Force and displacement mixed approach for interfaces connecting rigid bodies

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Abstract: - In this contribution, a simple mixed approach that considers as independent variables both displacements and interface stresses between rigid blocks is described. The work aims to propose an alternative computational tool for studying the static equilibrium of rigid blocks connected by elastic interfaces and subjected to in-plane actions, representing simple assemblies such as brittle or granular materials, and with particular reference to mortar and/or dry-jointed masonry. Assuming a piecewise constant distribution of normal and shear stresses along the generic interface, the numerical model converges to the typical stiffness matrix of the interface. The proposed tool is here applied to in-plane linear static analysis of rigid bodies connected by elastic interfaces, but will allow a further improvement for performing analyses in case of out-of-plane actions and also accounting for material nonlinearity.

Key-Words: - rigid blocks, elastic interfaces, mixed approach, masonry, dry or mortar joints, interface stiffness

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1 Introduction

Rigid body models for studying brittle material behavior are frequently adopted in civil engineering fields of research, namely for representing soil, rocks or granular materials [1,2,3], concrete potentially subjected to cracking [4], masonry structures [5], and general engineering structures [6].

The discrete or distinct element method (DEM) is the most important approach proposed in the past for studying this type of problem [1,2,5]. However, DEM often adopts large displacement hypothesis, requiring the solution of equations in dynamic field, and allows the possibility to consider the evolution of contacts between the elements, requiring the use of contact detection algorithms. Simpler models, characterized by small displacement hypothesis and, subsequently, not varying the topology of the contacts, turn out to be more effective for studying static equilibrium problems typical of masonry structures [7,8,9,10].

In the last decade, the author and his co-workers proposed a discrete or rigid block model for studying the in-plane behavior of regular masonry by means of a simple and effective static algorithm approach [11]. This model allowed to determine displacements and interface actions of different masonry specimens modelled as rigid body assemblies by introducing the stiffness matrix of the assemblage, considering interface stiffness and a compatibility matrix accounting for block relative positions. Such a matrix is frequently adopted in limit analysis approaches for masonry structures [12,13].

In this work, the discrete or rigid block model proposed by author and co-workers is further investigated by introducing a mixed formulation able to determine at the same time block displacements and interface actions, with the possibility to consider a piecewise constant discretization of interface normal and shear stresses. A mixed approach for studying masonry was already proposed in the past, without highlighting potential interface discretization, but considering joint elements [14]. The proposed approach follows the procedure already adopted in case of the more complex problem of a rigid indenter on elastic half-space [15]. Here, a linear relationship between interface stresses and strain, together with elastic behavior of interfaces and in-plane actions are considered.

After the formulation of the problem in section 2, several simple numerical tests are performed in section 3, by considering two staked rigid blocks connected by an interface. On one hand the tests show the convergence of the approach to the existing static algorithm, on the other hand the tests show model effectiveness in determining block displacements and interface stress distribution, with particular attention on normal stresses. Further developments of the work, highlighted in section 4 dedicated to conclusions, will focus on nonlinear interface behavior and will take into consideration more complex three-dimensional case studies.

2 Problem formulation

This contribution focuses on the elastic behavior of an interface between two rigid blocks or bodies (B_i, B_j) subjected to in-plane actions (Fig.1). Blocks are assumed to have a polygonal shape, characterized by flat edges. For simplicity, in this contribution, blocks are assumed to be quadrilateral. Plane stress or plane strain hypotheses are adopted, hence the same block and interface depth *s* or a unitary depth is considered. Small displacement hypothesis is also assumed. The flat interface has length *l* and, in case of not dry joints, a thickness parameter *e* is introduced, which is generally negligible with respect to block dimensions, and it is going to be considered as a parameter for determining interface stiffness, for instance in case of masonry mortar joints.

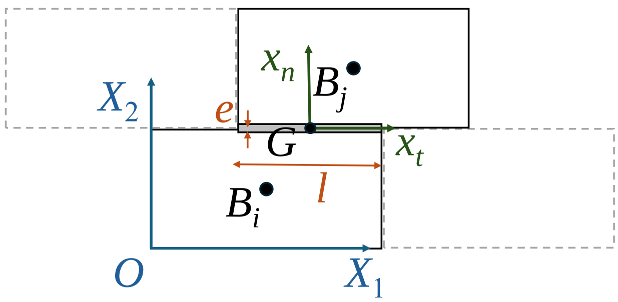

Fig.1 Interface between two rigid blocks, coordinate systems and geometric parameters

A global two-dimensional coordinate system OX_1X_2 is assumed. In this case, for simplicity, O is coincident with block B_i bottom-left corner, X_1 represents horizontal direction and coincides with block B_i base, whereas X_2 represents vertical direction, typical of gravitational loads, and it coincides with block B_i height. A local two-dimensional coordinate system Gx_tx_n is assumed at interface level, with G coincident with interface midpoint, x_t aligned with interface mid-plane and x_n orthogonal to interface mid-plane.

2.1 Basic relationships

The displacements of the model are given by rigid block in-plane translations u_1 u_2 and block rotation ω_3 with respect to its centre (Fig.2).

In case of two blocks, the vector collecting model degrees of freedom is:

$$\mathbf{q} = [u_1^i \quad u_2^i \quad \omega_3^i \quad u_1^j \quad u_2^j \quad \omega_3^j]^T \qquad (1)$$

Interface deformations along its length are given by relative displacements between the adjacent edges of the blocks connected by the interface, namely relative shear and normal displacements $d_t(x_t)$ and $d_n(x_t)$, which can be defined as function of relative tangential and normal displacements at interface centre:

$$\begin{cases} d_t(x_t) = d_{G,t} \\ d_n(x_t) = d_{G,n} + \delta_{G,3} x_t \end{cases} \qquad (2)$$

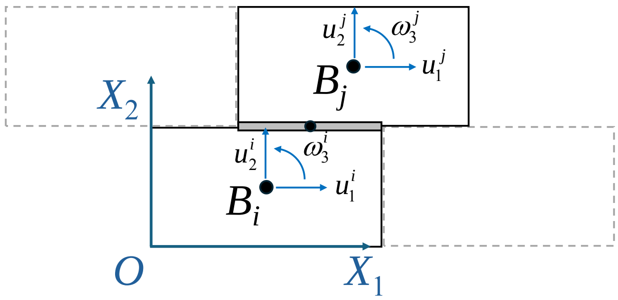

Fig.2 In-plane degrees of freedom of two blocks connected by an interface

In particular, relative displacements evaluated at interface centre G can be written in vector form as follows:

$$\mathbf{d}_G = [d_{G,t} \quad d_{G,n} \quad \delta_{G,3}]^T, \qquad (3)$$

and can be defined as function of block displacements in global coordinates (1) by means of a rotation matrix **R**, which considers interface orientation θ with respect to the global coordinate system (equal to zero in this case, Fig.1), and a compatibility matrix **B**, which considers relative distances between blocks and interface centres (Fig.3).

$$\mathbf{B} = [-\mathbf{B}_i \quad \mathbf{B}_j]$$

$$\mathbf{B}_i = \begin{bmatrix} 1 & 0 & -dx_n^i \\ 0 & 1 & dx_t^i \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{B}_j = \begin{bmatrix} 1 & 0 & dx_n^j \\ 0 & 1 & -dx_t^j \\ 0 & 0 & 1 \end{bmatrix} \qquad (4)$$

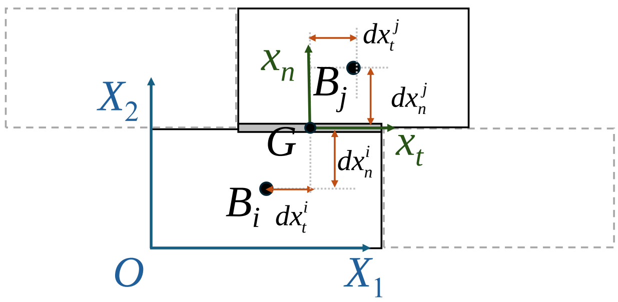

Fig. 3 Relative distances between blocks and interface centres

Then, interface relative displacements at its centre are related to block global displacements as follows:

$$\mathbf{d}_G = \mathbf{R}\mathbf{B}\mathbf{q} \tag{5}$$

2.1.1 Mortar joints

In case of not-dry joints between the blocks, interface shear and normal deformations are defined from interface relative displacements (2) by accounting for its thickness *e*:

$$\begin{cases} \varepsilon_t(x_t) = d_t(x_t)/e = d_{G,t}/e \\ \varepsilon_n(x_t) = d_n(x_t)/e = (d_{G,n} + \delta_{G,3}x_t)/e \end{cases} \tag{6}$$

Assuming an elastic interface behavior characterized by normal and shear elastic moduli, depending on the specific material of the interface (namely, mortar modelled as an isotropic material, having elastic modulus *E* and Poisson's ratio *ν*), normal and shear stresses along interface follow an elastic constitutive relationship, being linearly dependent on the corresponding strains:

$$\begin{cases} \sigma_t(x_t) = E/[2(1+\nu)]\,\varepsilon_t(x_1) \\ \sigma_n(x_t) = E\,\varepsilon_n(x_1) \end{cases} \tag{7}$$

Integrating interface stresses over its area, resultants in terms of shear forces, normal forces, and bending moment can be obtained:

$$\begin{cases} T = t\int_l \sigma_t(x_t)\,dx_t \\ N = t\int_l \sigma_n(x_t)\,dx_t \\ M = t\int_l \sigma_n(x_t)x_t\,dx_t \end{cases} \tag{8}$$

It is worth mentioning that these interface actions are defined in the local interface coordinate system, and can be collected in the following vector:

$$\mathbf{f} = [T \quad N \quad M]^T, \tag{9}$$

Accounting for the rotation matrix already introduced in equation (5), interface actions should be in equilibrium with external actions applied to the model, by considering the rotation matrix R already introduced in (5).

2.1.2 Dry joints

In case of dry joints, normal and shear strains along the interface length cannot be defined. However, normal and shear stresses can be directly related to interface relative displacements by means of stiffness parameters able to describe surface roughness and local deformations in case of dry contact

$$\begin{cases} \sigma_t(x_t) = k_t\,d_t(x_t) \\ \sigma_n(x_t) = k_n\,d_n(x_t) \end{cases} \tag{10}$$

Then, equation (8) remains valid also in case of dry contact.

Starting from the constitutive relationships in (7) and (10), the determination of interface actions (9) can be written in discrete form by highlighting the stiffness parameters of the interface:

$$\mathbf{f} = \mathbf{K}_G\,\mathbf{d}_G = \begin{bmatrix} K_t & & \\ & K_n & \\ & & K_m \end{bmatrix} \begin{bmatrix} d_{G,t} \\ d_{G,n} \\ \delta_{G,3} \end{bmatrix} \tag{11}$$

where

$$\begin{cases} K_t = k_t A = k_t(l\,s) \\ K_n = k_n A = k_n(l\,s) \\ K_m = k_n I = k_t(l^3\,s/12) \end{cases} \tag{12}$$

where *A* and *I* are interface area and second moment of area, respectively, and, in case of mortar joints:

$$\begin{cases} k_t = E/[2(1+\nu)]/e \\ k_n = E/e \end{cases} \tag{13}$$

2.2 Numerical model

In order to numerically study the behavior of a system of rigid blocks connected by interfaces, interface area is subdivided into equally-spaced *m* portions along its length (Fig.4).

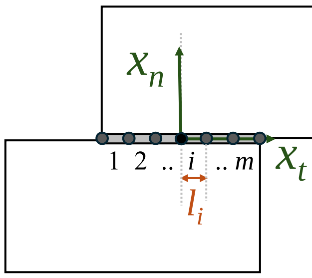

Fig.4 Interface subdivided into *m* equally-spaced portions

Then, a piecewise constant shear and normal stress distribution is considered, assuming the following base function

$$\rho(x_1) = \begin{cases} 1 & \text{on the } i\text{th element} \\ 0 & \text{elsewhere along } l \end{cases} \tag{14}$$

allowing to approximate both shear and normal interface stresses (7,10) of each interface portion as follows:

$$\begin{cases} \sigma_t^i(x_t) = \rho_i(x_t)r_{t,i} \\ \sigma_n^i(x_t) = \rho_i(x_t)r_{n,i} \end{cases} \tag{15}$$

where $r_{t,i}$ and $r_{n,i}$ are, respectively, shear and normal uniform stresses along the *i*-th interface element, which can be collected in the vector

$$\mathbf{r} = [\mathbf{r}_t \quad \mathbf{r}_n]^T \tag{16}$$

In the same manner, interface relative displacements (2) should be approximated with the base function (14)

$$\begin{cases} dx_t^i(x_t) = \rho_i(x_t) d_{t,i} \\ dx_n^i(x_n) = \rho_i(x_t) d_{n,i} \end{cases} \quad (17)$$

and the vector of discretized interface relative displacements is:

$$\mathbf{d} = [\mathbf{d}_t \quad \mathbf{d}_n]^T \quad (18)$$

Thanks to this discretization, equation (8) can be written in discrete form as follows:

$$\begin{bmatrix} T \\ N \\ M \end{bmatrix} = \mathbf{f} = \mathbf{Hr} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \\ \mathbf{0} & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{r}_t \\ \mathbf{r}_n \end{bmatrix} \quad (19)$$

where **A** is a matrix collecting interface portion areas and **S** is the matrix of first moments of area

$$\begin{cases} A_i = A / m = t\, l / m = t(x_t^{i+1} - x_t^i) \\ S_i = A_i (x_t^{i+1} + x_t^i) / 2 \end{cases} \quad (20)$$

whereas equation (2) in discrete form becomes:

$$\mathbf{d} = \mathbf{H}_0^T \mathbf{d}_G = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{X} \end{bmatrix} \begin{bmatrix} d_{G,t} \\ d_{G,n} \\ \delta_{G,3} \end{bmatrix} \quad (21)$$

where matrix **X** collects coordinates of interface portions mid points, and it can be demonstrated that

$$\mathbf{H} = A_i \mathbf{H}_0 \quad (22)$$

Equation (7), accounting for (13) and (10), can be written as

$$\begin{bmatrix} \mathbf{r}_t \\ \mathbf{r}_n \end{bmatrix} = \mathbf{r} = \mathbf{K}\,\mathbf{d} = \begin{bmatrix} k_t \mathbf{I} & \\ & k_n \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{d}_t \\ \mathbf{d}_n \end{bmatrix} \quad (23)$$

where the stiffness matrix **K** of the discretized interface is highlighted. Considering expressions (19), (21), and (23), the following system of equations is obtained

$$\begin{bmatrix} \mathbf{0} & \mathbf{H} \\ \mathbf{H}_0^T & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{d}_G \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \quad (24)$$

where $\mathbf{G} = \mathbf{K}^{-1}$ represents the compliance or flexibility matrix of the discretized interface. The system of equations above turns out to be in agreement with the system of equations obtained in [15] for a rigid indenter on elastic half-space. In this case, however, matrix **G** turns out to be diagonal.

Expression (24) represents a mixed approach that considers together interface stresses and relative displacements. Its numerical solution allows to determine stresses and relative displacements with assigned interface resultants.

The first relation of system (24)

$$\mathbf{Hr} = \mathbf{f} \quad (25)$$

represents the equilibrium of interface stresses and its resultants, whereas the solution of second relation of (24)

$$\mathbf{r} = \mathbf{G}^{-1} \mathbf{H}_0^T \mathbf{d}_G, \quad (26)$$

if substituted in (25), returns equation (11), allowing to highlight the stiffness matrix of the interface as function of its discretization

$$\mathbf{K}_G = \mathbf{H}\mathbf{G}^{-1}\mathbf{H}_0^T. \quad (27)$$

Furthermore, relations in (24) can be re-written in terms of block global displacements accounting for (5) as follows

$$\begin{bmatrix} \mathbf{0} & \mathbf{BRH} \\ \mathbf{BRH}_0^T & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} \quad (28)$$

Similarly to (24), expression (28) represents a mixed approach that considers together interface stresses and the global displacements of the rigid block model. Its numerical solution allows to determine together stresses and global displacements with assigned external actions **F**.

3 Numerical tests

Several numerical tests are proposed in order to evaluate the simplicity and effectiveness of the model. On one hand, a convergence test is performed in order to evaluate the influence of interface discretization in the determination of the stiffness parameters, hence by verifying relation (27). Then, further static analysis tests are performed by determining displacements and interface stresses caused by external actions.

3.1 Convergence test for the determination of interface stiffness

It is worth mentioning that the results of proposed convergence test do not depend on the specific interface geometric and mechanical parameters. However, an interface having l = 0.2 m, s = 0.1 m, and e = 0.01 m is considered. Furthermore, mortar joint case is considered, assuming E = 1 GPa and ν = 0.2.

Numerical tests are performed by determining the values of $\mathbf{K}_G$ coefficients by applying equation (27) and comparing them with respect to equation (12). Fig.5 shows relative differences for each stiffness parameter. On one hand, in case of normal and shear stiffness, results determined with the mixed approach are equal to reference solutions. On the other hand, a fast convergence to the reference solution is obtained for the bending stiffness parameter, with difference smaller than 1% with $m \geq 16$.

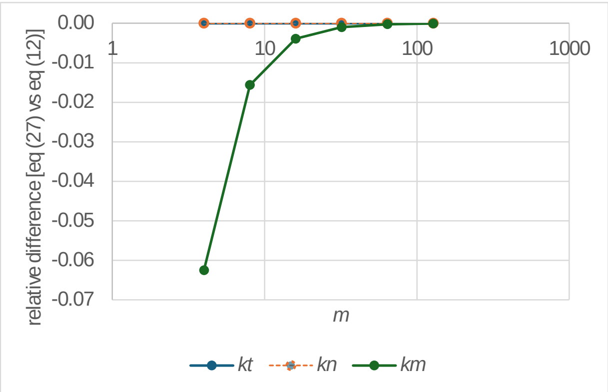

Fig.5 Convergence test for the determination of interface stiffness parameters with the proposed mixed approach

3.1 Static analysis of rigid blocks connected by elastic interfaces

Keeping the mechanical parameters of the previous sub-section, together with model depth and interface thickness, a numerical test is here proposed by considering a set of two stacked rigid blocks (Fig.6a) with varying interface length l, as depicted in Tab.1. The first block is fixed, whereas the second one is subjected to varying vertical (F_2) (gravitational) and fixed horizontal (F_1) loads. It is worth mentioning that such a model, since the first block is fixed, can represent the behavior of a monolithic block, namely the second one, resting on an elastic support (Fig.6b). For this reason, three different values of the height a of the blocks are also considered.

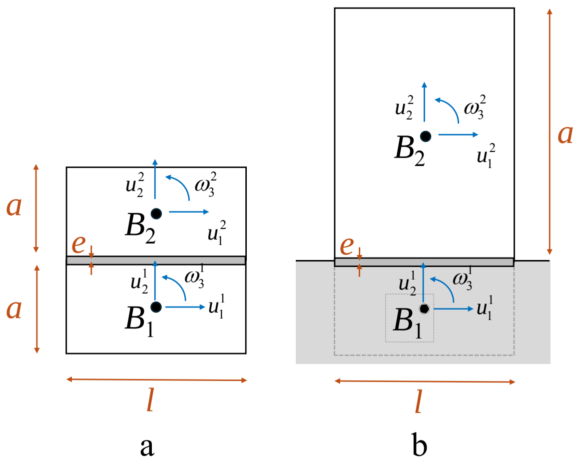

Fig.6 (a) two stacked rigid blocks; (b) one rigid block on elastic support

Considering the convergence test proposed in the previous section, m = 16 subdivisions are assumed along interface. The following table summarizes the varying parameters adopted (geometric and loads).

Case	l [m]	a [m]	F_1 [kN]	F_2 [kN]
1	0.2	0.1	1000	0
2	0.2	0.1	1000	-1000
3	0.2	0.2	1000	-1000
4	0.2	0.4	1000	-1000
5	0.4	0.1	1000	0
6	0.4	0.1	1000	-1000
7	0.4	0.2	1000	-2000
8	0.4	0.4	1000	-4000

Tab.1 Case studies considered with varying geometric parameter and applied loads

Results in terms of translations and rotations of the second block with respect to the first one are collected in Tab.2, whereas normal stresses along the interface between the blocks are proposed in Fig.7 for the 8 different case studies taken into consideration. Shear stresses are not presented, since they turn out to be uniform along interface length and equal to $F_1/(ls)$. Due to rigid block hypothesis, the applied horizontal force generates a bending moment over the interface, depending on the distance between interface and block 2 center.

It is worth noting that cases 1 and 5, characterized by nil vertical actions, show asymmetric normal stresses varying linearly along interface length and typical of a section subjected to pure bending. The other case studies, characterized by bending moment and compressive force, show linearly varying normal stresses, with small traction values on the left side and large compression values on the right side. Case 6 is characterized by the interface fully subjected to increasing compressive stress, due to the small block height, and subsequent small bending moment applied to the interface.

Case	u_1^2 [m]	u_2^2 [m]	ω_3^2 [rad]
1	1.66E-06	0	-8.28E-06
2	1.66E-06	-5.00E-07	-8.28E-06
3	2.86E-06	-5.00E-07	-1.58E-05
4	7.53E-06	-5.00E-07	-3.09E-05
5	6.57E-07	0	-1.04E-06
6	6.57E-07	-2.50E-07	-1.04E-06
7	8.08E-07	-2.50E-07	-1.98E-06
8	1.39E-06	-2.50E-07	-3.86E-06

Tab.2 Horizontal translation, vertical translation, and rotation of the second block

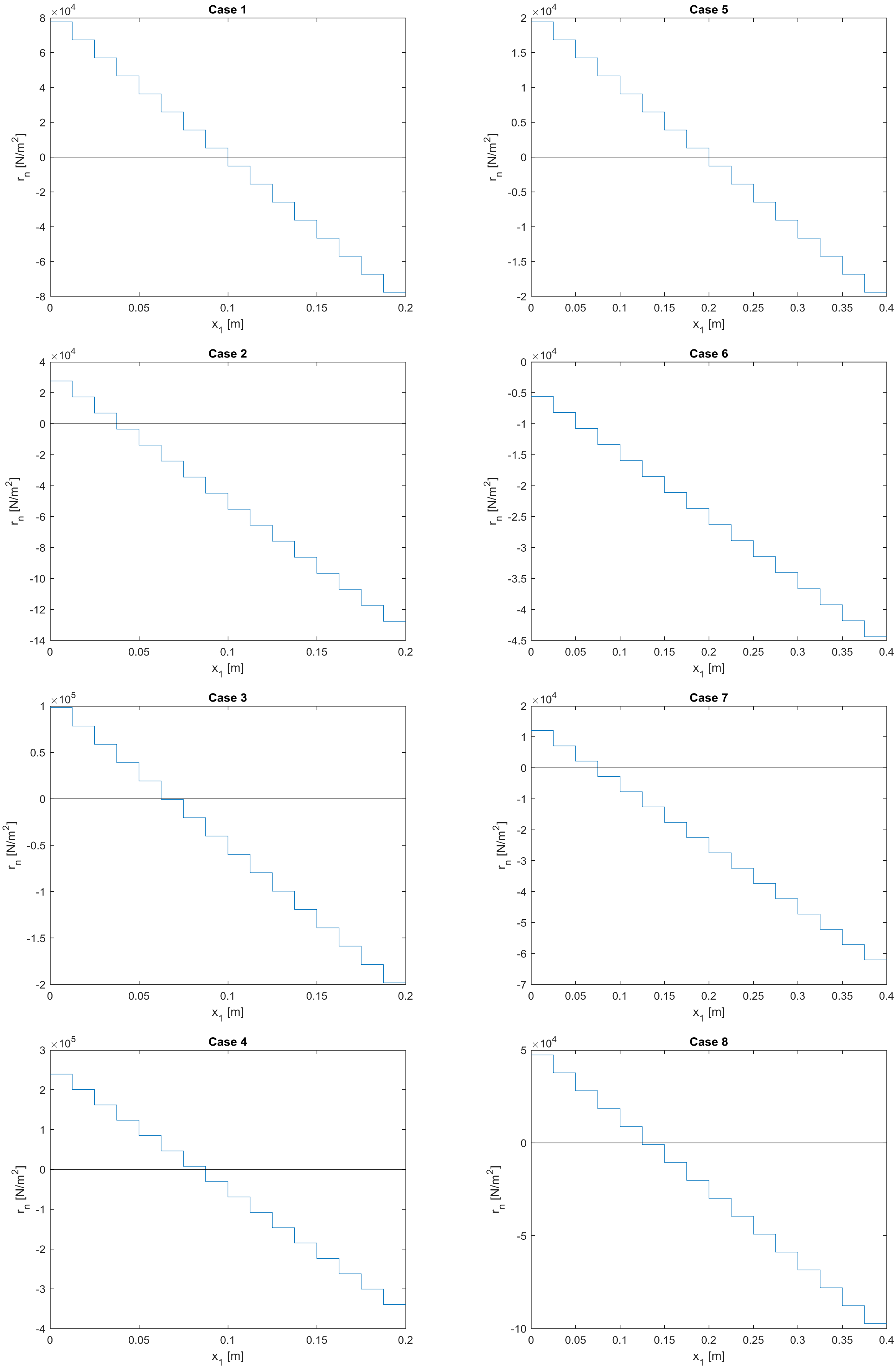

Fig.8 Normal stress piecewise distributions along interface length for the 8 case studies considered (Tab.1)

4 Conclusion

This work has proposed a simple mixed approach for studying rigid blocks connected by elastic interfaces subjected to in-plane actions. The model is able to consider as independent variables both block displacements and interface stresses between blocks. The proposed approach has turned out to be an alternative computational tool for studying the static equilibrium of rigid blocks, with a fast convergence to the results obtained with a standard approach.

The numerical tests performed have also shown the effectiveness of the model, in particular by allowing to determine the discretized distribution of normal stresses along interfaces. This aspect will be useful for the further extension of the approach accounting for the nonlinear material behavior of the interfaces. For instance, it will be possible to account for tensile and compressive strength of mortar and, in particular, to consider the shear strength following an appropriate strength criterion, namely Mohr-Coulomb or Drucker-Prager ones.

Further developments of the approach will also consider the three-dimensional case, by increasing model complexity to six degrees of freedom for each block, namely three translations and three rotations, and by considering two components of shear stresses along the interface together with the normal stress. Consequently, the determination of interface stresses and the subsequent application of a strength criterion, will allow to better determine the six resultants of interface stresses [16].

The proposed approach will also be tested on more complex case studies, namely masonry walls, concrete elements, soil portions, rock masses.

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