided matrix problem arises from encryption/decryption of color images, control theory, and scientific computing. Depending on the applications, the matrices R and S can be block Toeplitz, circulant, or unitary. The matrix nearness problem considered here is more general and we assume that the system (7) is consistent, by which a system of equations is said to be consistent if it has at least one solution. The implicit form of the best least squares solution of the problem over the set of symmetric and skew-symmetric solution space of the problem (7) will be presented in this section. Let $R^{n \times 2n}$, $S^{2n \times n}$, and $SSR^{2n \times 2n}$ be the set of all $n \times 2n$ real matrices, the set of all $2n \times 2n$ real symmetric matrices, and the set of all $2n \times 2n$ skew-symmetric matrices, respectively. We denote that A^T , A^+ , and $\|A\|$ be used in the usual sense, namely, the transpose, Moore-Penrose inverse, and the Frobenius 2-norm of the matrix A . We also adopt the symbol, $\text{vect}(\cdot)$, for the vector operator, .e.g. $\text{vect}(A) = (A_1^T, A_2^T, \dots, A_n^T)$ for vectorizing the matrix $A = (A_1, A_2, \dots, A_n) \in R^{n \times n}$. Furthermore, the Kronecker product of matrices R and S , $(R \otimes S)$, is defined as

$$R \otimes S = R_{ij}S, \quad i, j = 1, 2, \dots, n.$$

If R is an $m \times n$ matrix and S is a $p \times q$ matrix, then $(R \otimes S)$ is an $mp \times nq$ matrix. In the literature, many authors assumed that the system of equations (7) is consistent. But, equation (7) rarely satisfies the consistent condition, since matrices R , S , and T in practice are usually constructed from experiment data. In this section, we shall be concerned with the best least squares solution of (7) in a heuristic way. Our interest is in the matrix nearness problems; as such we have the following two cases:

Case 1: For given matrices $R \in \mathbb{R}^{m \times 2n}$, $S \in \mathbb{R}^{2n \times n}$, and $T \in \mathbb{R}^{n \times n}$, let E_G be the set of all solutions of the consistent system (7). For a given matrix $X_0 \in \mathbb{R}^{2n \times 2n}$, we find $\hat{X} \in E_G$ such that

$$\|\hat{X} - X_0\| = \min_{X \in E_G} \|X - X_0\|.$$

Case 2: For given matrices R , S , and T , let E_E be the set of all least squares solutions of the minimum residual problem

$$\min_{X \in \mathbb{R}^{2n \times 2n}} \|RXS - T\|.$$

For a given matrix $X_0 \in \mathbb{R}^{2n \times 2n}$, find $\hat{X} \in E_E$ such that

$$\|\hat{X} - X_0\| = \min_{X \in E_E} \|X - X_0\|$$

In fact, what we are interested in among the two cases is to find the best least squares solutions for a given matrix $X_0 \in \mathbb{R}^{2n \times 2n}$ over the set of general solutions E_G and the set of least squares solutions E_E of the matrix equation (7), respectively. Therefore, it is essential to find the best approximate solutions of system (7) in an implicit form. In order to relate our results to existing works in this area, we refer to the papers of Peng [13, 14]. They established an iterative method to solve the system of equations of a similar type with the Frobenius norm residual problem. The two cases we mentioned above are to find the best approximate or least squares solution of (7). These problems are known as the matrix nearness problems in the literature. The matrix nearness problem is important in solving two-point boundary value problems. In

general, numerical algorithms or iterative methods for solving these problems are available in the area. Using the Moore-Penrose inverse, the implicit form of the solution is obtained in this section. We need the following preliminary results to establish our main results.

The vector $x_0 \in \mathbb{R}^{n \times 1}$ is said to be the best approximate solution of the linear system of equations, $Ax = b$, where $A \in \mathbb{R}^{m \times n}$, if and only if

$$(a) (Ax - b)^T(Ax - b) \geq (Ax_0 - b)^T(Ax_0 - b), \quad \forall x \in \mathbb{R}^{n \times 1}$$

$$(b) x^T x > x_0^T x_0 \quad \text{for all } x \in \mathbb{R}^{n \times 1} \setminus \{x_0\} \quad \text{satisfying} \\ (Ax - b)^T(Ax - b) = (Ax_0 - b)^T(Ax_0 - b).$$

The vector $x_0 \in \mathbb{R}^{n \times 1}$ is a least squares solution to the matrix inconsistent system $Ax = b$ if and only if $(A \in \mathbb{R}^{m \times n})(Ax - b)^T(Ax - b) \geq (Ax_0 - b)^T(Ax_0 - b)$ for all $x \in \mathbb{R}^{n \times 1}$.

Let A be an $(m \times n)$ matrix. Then the generalized inverse of the matrix A^+ of $(n \times m)$ is defined as

(i) $AA^+A = A$, (ii) $A^+AA^+ = A^+$, (iii) $(AA^+)^* = AA^+$, and (iv) $(A^+A)^* = A^+A$. It is known that the matrix A^+ with the above properties is unique and is known as the Moore-Penrose inverse. The following results will be used to derive the least squares solution to (7).

Let E_G be the set of all solutions to the consistent system $Ax = b$. For a given $x_0 \in \mathbb{R}^{n \times 1}$, the vector $\hat{x} \in E_G$ satisfying $\|\hat{x} - x_0\| = \min_{x \in E_G} \|x - x_0\|$ is given by

$$\hat{x} = A^+b + (I - A^+A)x_0 \quad (8)$$

Similar result can be obtained for inconsistent systems $Ax = b$ over the set of least squares solutions, E_E , to the system.

In what follows, we discuss the best least squares solution of (7). It is assumed that the matrix equation $RXS = T$ is an inconsistent system. The fundamental question that arises is if we can find a matrix X such that $\|RXS - T\|$ is minimal. A matrix X

that minimizes $\|RXS - T\|$ is known as an approximate solution to the system (7). The matrix $\hat{X} \in \mathbb{R}^{2n \times 2n}$ is called the best approximate solution to (7) if it satisfies the following two conditions:

- (i) $\|RXS - T\| \geq \|R\hat{X}S - T\|$ for all $X \in \mathbb{R}^{2n \times 2n}$ and
- (ii) $\|X\| \geq \|\hat{X}\|$ for all matrices $X \in \mathbb{R}^{2n \times 2n} \setminus \{\hat{X}\}$ satisfying $\|RXS - T\| = \|R\hat{X}S - T\|$.

We note that a vector of order $x \in \mathbb{R}^{4mn \times 1}$ stands for the vector(X) since $X \in \mathbb{R}^{2m \times 2n}$. Using this notation we can always write system of equations $RXS = T$ in the form

$$(S^T \otimes R)x = t \quad (9)$$

where $S^T \otimes R$ is the Kronecker product of the two matrices S^T and R , and $t = \text{vector}(T)$. Consequently, the solution of the system of equations (7) can be obtained via solving the linear system of equations (9). Now, we have the following theorem for the best least squares solution of (7).

Theorem 4.1: Let the system of equations (7) be consistent. Then, for a given matrix $X_0 \in \mathbb{R}^{2n \times 2n}$, the matrix $\hat{X} \in E_G$ satisfying

$$\|\hat{X} - X_0\| = \min_{X \in E_G} \|X - X_0\|$$

is given by

$$\hat{X} = R^+TS^+ + X_0 - R^TRX_0SS^T$$

Proof: For any $X \in E_G$, it can be written as

$$X = R^+TS^+ + H - R^TRHSS^T \quad (10)$$

for some $H \in \mathbb{R}^{2n \times 2n}$. Also, note that equation (7) is equivalent (9). Then, the equation (10) is equivalent to the following equation

$$x = (S^T \otimes R)^+t + [I - (S^T \otimes R)^T(S^T \otimes R)]h.$$

The problem is turned into finding the best least squares solution of the usual linear system of

equations of the form $(S^T \otimes R)x = t$, and according to (8), the best least squares solution that satisfies nearness condition is given by

$$\hat{x} = (S^T \otimes R)^+t + [I - (S^T \otimes R)^T(S^T \otimes R)]x_0 \quad (11)$$

In the matrix form, correspondingly, the least squares solution (11) is given by

$$\hat{X} = R^+TS^+ + X_0 - R^TRX_0SS^T.$$

□

4 Conclusion

In this paper, we applied the variation of parameters technique to derive a closed form solution of the first order non-homogeneous Lyapunov system of differential equations associated with general two-point boundary conditions. The uniqueness of the solution is established through the least squares solution to a double-sided matrix equation that satisfies the nearness constraint. Our future work includes exploring applications of the proposed algorithm in image processing, new preconditioning techniques in solving large sparse systems of linear equations, and developing robust methods in eigenstructure assignment of closed loop systems.

Acknowledgement:

The author thanks the anonymous reviewers for their in-depth comments, constructive suggestions, and corrections to improve this paper.

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Contribution of Individual Author(s) to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflicts of interest to declare that are relevant to the content of this article.

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