# On Divisions and Matrices of Special Sequences

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*Abstract:* This article is about new properties of some special matrices. Matrices of special sequences such as Naryana, Fibonacci and Lucas sequences are studied. New properties are obtained on the Binet formulas of these sequences. The three-row column matrix existing in the literature is analyzed. This column matrix is moved to a new regular matrix order $3^{rd}$. Comparisons are made between square matrices formed by special sequences. The basic relationships between products and divisions are given. Solution methods of linear matrix equations are considered. Some results on new properties with this approach are given. Studies of Binet's formula between special sequences are presented.

*Key–Words:* Sequence, Narayana, Fibonacci, Lucas, Matrices, Binet formulas, Division.

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## 1 Introduction

Pandit made an important contribution to the study of sequences in the 14th century by solving the natural problem. He obtained the number series of this natural problem. The problem describes the natural relationship between a cow and her calf. In short, the story is "A cow gives birth to a calf every year [1]. Pesovic and Pucanović studied generalized Narayana numbers. They studied the conditions under which the circulant matrix and the skew circulant matrix are invertible [2]. Bensella and Behloul generalized Narayana's cow numbers and calculated Fibonacci numbers. The Narayana sequence is characterized by a third-order recurrence relation as follows [3]

$$\mathcal{N}_n = \mathcal{N}_{n-1} + \mathcal{N}_{n-3}, n \geq 3, \tag{1}$$

where $\mathcal{N}_0 = 0, \mathcal{N}_1 = 1$, and $\mathcal{N}_2 = 1$.

In addition to the Naryana sequences, Jacobsthal, -Oresme-Lucas, Leonardo, Gaussian Fibonacci and Mersenne sequences are known [4–8]. Altıparmak, Akkuş and Özkan studied the relationship between Fibonacci and Lucas sequences and calculated new properties [9].

For $n \in \mathbb{N}$, Fibonacci numbers $\mathcal{F}_n$, Lucas numbers $\mathcal{L}_n$, respectively, are

$$\mathcal{F}_{n+2} = \mathcal{F}_{n+1} + \mathcal{F}_n \tag{2}$$

where $\mathcal{F}_0 = 1, \mathcal{F}_1 = 1$.

$$\mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n \tag{3}$$

with $\mathcal{F}_0 = 2, \mathcal{F}_1 = 1$.

The main applications of these sequences are Graph Theory, Biomathematics, Chemistry and Engineering.

## 2 Matrix of Sequences and Notation

Let us start with matrix of Naryana sequence.

**Definition 1** *Matrix of Narayana sequence (MNS) is defined by*

$$\{\mathfrak{N}_n\}_{n=0}^{\infty} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \ldots, \mathfrak{N}_n, \ldots \right\} \tag{4}$$

*with being the initial matrices*

$$\mathfrak{N}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathfrak{N}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathfrak{N}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{5}$$

*and*

$$\mathfrak{N}_n = \begin{bmatrix} \mathcal{N}_{n+1} & \mathcal{N}_{n-1} & \mathcal{N}_n \\ \mathcal{N}_n & \mathcal{N}_{n-2} & \mathcal{N}_{n-1} \\ \mathcal{N}_{n-1} & \mathcal{N}_{n-3} & \mathcal{N}_{n-2} \end{bmatrix}. \tag{6}$$

In this new definition, any path of the matrix traced partially to the right and down, such as a stair step with broken lines, is given below, representing a Fibonacci sequence of matrices.

**Definition 2** *Matrix of Fibonacci sequence(MFS) is defined by*

$$\mathfrak{F}_n = \begin{bmatrix} \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \cdots & \mathcal{F}_n \\ \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_4 & \cdots & \mathcal{F}_{n+1} \\ \mathcal{F}_3 & \mathcal{F}_4 & \mathcal{F}_5 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{F}_n & \mathcal{F}_{n+1} & \mathcal{F}_{n+2} & \cdots & \mathcal{F}_{2n-1} \end{bmatrix} \tag{7}$$

*with* $f_{ij} = \mathcal{F}_{i+j-1}, i, j = 1, \cdots, n$. *That is*

$$\mathfrak{F}_n = [f_{ij}]_n. \tag{8}$$

The MFS and the MLS are investigated using the product and division properties of matrices in the MNS [14]. For any $p \in \mathbb{R}$, MFS is

$$\mathfrak{F}_n(p) = \begin{bmatrix} p & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p-1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_n. \tag{9}$$

If $p = 2, n = 3$, then

$$\mathfrak{F}_3(p) = \begin{bmatrix} p & 1 & 0 \\ p-1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{10}$$

**Definition 3** *( [10]) Matrix of Lucas sequence(MLS) is defined by*

$$\mathfrak{L}_n = \begin{bmatrix} \mathcal{L}_{n+1} & \mathcal{L}_{n-1} & \mathcal{L}_n \\ \mathcal{L}_n & \mathcal{L}_{n-2} & \mathcal{L}_{n-1} \\ \mathcal{L}_{n-1} & \mathcal{L}_{n-3} & \mathcal{L}_{n-2} \end{bmatrix}, n \in \mathbb{Z}^+. \tag{11}$$

If $n = 0$, then

$$\mathfrak{L}_0 = \begin{bmatrix} 0 & 3 & -2 \\ -2 & 2 & 3 \\ 3 & -5 & 2 \end{bmatrix}. \tag{12}$$

Some notations are given below.

- The set of all $n \times n$ diagonal matrices over a field $\mathbb{F}$ is denoted by $\mathbb{D}_n(\mathbb{F}) = \{[a_{ij}] | a_{ij} \in \mathbb{F}, a_{ii} \neq 0, a_{ij} = 0 \text{ for } i \neq j\}$.
- The set of all regular matrices order $n$ over a field $\mathbb{F}$ is denoted by $\mathbb{M}_n(\mathbb{F}) = \{[a_{ij}]_n | a_{ij} \in \mathbb{F}\}$.
- The $(ij)^{th}$ co-factor of matrix A is denoted by $C(A)_{ij}$.

The row and column co-divisors given below are expressed in different notation.

**Definition 4** *([Keleş [11–13]]) Let* $A, B \in \mathbb{M}_n(\mathbb{F})$. *Then,*

*(i) The determinant of the new matrix obtained by writing the* $i^{th}$ *column of the matrix* $B$ *on the* $j^{th}$ *column of the matrix* $A$ *is called* $ij^{th}$ *the column co-divisor of the matrix* $B$ *on the matrix* $A$ *and denoted by* $d^c_{ij}\begin{pmatrix} B \\ A \end{pmatrix}$. *That is,*

$$d^c_{ij}\begin{pmatrix} B \\ A \end{pmatrix} = \sum_{j=1}^{n} a_{ij} C(A)_{ij}, \textit{for some } i = 1, ..., n.$$

*For the two matrices satisfying the above conditions, the matrix division is also given by*

$$\frac{B}{A} := \frac{1}{|A|}\left[\left(d^c_{ij}\begin{pmatrix} B \\ A \end{pmatrix}\right)_{ji}\right]. \tag{13}$$

*and the solution of the equation* $AX = B$ *is* $X = \dfrac{B}{A}$.

*(ii) The determinant of the new matrix obtained by writing the* $i^{th}$ *row of the matrix* $B$ *on the* $j^{th}$ *row of the matrix* $A$ *is called* $ij^{th}$ *the row co-divisor of the matrix* $B$ *on the matrix* $A$ *and denoted by* $d^r_{ij}(BA)$. *That is,*

$$d^r_{ij}(BA) = \sum_{j=1}^{n} a_{ij} C(A)_{ij}, \textit{ for some } i = 1, ..., n.$$

*The solution of the linear matrix equation* $XA = B$ *is*

$$X = \frac{1}{|A|}\left[\left(d^r_{ij}(BA)\right)_{ij}\right] \tag{14}$$

Their number of co-divisor of columns and rows is $n^2$.

**Corollary 5** *If* $A \in \mathbb{M}_n(\mathbb{F})$, *then there exists* $B \in \mathbb{M}_n(\mathbb{F})$ *such that* $A = BA_1$, *for some* $A_1 \in \mathbb{M}_n(\mathbb{F})$.

**Proof:** If $A \in \mathbb{M}_n(\mathbb{F})$, then for any $B \in \mathbb{M}_n(\mathbb{F})$, $B|A$, so $A = BA_1$, for some $A_1 \in \mathbb{M}_n(\mathbb{F})$.

Two different regular matrices of the same order that are different from the unit matrix always have a common factor.

**Lemma 6** *If $A, B \in \mathbb{M}_n(\mathbb{F})$, then there exists $C_1 \in \mathbb{M}_n(\mathbb{F})$ such that $A = C_1A_1, B = C_1B_1$, for some $A_1, B_1 \in \mathbb{M}_n(\mathbb{F})$.*

**Proof:** If $A, B \in \mathbb{M}_n(\mathbb{F})$, then for any $C_1 \in \mathbb{M}_n(\mathbb{F})$

$$C_1|A, A = C_1A_1 \text{ for some } A_1 \in \mathbb{M}_n(\mathbb{F}).$$

and

$$C_1|B, B = C_1B_1 \text{ for some } B_1 \in \mathbb{M}_n(\mathbb{F}).$$

**Corollary 7** *If Let $A \in \mathbb{M}_n(\mathbb{F})$ then*

*$A = C_1C_2...C_k$, for some $, C_1, ..., C_k \in \mathbb{M}_n(\mathbb{F}), k \in \mathbb{Z}^+$.*

**Proof:** The proof of this corollary is clear by Corollary 5.

**Theorem 8 (Keleş [11,13])** *Let $A, B \in \mathbb{M}_n(\mathbb{F})$. Then there exist $A_1, A_2, B_2 \in \mathbb{M}_n(\mathbb{F})$ such that $A = A_1A_2$ and $B = A_1B_2$. Therefore.*

$$\frac{A}{B} = \frac{A_1A_2}{A_1B_2} = \frac{A_2}{B_2}.$$

**Proof:** This proof is clear by Lemma 6.

**Theorem 9 ( [13])** *Let $A, B \in \mathbb{M}_n(\mathbb{F})$ be any two elements and the linear matrix equation is $XA = B$. The following holds.*

$$X = \left(\frac{B^T}{A^T}\right)^T.$$

**Theorem 10** *( [10]) Binet's formula of MNS is $A\alpha^n + B\beta^n + C\gamma^n$ where $\alpha, \beta, \gamma$ are the roots of the equation $x^3 - x^2 - 1 = 0$ and $A, B, C$ are the matrices given by*

$$A = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\begin{bmatrix} (1-\beta)(1-\gamma) & 1 & 1-\beta-\gamma \\ 1-\beta-\gamma & \beta\gamma & 1 \\ 1 & -(\beta+\gamma) & \beta\gamma \end{bmatrix}. \tag{15}$$

$$B = \frac{1}{(\beta-\alpha)(\beta-\gamma)}\begin{bmatrix} (1-\alpha)(1-\gamma) & 1 & 1-\alpha-\gamma \\ 1-\alpha-\gamma & \alpha\gamma & 1 \\ 1 & -(\alpha+\gamma) & \alpha\gamma \end{bmatrix}. \tag{16}$$

$$C = \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\begin{bmatrix} (1-\alpha)(1-\beta) & 1 & 1-\alpha-\beta \\ 1-\alpha-\beta & \alpha\beta & 1 \\ 1 & -(\alpha+\beta) & \alpha\beta \end{bmatrix}. \tag{17}$$

$$A = \frac{(\alpha-1)\mathfrak{N}_1 + \mathfrak{N}_2 + \beta\gamma\mathfrak{N}_0}{(\alpha-\beta)(\alpha-\gamma)}, \tag{18}$$

$$B = \frac{(\beta-1)\mathfrak{N}_1 + \mathfrak{N}_2 + \alpha\gamma\mathfrak{N}_0}{(\beta-\alpha)(\beta-\gamma)}, \tag{19}$$

$$C = \frac{(\gamma-1)\mathfrak{N}_1 + \mathfrak{N}_2 + \alpha\beta\mathfrak{N}_0}{(\gamma-\beta)(\gamma-\alpha)}. \tag{20}$$

# 3 Divisions and Matrices of Special Sequences

In this section we work with matrices at most order $3^{rd}$.

We write

$$\mathfrak{F}_3(p) = \mathfrak{N}_2\mathfrak{F}_3^1(p) = \begin{bmatrix} p & 1 & 0 \\ p-1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \tag{21}$$

where $\mathfrak{F}_3^1(p) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ p-2 & 0 & 1 \end{bmatrix}$,

$$\frac{\mathfrak{F}_3(p)}{\mathfrak{N}_2} = \mathfrak{F}_3^1(p) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ p-2 & 0 & 1 \end{bmatrix}. \tag{22}$$

And

$$\mathfrak{F}_3(p) = \mathfrak{L}_0\mathfrak{F}_3^2(p) = \begin{bmatrix} p & 1 & 0 \\ p-1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \tag{23}$$

where $\mathfrak{F}_3^2(p) = \frac{1}{31}\begin{bmatrix} 13p-3 & 4 & 9 \\ 4p+11 & 6 & -2 \\ 6p+1 & 9 & -3 \end{bmatrix}$.

$$\frac{\mathfrak{F}_3(p)}{\mathfrak{L}_0} = \mathfrak{F}_3^2(p). \tag{24}$$

$$\frac{\mathfrak{L}_0}{\mathfrak{N}_2} = \begin{bmatrix} 0 & 3 & -2 \\ -2 & 2 & 3 \\ 3 & -5 & 2 \end{bmatrix}. \tag{25}$$

**Proposition 11** *If $\mathfrak{F}_3(p)\mathfrak{X}_n(\mathcal{F}, \mathcal{L}) = \mathfrak{L}_n$, then*

$$\mathfrak{X}_n(\mathcal{F}, \mathcal{L}) = \frac{1}{det(\mathfrak{F}_3(p))}\left[d_{ij}^c\left(\frac{\mathfrak{L}_n}{\mathfrak{F}_3(p)}\right)\right]. \tag{26}$$

**Proof:** If $\mathfrak{F}_3(p)\mathfrak{X}_n(\mathcal{F}, \mathcal{L}) = \mathfrak{L}_n$, then by Equation 13, we have

$$\mathfrak{X}_n(\mathcal{F}, \mathcal{L}) = \frac{1}{det(\mathfrak{F}_3(p))}\left[d_{ij}^c\left(\frac{\mathfrak{L}_n}{\mathfrak{F}_3(p)}\right)\right]. \tag{27}$$

**Proposition 12** *If $\mathfrak{N}_n\mathfrak{X}_n(\mathcal{N}, \mathcal{L}) = \mathfrak{L}_n$, then*

$$\mathfrak{X}_n(\mathcal{N}, \mathcal{L}) = \frac{1}{det(\mathfrak{N}_n)}\left[d_{ij}^c\left(\frac{\mathfrak{L}_n}{\mathfrak{F}_3(p)}\right)\right]. \tag{28}$$

**Proof:** If $\mathfrak{N}_n\mathfrak{X}_n(\mathcal{N}, \mathcal{L}) = \mathfrak{L}_n$, then by Equation 13, we write

$$\mathfrak{X}_n(\mathcal{N}, \mathcal{L}) = \frac{1}{det(\mathfrak{N}_n)}\left[d_{ij}^c\left(\frac{\mathfrak{L}_n}{\mathfrak{N}_n}\right)\right]. \tag{29}$$

**Proposition 13** *If $\mathfrak{F}_3(p)\mathfrak{X}_n(\mathcal{F},\mathcal{N}) = \mathfrak{N}_n$, then*

$$\mathfrak{X}_n(\mathcal{F},\mathcal{N}) = \frac{1}{det(\mathfrak{F}_3(p))}\left[d^c_{ij}\begin{pmatrix}\mathfrak{N}_n\\ \mathfrak{F}_3(p)\end{pmatrix}\right]. \quad (30)$$

**Proof:** If $\mathfrak{F}_3(p)\mathfrak{X}_n(\mathcal{F},\mathcal{N}) = \mathfrak{N}_n$, then by Equation 13, we calculate

$$\mathfrak{X}_n(\mathcal{F},\mathcal{N}) = \frac{1}{det(\mathfrak{F}_3(p))}\left[d^c_{ij}\begin{pmatrix}\mathfrak{N}_n\\ \mathfrak{F}_3(p)\end{pmatrix}\right]. \quad (31)$$

**Proposition 14** *If $\mathfrak{X}_n(\mathcal{F},\mathcal{L})\mathfrak{F}_3(p) = \mathfrak{L}_n$, then*

$$\mathfrak{X}_n(\mathcal{F},\mathcal{L}) = \frac{1}{|\mathfrak{F}_3(p)|}\left[\left(d^r_{ij}(\mathfrak{L}_n\mathfrak{F}_3(p))\right)_{ij}\right]. \quad (32)$$

**Proof:** If $\mathfrak{X}_n(\mathcal{F},\mathcal{L})\mathfrak{F}_3(p) = \mathfrak{L}_n$, then by Equation 14, we write

$$\mathfrak{X}_n(\mathcal{F},\mathcal{L}) = \frac{1}{|\mathfrak{F}_3(p)|}\left[\left(d^r_{ij}(\mathfrak{L}_n\mathfrak{F}_3(p))\right)_{ij}\right]. \quad (33)$$

**Proposition 15** *If $\mathfrak{X}_n(\mathcal{F},\mathcal{N})\mathfrak{F}_3(p) = \mathfrak{N}_n$, then*

$$\mathfrak{X}_n(\mathcal{F},\mathcal{N}) = \frac{1}{|\mathfrak{F}_3(p)|}\left[\left(d^r_{ij}(\mathfrak{N}_n\mathfrak{F}_3(p))\right)_{ij}\right]. \quad (34)$$

**Proof:** If $\mathfrak{X}_n(\mathcal{F},\mathcal{N})\mathfrak{F}_3(p) = \mathfrak{N}_n$, then by Equation 14, we have

$$\mathfrak{X}_n(\mathcal{F},\mathcal{N}) = \frac{1}{|\mathfrak{F}_3(p)|}\left[\left(d^r_{ij}(\mathfrak{N}_n\mathfrak{F}_3(p))\right)_{ij}\right]. \quad (35)$$

**Proposition 16** *If $\mathfrak{X}_n(\mathcal{L},\mathcal{N})\mathfrak{L}_n = \mathfrak{N}_n$, then*

$$\mathfrak{X}_n(\mathcal{L},\mathcal{N}) = \frac{1}{|\mathfrak{L}_n|}\left[\left(d^r_{ij}(\mathfrak{N}_n\mathfrak{L}_n)\right)_{ij}\right]. \quad (36)$$

**Proof:** If $\mathfrak{X}_n(\mathcal{L},\mathcal{N})\mathfrak{L}_n = \mathfrak{N}_n$, then by Equation 14, we calculate

$$\mathfrak{X}_n(\mathcal{L},\mathcal{N}) = \frac{1}{|\mathfrak{L}_n|}\left[\left(d^r_{ij}(\mathfrak{N}_n\mathfrak{L}_n)\right)_{ij}\right]. \quad (37)$$

**Theorem 17** *The followings hold.*

*(i)*

$$\left[d^c_{ij}\begin{pmatrix}\mathfrak{L}_n\\ \mathfrak{F}_3(p)\end{pmatrix}\right] = \left[\left(d^r_{ij}(\mathfrak{L}_n\mathfrak{F}_3(p))\right)_{ij}\right]. \quad (38)$$

*(ii)*

$$\left[d^c_{ij}\begin{pmatrix}\mathfrak{N}_n\\ \mathfrak{F}_3(p)\end{pmatrix}\right] = \left[\left(d^r_{ij}(\mathfrak{N}_n\mathfrak{F}_3(p))\right)_{ij}\right]. \quad (39)$$

*(iii)*

$$\left[d^c_{ij}\begin{pmatrix}\mathfrak{L}_n\\ \mathfrak{N}_n\end{pmatrix}\right] = \left[\left(d^r_{ij}(\mathfrak{L}_n\mathfrak{N}_n)\right)_{ij}\right]. \quad (40)$$

**Proof:**

(i) By Proposition 11 and Proposition 11, we calculate

$$\left[d^c_{ij}\begin{pmatrix}\mathfrak{L}_n\\ \mathfrak{F}_3(p)\end{pmatrix}\right] = \left[\left(d^r_{ij}(\mathfrak{L}_n\mathfrak{F}_3(p))\right)_{ij}\right]. \quad (41)$$

Similarly $(ii)$ and $(iii)$ are clear.

**Theorem 18** *The following equations hold.*

*(i)*

$$\frac{1}{det(\mathfrak{F}_3(p))} = \left[\left(d^r_{ij}(\mathfrak{L}_n\mathfrak{F}_3(p))\right)_{ij}\right]^T = \frac{\mathfrak{L}_n^T}{\mathfrak{F}_3(p)^T}. \quad (42)$$

*(ii)*

$$\frac{1}{det(\mathfrak{F}_3(p))} = \left[\left(d^r_{ij}(\mathfrak{N}_n\mathfrak{F}_3(p))\right)_{ij}\right]^T = \frac{\mathfrak{N}_n^T}{\mathfrak{F}_3(p)^T}. \quad (43)$$

*(iii)*

$$\frac{1}{det(\mathfrak{N}_n)} = \left[\left(d^r_{ij}(\mathfrak{L}_n\mathfrak{N}_n)\right)_{ij}\right]^T = \frac{\mathfrak{L}_n^T}{\mathfrak{N}_n^T}. \quad (44)$$

**Proof:**

(i) By Proposition 11 and Theorem 9, we write

$$\left[d^c_{ij}\begin{pmatrix}\mathfrak{L}_n\\ \mathfrak{F}_3(p)\end{pmatrix}\right] = \left[\left(d^r_{ij}(\mathfrak{L}_n\mathfrak{F}_3(p))\right)_{ij}\right]. \quad (45)$$

By Proposition 12, Proposition 13 and Theorem 9, similarly $(ii)$ and $(iii)$ are clear.

**Theorem 19** *Let $A, B$ and $C$ be matrices given Theorem 10. The following equations hold.*

*(i)*

$$\frac{A}{\mathfrak{F}_3(p))} = \frac{1}{det(\mathfrak{F}_3(p))}\left[d^c_{ij}\begin{pmatrix}A\\ \mathfrak{F}_3(p)\end{pmatrix}\right]. \quad (46)$$

*(ii)*

$$\frac{B}{\mathfrak{F}_3(p))} = \frac{1}{det(\mathfrak{F}_3(p))}\left[d^c_{ij}\begin{pmatrix}B\\ \mathfrak{F}_3(p)\end{pmatrix}\right]. \quad (47)$$

*(iii)*

$$\frac{C}{\mathfrak{F}_3(p))} = \frac{1}{det(\mathfrak{F}_3(p))}\left[d^c_{ij}\left(\frac{C}{\mathfrak{F}_3(p)}\right)\right]. \quad (48)$$

*Similar properties to the above between MNS $(\mathfrak{N}_n)$ and MLS $(\mathfrak{L}_n)$ are written down with Binet's formulas.*

**Proof:** *By Theorem 10 and Definition 4, They are clear.*

*(i)*

$$\frac{A}{\mathfrak{F}_3(p))} = \frac{1}{det(\mathfrak{F}_3(p))}\left[d^c_{ij}\left(\frac{A}{\mathfrak{F}_3(p)}\right)\right]. \quad (49)$$

*(ii)*

$$\frac{B}{\mathfrak{F}_3(p))} = \frac{1}{det(\mathfrak{F}_3(p))}\left[d^c_{ij}\left(\frac{B}{\mathfrak{F}_3(p)}\right)\right]. \quad (50)$$

*(iii)*

$$\frac{C}{\mathfrak{F}_3(p))} = \frac{1}{det(\mathfrak{F}_3(p))}\left[d^c_{ij}\left(\frac{C}{\mathfrak{F}_3(p)}\right)\right]. \quad (51)$$

# 4 Conclusion

The study revealed that there are many relationships between MNS, MFS and MLS Coroollary 5,7. These sequences are written in terms of each other. At least one of the factors of MNS is MFS and at least one is MLS by Theorem 9. The same property applies to each of them.

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