Estimation of the Pareto Distribution of the Second Kind Parameters Based on Picard's Method

MOHAMED MASWADAH
Department of Mathematics, Faculty of Science
Aswan University
Aswan, EGYPT

Abstract: - Bayes' method is the most applicable for estimating the distribution parameters in the literature. Thus, the main objective of this work is to introduce an optimal numerical iteration technique, such as Picard's method. This method has been used to estimate the Pareto distribution of the second kind of parameters and compare it with Bayes' method via a Monte Carlo simulation study. The simulation results indicated that Picard's method is more efficient than the Bayes method based on the generalized progressive hybrid censored scheme. Numerical examples are given for illustration and comparison of the proposed methods.

Key-Words: - Bayesian inference; Characteristic prior; Informative prior; Kernel prior; Picard' method.

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1 Introduction

The Pareto distribution of the second kind, called the Pareto-II distribution, was introduced as a model for income. In recent years, several authors have studied several different forms of this distribution, including [10] and [16], among others. As researchers continue to explore its parameters and characteristics, the distribution remains a vital tool for understanding wealth distribution and inequality. The Pareto-II distribution, also known as the Lomax or Pearson's Type VI distribution, see [19, 20]. [5] provided the Pareto-II as a good model in biomedical problems, such as survival time following a heart transplant. [12] studied annual wage data of production line workers in a large industrial firm. Several authors have studied the Pareto distribution based on complete as well as censored samples, including [2], [3, 4], [11], [14, 15], [17], [18], [22], [23], [33], [34], and [35]. The probability density function (pdf) and the cumulative distribution function (cdf) of the Pareto-II distribution are given, respectively, as follows:

$$f(x|\alpha,\beta)=\frac{\alpha}{\beta}(1+\frac{x}{\beta})^{-(\alpha+1)}, \quad x,\alpha,\beta>0 \quad (1)$$

$$F(x)=1-(1+\frac{x}{\beta})^{-\alpha}, \qquad x,\alpha,\beta>0. \quad (2)$$

$\alpha,\beta>0$ are shape and scale parameters, respectively.

In economics, special attention is given to the determination of the parameter, which measures the degree of inequality of income. The hazard or instantaneous failure rate of the distribution, which is defined as follows:

$$h(t)=\frac{\alpha}{\beta}(1+\frac{x}{\beta})^{-1},\ t>0,\ \ \alpha,\beta>0, \quad (3)$$

is a decreasing function of time making it a suitable model for correlated the age with time.

In reliability analysis, [21] proposed a censoring scheme called the Type-II progressive hybrid censoring scheme, which is a mixture of Type-II progressive and hybrid censoring schemes. However, the drawback of the progressive hybrid censoring scheme is that very few failures may occur before the time point **T**. To provide assurance of the number of observed failures as well as the time to complete the test, [8, 9] proposed the generalized progressive hybrid censoring scheme (GPHCS). This scheme modifies the progressive hybrid censoring scheme by allowing the experiment to continue beyond time **T** if the number of failures is less than **m**, which allows the experimenter to observe at least **k** failures. This scheme can be described as follows:

Consider n identical items are placed on a test with considering $\mathrm{R_1,R_2,\dots,R_m}$ are the random removal units, which are fixed at the beginning of the experiment with $\mathrm{m<n}$ such that $n=\mathrm{m}+\sum_{\mathrm{i=1}}^{\mathrm{m}}\mathrm{R_i}$. The terminated time T is also fixed beforehand with the integers k and m are pre-fixed such that $\mathrm{k<m}$. In general, at the time of the $\mathrm{i^{th}}$ failure, $\mathrm{R_i}$ units will

be removed randomly from the remaining surviving units $S_i = n - i - \sum_{j=1}^{i-1} R_j$, where $i \in [1, m]$. Thus, we have three scenarios:

i. If the time of the m^{th} failure occurs before the time point T, then the experiment will stop at the time point $X_{m:m:n}$ and all the remaining surviving units $R_m = n - m - \sum_{j=1}^{m-1} R_j$ will be removed.

ii. If the m^{th}failure does not occur before the time point T and only k failures occur after the time point T, where $X_{m:m:n} > T$. Then, at the time point $X_{k:mn}$ the experiment terminates, and all the remaining surviving units $R_k = n - k - \sum_{j=1}^{k-1} R_j$ will be removed.

iii. If the m^{th} failure does not occur before the time point T and only D failures occur at the time point T, where $X_{k:m:n} < T < X_{m:m:n}$. Then, at the time point $X_{D:m:n}$ the experiment terminates, and all the remaining surviving units $R_T^* = n - D - \sum_{j=1}^{D} R_j$ will be removed.

Thus, given a generalized progressive hybrid censored sample, the likelihood function for the three different cases can be written in a unified form as follows:

$$L(\underline{X}; \theta) = C \prod_{i=1}^{n} f(x_i) [1 - F(x_i)]^{R_i} [1 - F(T)]^{\delta R_T^*}, \tag{3}$$

where $C = \prod_{i=1}^{n} \sum_{j=i}^{m} (R_j + 1)$,

$$n = \begin{cases} m, & \delta = 0, \quad \text{if } X_{k:m:n} \leq X_{m:m:n} < T \\ k, & \delta = 0, \quad \text{if } T < X_{K:mn} \leq X_{m:m:n}, \\ D, & \delta = 1, \quad \text{if } X_{k:mn} < T < X_{m:mn} \end{cases}$$

where $\underline{X} = (X_1, X_2, \dots, X_n)$ and R_T^* is the number of surviving units that are removed at the stopping time $T = \max\{X_{k:m:n}, \min\{X_{m:m:n}, T\}\}$.

The GPHCS has been applied for some distributions, such as the Weibull distribution, see [9], the inverse Weibull distribution, see [25] and [32]. The exponential distribution, see [8], and the Rayleigh distribution, see [7]. Several lifetime distributions have been studied based on the GPHCS, see [25 - 31].

2 Estimation Methods

2.1 Picard's Method

Theoretically, it is known that the traditional log-likelihood function, $H(\theta; x)$, depends on the unknown parameter $\theta = (\alpha, \beta)$ and the sample data X, which can be used to derive the maximum likelihood estimator (MLE) $\hat{\theta}(x)$ of θ, by solving the stationary equation $\frac{\partial H(\theta;x)}{\partial \theta}|_{\hat{\theta}(x)} = 0$. Thus, based on the dependence of the MLE on the sample data, we can apply the implicit function theorem to the stationary equation by taking the total derivative with respect to any $x \in X$, with considering all partial derivatives as well as the total derivatives are assumed to be evaluated at some known value of $\hat{\theta}(x_0) = \theta_0$, we obtain the following equation:

$$\frac{d}{dx}\left(\frac{\partial H(\theta|X)}{\partial \theta}\right) = \frac{\partial^2 H(\theta|X)}{\partial \theta\, \partial x}\Big|_{\theta=\hat{\theta}} + \frac{\partial^2 H(\theta|X)}{\partial \theta^2}\Big|_{\theta=\hat{\theta}} \frac{d\hat{\theta}}{dx} = 0. \tag{4}$$

Solving (4) we obtain the first derivative of $\hat{\theta}$ with respect to any $x \in X$ at $\theta = \hat{\theta}$ as follows:

$$\frac{d\hat{\theta}(x)}{dx} = -\left(\frac{\partial^2 H(\theta|X)}{\partial \theta^2}\Big|_{\theta=\hat{\theta}}\right)^{-1} \frac{\partial^2 H(\theta|X)}{\partial \theta\, \partial x}\Big|_{\theta=\hat{\theta}}. \tag{5}$$

Thus, we can write (5) as the following form:

$$\frac{d\hat{\theta}(x)}{dx} = f(x, \hat{\theta}), \text{ at } \hat{\theta}(x_0) = \theta_0, \tag{6}$$

where

$$f(x, \hat{\theta}) = -\left(\frac{\partial^2 H(\theta|X)}{\partial \theta^2}\Big|_{\theta=\hat{\theta}}\right)^{-1} \frac{\partial^2 H(\theta|X)}{\partial \theta\, \partial x}\Big|_{\theta=\hat{\theta}}.$$

The equation (6) is a first-order ordinary differential equation in $\hat{\theta}$. A numerical method, such as Picard's method, can be used for finding the approximate solution given a trial set of parameter values and initial conditions. If the initial conditions are unavailable, they must be appended to the parameter $\hat{\theta}$ as quantity, which the fit is optimized.

The general iteration rule for Picard's method for the parameter α (say) for any $x \in \underline{X}$, can be derived by integrating (6) with respect to x from x_0 to x^* as follows:

$$\hat{\alpha}(x^*) = \hat{\alpha}(x_0) + \int_{x_0}^{x^*} f(x, \hat{\alpha}_0, \hat{\beta}) dx$$
$$= \alpha_0 + \int_{\alpha_0}^{\alpha^*} f(x, \hat{\alpha}_0, \hat{\beta}) \frac{dx}{d\hat{\beta}} d\hat{\beta}$$

Using the differential equation for the parameter β, which is $\frac{d\hat{\beta}}{dx} = g(x, \hat{\alpha}_0, \hat{\beta})$, we get the recurrence relation for the parameter α as follows:

$$\alpha_{i+1}^{*} = \alpha_0 + \int_{\alpha_0}^{\alpha^*} \left[\frac{f(x,\hat{\alpha}_i,\hat{\beta})}{g(x,\hat{\alpha}_i\hat{\beta})}\right] d\hat{\beta} \ , \qquad (7)$$

for $i = 0,1,2,3, ..$

where α_0 is the initial point and α^* is the value for which the desired solution should be optimized.

Similarly, the Picard estimator for the parameter β can be derived from the following recurrence relation:

$$\beta_{i+1}^{*}(x) = \beta_0 + \int_{\beta_0}^{\beta^*} [\frac{f(x,\hat{\alpha},\hat{\beta}_i)}{f(x,\hat{\alpha},\hat{\beta}_i)}] d\hat{\alpha}, \qquad (8)$$

for $i = 0,1,2,3, ..$

The iterative processes in (7) and (8) are continued until two consecutive numerical solutions are almost the same, that is if $|\theta_{i+1}^{*} - \theta_i^{*}| < 10^{-5}$, for $i = 0,1,2,3, ..$.

It is important to notice that the Picard's method has been used for estimating the three-parameter Burr-XII, see [31].

2.2 Bayes' Method

In this section, the Bayes estimators for the parameters α and β will be derived using the informative gamma prior, the non-parametric kernel prior, and non-parametric characteristic prior.

2.2.1 The Informative Gamma Prior

We consider the unknown parameters α and β have independent gamma prior distributions with the joint probability density function, which is given by:

$$h(\alpha, \beta) \propto \alpha^{a-1}\beta^{c-1}e^{-b\alpha-d\beta}, \qquad (9)$$

where the hyperparameters a, b, c and d are assumed to be known non-negative real numbers and chosen to reflect the prior belief about the unknown parameters.

2.2.2 The Kernel Prior

For deriving the kernel prior, we introduce the bivariate kernel density estimator for the unknown probability density function $g(\alpha, \beta)$ with support on $(0, \infty)$, which is defined as

$$\hat{g}(\alpha, \beta) = \frac{1}{nh_1h_2}\sum_{i=1}^{n} K\left(\frac{\alpha-\alpha_i}{h_1}, \frac{\beta-\beta_i}{h_2}\right), \qquad (10)$$

where h_i , for i=1,2 are called the bandwidths or smoothing parameters, which are chosen such that $h_i \to 0$ and $nh_i \to \infty$ as $n \to \infty$. The optimal choice for h_i are not known in general, but for a large amount of data, the mean integrated squared error of $\hat{g}(\alpha, \beta)$ is minimized when $h_i = 1.06S_i n^{-0.2}$, S_i are the sample standard deviation of the samples. The algorithm for deriving $\hat{g}(\alpha, \beta)$ has been clarified in [1] and [24].

2.2.3 The Characteristic Prior

The characteristic function (CF) is the Fourier transformation of the cumulative distribution function (CDF), and hence there is a one–to-one correspondence between the CF and the CDF. Thus, the CF fully characterizes the distribution of the underlying random variable. Since the CF can be estimated using the empirical characteristic function (ECF), which retains all the information in the sample, it has played an increasing and important role in econometrics and finance, see [13]. Based on the CF, we can derive the characteristic prior function for α and β as follows:

$$\hat{q}(\alpha, \beta) = \frac{1}{4\pi^2 n}\sum_{i=1}^{n} \frac{1}{|(\alpha-\hat{\alpha}_i)(\beta-\hat{\beta}_i)|}, \qquad (11)$$

see [31].

Thus, using the joint priors (9), (10) and (11) with the likelihood function of the GPHCS (3) the posterior density for the parameters α and β can be written in a unified form as follows:

$$f(\alpha, \beta|\underline{x}) = Kl(\alpha, \beta)L(\bar{X}; \theta), \text{ where}$$

$l(\alpha,\beta) = h(\alpha,\beta)\hat{g}(\alpha,\beta)\hat{q}(\alpha,\beta) \propto$

$\alpha^{a-1}\beta^{c-1}e^{-b\alpha-d\beta}\hat{g}_1^{p_1}(\alpha)\,\hat{g}_2^{p_2}(\beta)\hat{q}_1^{r_1}(\alpha)\,\hat{q}_2^{r_2}(\beta),$

is the general prior distribution function with the following special cases:

i. $p_1 = p_2 = r_1 = r_2 = 0$ for the informative prior (9).

ii. $p_1 = p_2 = 1,\ a = c = 1,$ and $b = d = r_1 = r_2 = 0$ for the kernel prior (10).

iii. $p_1 = p_2 = 0,\ a = c = 1,\ b = d = 0$ and $r_1 = r_2 = 1$ for the characteristic prior (11).

Thus, the posterior density can be written as follows:

$$f(\alpha,\beta|\underline{X}) = C\hat{g}_1^{p_1}(\alpha)\,\hat{g}_2^{p_2}(\beta)\hat{q}_1^{r_1}(\alpha)\,\hat{q}_2^{r_2}(\beta)\alpha^{n+a-1}\beta^{-n+c-1}$$

$$\times \exp\{-\alpha b - \beta d - \sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)$$

$$-\alpha\left[\sum_{i=1}^{n}(1+R_i)\ln\left(1+\frac{x_i}{\beta}\right) + \delta R_T^*\ln(1+\frac{T}{\beta})\right]\Big\}. \quad (12)$$

The marginal posterior densities for the parameters based on the kernel and characteristic priors can't be derived analytically. However, for the informative gamma prior, the posterior densities for the parameters can be derived as follows:

$$f(\beta|\underline{X}) = K\Gamma(n+a)\beta^{-n+c-1}$$

$$[b + \sum_{i=1}^{n}(1+R_i)\ln(\frac{x_i}{\beta}) + \delta R_T^*\ln(1+\frac{T}{\beta})]]^{-(n+a)}$$

$$\times \exp[-\beta d - \sum_{i=1}^{n}\ln(1+x_i/\beta)$$

The normalized constant K can be derived as follows:

$$K^{-1} = \Gamma(n+a)\int_0^\infty \beta^{-n+c-1}$$

$$[b + \sum_{i=1}^{n}(1+R_i)\ln(1+\frac{x_i}{\beta}) + \delta R_T^*\ln(1+\frac{T}{\beta})]]^{-(n+a)}$$

$$\times \exp[-\beta d - \sum_{i=1}^{n}\ln(1+x_i/\beta)d\beta.$$

3 Simulation Study

In this section, we study the performance of the Picard and Bayes methods, through two criteria, the average bias (AVB) and the mean squared error (MSE) are given respectively, as follows:

$AVB = \frac{1}{L}\sum_{i=1}^{L}|\hat{\theta}_i - \theta|, \quad MSE = \sum_{i=1}^{L}(\hat{\theta}_i - \theta)^2/L$

$\hat{\theta}$ is the estimate of θ and L is the number of replications.

In our simulation study we choose the distribution hyperparameters of α and β as:
$a = c = 5$ and $b = d = 3$ the values for the parameters $\alpha = (0.75, 1.5)$ and $\beta = (1.5, 2.5)$. Using the above values of the parameters for generating different samples from the Pareto-II distribution with sizes $n = 20, 40,$ and 60 to represent small, moderate, and large sizes. To assess the performance of these estimates the AVB and MSEs for each one were calculated using 1000 replications.

From the simulation results in Tables 3, 4, 5 and 6, some of the points are quite clear based on these estimates, and the others have been summarized in the following main points:

1- In general, the point estimates for the parameters α and β based on Picard's method have the smallest estimated AVB and MSE values as compared with the Bayesian estimates based on the informative gamma, kernel, and characteristic priors.

2- The estimated values of MSE based on the kernel prior are smaller than those based on the informative gamma and characteristic priors.

3- The estimated values of MSE increase as the value of α increases and decrease as the value of β increases.

4- The estimated values of MSE decrease as the termination time of the experiment, T, and the sample size increase as expected.

In conclusion, it appears that the point estimates based on Picard's method compete with and outperform the Bayes' method based on the informative gamma, kernel, and characteristic priors.

4 Real Data Analysis

In this section, we studied two real data sets to demonstrate the performance of the proposed methods on the Pareto type-II model in practice and to illustrate that this distribution can be considered as a good lifetime model for fitting some new areas of applications. We have fitted these data sets using some goodness-of-fit tests, such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and Chi-Square (CH2) tests for a significance level equal to 0.05, see Table 1.

4.1 The Reactor Pumps Data

In this section, real data sets for the secondary nuclear pumps have been analyzed to illustrate the proposed methods. One of the most severe accidents in nuclear power generation is the loss of coolant, where the recirculating coolant of the pressurized water reactor may flash into steam. Under such conditions, the reactor cooling pumps become unable to generate the same head as those in the single-phase flow case. Thus, the secondary reactor pump is a feed-water pump that takes feed water from the desiccant storage tank, pressurized by the booster pump, and pushes it into the steam generator through the high-pressure heater. Accordingly, the main feed pump must be a high-temperature and high-pressure pump since it requires a head larger than the pressure inside the steam generator. The secondary circulation pump differs slightly in design and has been developed specifically for cooling at higher temperatures. The following data set represents the time between the failures of the secondary reactor pumps. [36] has discussed the classical and Bayesian estimation methods under the Type-II censoring scheme of this data set. The times between failures of 23 secondary reactor pumps are as follows:

2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347, 0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921, 4.082, 0.199, 0.605, 0.273, 0.070, 0.062, 5.320.

We found the Pareto-II model to be a good fit for this dataset, as shown in Table 1 and Fig. (1 a). For studying the reliability of these reactor pumps based on this dataset, we found the estimates for the parameters that represent the shape of the failures between pumps using our model to determine the behavior of the failed pumps. We noticed that the Picard and Bayes estimates for $\boldsymbol{\alpha}$ lie in the interval [2, 2.2] and for $\boldsymbol{\beta}$ lie in the interval [1.5, 2.2] indicating that the above dataset is heavily right-skewed. Thus, the failure rate decreases with increasing time, see Fig. (1b), and that means decreasing the reliability of safety mechanisms with increasing time.

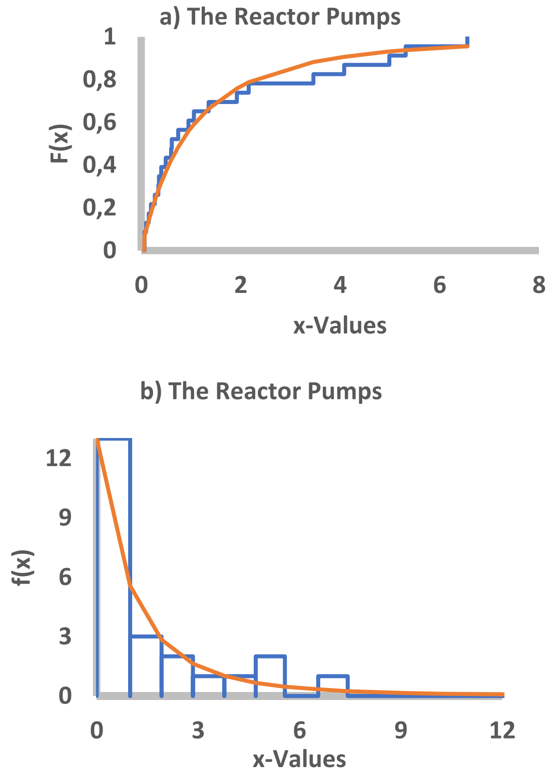


Fig.1: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF.

4.2 The Repair Times of Airborne Data

Airborne satellite communications on the move is a technology used to provide Satellite communication services to commercial and military aircraft and UAVs (unmanned aerial vehicles) while they are in motion. This technology is becoming increasingly important as the demand for real-time data and connectivity in the aviation industry grows. [6] used 46 observations for the repair times of airborne communication transceivers as given below:

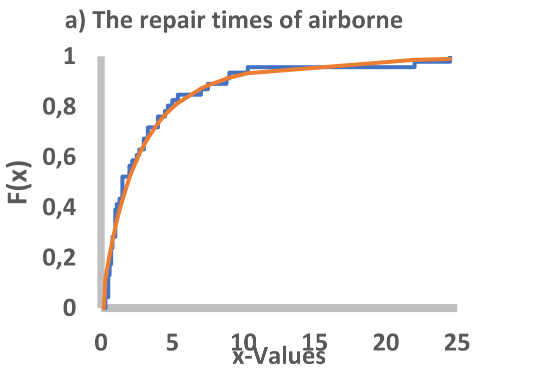
0.20, 0.30, 0.50, 0.50, 0.50, 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 7.00, 7.50, 8.80, 9.00, 10.3, 22.0, 24.5.

The validity of the Pareto type-II model was checked by using the Kolmogorov-Smirnov (K-S) test, Anderson-Darling (A-D) and Chi-Square (C-S) tests, see Table 1. It is observed that the calculated values of these tests are less than the critical values, and the power of the tests is greater than the significance level, which equals 0.05. This shows that the Pareto-

II model provides a good fit to the above data, see Figure (2 a). For studying the reliability of the airborne communication transceiver based on this dataset, we find the estimates for the parameters that represent the shape and scale of the failures between the repaired times using our model to determine the behavior of the airborne communication transceiver. We noticed that the Picard and Bayes estimates for **α** lie in the interval [2.3, 3.2] and for **β** lie in the interval [6.5, 7.8]. These estimates indicate that the above dataset is heavily right-skewed. Thus, the repair times decrease with increasing time, see Figure (2 b), and that means the repair times of the airborne communication transceiver are very significant.

Table 1: The K-S, A-D, and CH2 tests and their powers (p-values). .

Data	The Tests	Critical values	Calculated values	p-values	MLES	
					$\hat{\alpha}$	$\hat{\beta}$
Reactor Pumps N=23	K-S	0.8109	0.4784	0.7895	2.2406	2.1669
	A-D	0.8131	0.2917	0.6725		
	CH2	9.1989	8.7408	0.1484		
Repair time N=46	K-S	0.8200	0.9248	0.0322	3.2519	7.9541
	A-D	0.7725	0.7326	0.0974		
	CH2	24.1995	8.6155	0.1487		



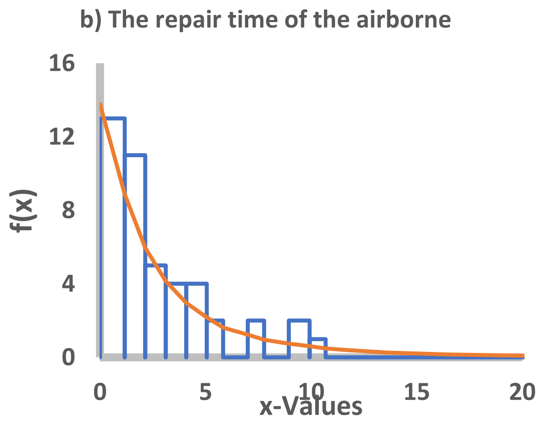


Fig.2: a) The Empirical CDF and the fitted CDF. b) The Histogram and the fitted PDF.

5 Conclusions

In reliability theory, it is known that the Bayes' method is the most applicable in estimation methods. Thus, in this work we introduced an optimal numerical estimation method, Picard's method, which is more reliable and outperforms the Bayes method based on the informative gamma prior, the non-parametric kernel, and the characteristic priors. The simulation results indicated that the point estimates based on the Picard's method are strongly unbiased and more efficient than Bayes estimates based on the generalized progressive hybrid censored data.

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Table 2: The estimate and the mean squared errors (MSE) for the parameters α and β using the Picard and Bayes methods based on the GHPCS: for $m = n/2, k = m/2$.

Samples	T	Param	Picard Estimates		Bayes estimates					
					Gamma prior		Kernel prior		Characteristic prior	
			Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
Reactor Pumps data N=23	0.5	α	2.1739	0.0044	1.9247	0.0998	2.1413	0.0986	2.0847	0.0243
		β	2.1993	0.0014	1.3346	0.6926	1.5149	0.4251	1.5176	0.4215
	5.5	α	2.1724	0.0047	2.0019	0.0569	2.1901	0.00255	2.2324	0.0067
		β	2.2129	0.0021	1.3318	0.6973	1.5063	0.4364	1.6551	0.2619
Repair time data N=46	1.5	α	3.1740	0.0611	3.0237	0.8699	2.5833	0.8961	2.3904	0.9076
		β	7.8084	0.0212	6.9847	0.0939	6.3877	0.2454	6.6487	0.1703
	4.5	α	3.1269	0.0156	3.0694	0.8672	2.7785	0.8844	2.5599	0.8975
		β	7.8058	0.0223	7.0017	0.0907	6.5287	0.2032	6.6867	0.01606

Table 3: The Average bias (ABS) and Mean Squared Errors (MSEs) in parentheses for the parameter α using Picard and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75.

n	m	k	α	β	Picard Estimations	Bayes Estimations		
						Chara-Prior	Gamma Prior	Kernel Prior
20	10	5	0.75	1.5	0.1268(0.0161)	0.1777(0.0333)	0.1948(0.0417)	0.3119(0.0991)
				2.5	0.0983(0.0097)	0.2880(0.0866)	0.3659(0.1355)	0.3697(0.1381)
			1.5	1.5	0.4106(0.1699)	0.7686(0.5969)	0.8672(0.7534)	0.9373(0.8796)
				2.0	0.2578(0.0665)	0.6583(0.4985)	0.7953(0.6584)	0.8795(0.7854)
		8	0.75	1.5	0.0867(0.0075)	0.0870(0.0089)	0.0877(0.0116)	0.1878(0.0399)
				2.5	0.0687(0.0047)	0.1694(0.0329)	0.2307(0.0577)	0.2510(0.0665)
			1.5	1.5	0.2269(0.0516)	0.4489(0.2087)	0.5939(0.3572)	0.7258(0.5294)
				2.0	0.1931(0.0373)	0.9084(0.8277)	0.8761(0.7694)	0.8397(0.7070)
	15	8	0.75	1.5	0.0864(0.0075)	0.0887(0.0097)	0.0890(0.0125)	0.1896(0.0410)
				2.5	0.0687(0.0047)	0.1706(0.0339)	0.2308(0.0581)	0.2512(0.0669)
			1.5	1.5	0.2271(0.0517)	0.4506(0.2104)	0.5952(0.3589)	0.7268(0.5309)
				2.0	0.1932(0.0374)	0.9064(0.8242)	0.8746(0.7668)	0.8381(0.7045)
		11	0.75	1.5	0.0704(0.0050)	0.0453(0.0033)	0.1380(0.0291)	0.1126(0.0184)
				2.5	0.0613(0.0038)	0.1165(0.0178)	0.1441(0.0280)	0.1738(0.0363)
			1.5	1.5	0.1973(0.0390)	0.2609(0.0740)	0.3651(0.1451)	0.5633(0.3229)
				2.0	0.1564(0.0245)	0.7396(0.5543)	0.7085(0.5062)	0.6868(0.4757)
			0.75	1.5	0.1054(0.0111)	0.1810(0.0342)	0.2182(0.0495)	0.3425(0.1183)
				2.5	0.0803(0.0065)	0.3915(0.1545)	0.3790(0.1444)	0.3954(0.1570)

40	20	10	1.5	1.5	0.2641(0.0699)	0.9183(0.8453)	0.8898(0.7925)	0.9901(0.9807)
				2.0	0.2113(0.0447)	0.7865(0.7943)	0.7854(0.5732)	0.8657(0.6743)
		15	0.75	1.5	0.0776(0.0060)	0.0720(0.0061)	0.0732(0.0085)	0.2235(0.0524)
				2.5	0.0643(0.0041)	0.2650(0.0737)	0.2576(0.0686)	0.2881(0.0847)
			1.5	1.5	0.2117(0.0449)	0.5172(0.2746)	0.6443(0.4172)	0.7981(0.6382)
				2.0	0.1696(0.0288)	0.9998(0.6574)	0.9106(0.8301)	0.9072(0.8238)
	30	15	0.75	1.5	0.0776(0.0060)	0.0714(0.0059)	0.0725(0.0082)	0.2233(0.0522)
				2.5	0.0643(0.0041)	0.2672(0.0747)	0.2594(0.0693)	0.2896(0.0855)
			1.5	1.5	0.2117(0.0449)	0.5184(0.2756)	0.6451(0.4182)	0.7987(0.6391)
				2.0	0.1693(0.0287)	0.9990(0.9987)	0.9098(0.8285)	0.9063(0.8222)
		23	0.75	1.5	0.0639(0.0041)	0.0270(0.0014)	0.1185(0.0215)	0.1136(0.0175)
				2.5	0.0552(0.0031)	0.1629(0.0312)	0.1443(0.0258)	0.1857(0.0384)
			1.5	1.5	0.1646(0.0271)	0.1728(0.0333)	0.3232(0.1116)	0.5567(0.3133)
				2.0	0.1376(0.0189)	0.8100(0.6583)	0.6831(0.4694)	0.6967(0.4877)
60	30	15	0.75	1.5	0.0897(0.0081)	0.1625(0.0277)	0.2175(0.0487)	0.3483(0.1220)
				2.5	0.0702(0.0049)	0.4210(0.1778)	0.3813(0.1459)	0.4041(0.1638)
			1.5	1.5	0.2336(0.0546)	0.9360(0.8768)	0.8895(0.7917)	0.4859(0.2425)
				2.0	0.1967(0.0387)	0.7864(0.5473)	06835(0.5735)	0.5342(0.1435)
		23	0.75	1.5	0.0679(0.0046)	0.0437(0.0024)	0.0593(0.0055)	0.2158(0.0483)
				2.5	0.0593(0.0035)	0.2914(0.0872)	0.2527(0.0654)	0.2892(0.0849)
			1.5	1.5	0.1856(0.0345)	0.5018(0.2572)	0.6195(0.3853)	0.7935(0.6305)
				2.0	0.1505(0.0227)	0.4756(0.2537)	0.8950(0.8016)	0.9069(0.8231)
	45	23	0.75	1.5	0.0678(0.0046)	0.0441(0.0025)	0.0591(0.0056)	0.2162(0.0485)
				2.5	0.0593(0.0035)	0.2893(0.0859)	0.2507(0.0643)	0.2875(0.0838)
			1.5	1.5	0.1857(0.0345)	0.5088(0.2649)	0.6232(0.3899)	0.7962(0.6349)
				2.0	0.1507(0.0227)	0.3542(0.3426)	0.8949(0.8014)	0.9068(0.8229)
		34	0.75	1.5	0.0597(0.0036)	0.0254(0.0011)	0.1116(0.0180)	0.1074(0.0153)
				2.5	0.0515(0.0026)	0.1771(0.0350)	0.1414(0.0234)	0.1866(0.0374)
			1.5	1.5	0.1505(0.0227)	0.1648(0.0301)	0.3254(0.1109)	0.5657(0.3224)
				2.0	0.1244(0.0155)	0.8208(0.6751)	0.6810(0.4656)	0.7061(0.5001)

Table 4: The Average bias (ABS) and Mean Squared Errors (MSEs) in parentheses for the parameter α using Picard and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5.

n	m	k	α	β	Picard Estimations	Bayes Estimations		
						Chara-Prior	Gamma Prior	Kernel Prior
20	10	5	0.75	1.5	0.1572(0.0247)	0.2129(0.0461)	0.1948(0.0417)	0.3379(0.1157)
				2.5	0.1303(0.0170)	0.2829(0.0822)	0.3659(0.1355)	0.3947(0.1570)
			1.5	1.5	0.4169(0.1758)	0.7808(0.6130)	0.8672(0.7534)	0.9838(0.9686)
				2.0	0.3228(0.1043)	0.6573(04746)	0.7943(0.6845)	0.8463(0.7684)
		8	0.75	1.5	0.1198(0.0144)	0.1450(0.0216)	0.0877(0.0116)	0.2190(0.0520)
				2.5	0.1038(0.0108)	0.1908(0.0381)	0.2307(0.0577)	0.2823(0.0826)
			1.5	1.5	0.2903(0.0844)	0.4968(0.2498)	0.5939(0.3572)	0.7880(0.6231)
				2.0	0.2577(0.0665)	0.8312(0.6953)	0.8761(0.7694)	0.8942(0.8012)
	15	8	0.75	1.5	0.1195(0.0143)	0.1462(0.0222)	0.0890(0.0125)	0.2207(0.0530)
				2.5	0.1037(0.0108)	0.1918(0.0387)	0.2308(0.0581)	0.2825(0.0830)
			1.5	1.5	0.2904(0.0844)	0.4977(0.2508)	0.5952(0.3589)	0.7889(0.6245)
				2.0	0.2578(0.0665)	0.8286(0.6914)	0.8746(0.7668)	0.8928(0.7988)
		11	0.75	1.5	0.1051(0.0111)	0.1109(0.0130)	0.1380(0.0291)	0.1389(0.0253)
				2.5	0.0967(0.0094)	0.1528(0.0247)	0.1441(0.0280)	0.2071(0.0481)
			1.5	1.5	0.2617(0.0685)	0.3576(0.1300)	0.3651(0.1451)	0.6346(0.4072)
				2.0	0.2265(0.0513)	0.6199(0.3920)	0.7085(0.5062)	0.7515(0.5678)
		10	0.75	1.5	0.1368(0.0187)	0.1987(0.0400)	0.2182(0.0495)	0.3649(0.1340)
				2.5	0.1139(0.0130)	0.3420(0.1186)	0.3790(0.1444)	0.4178(0.1751)
			1.5	1.5	0.3323(0.1105)	0.8575(0.7376)	0.8898(0.7925)	0.3764(0.1045)

40	20			2.0	0.2876(0.0828)	0.7574(0.6573)	0.7946(0.6835)	0.3524(0.1342)
		15	0.75	1.5	0.1115(0.0124)	0.1266(0.0164)	0.0732(0.0085)	0.2503(0.0647)
				2.5	0.1009(0.0102)	0.2153(0.0481)	0.2576(0.0686)	0.3154(0.1009)
			1.5	1.5	0.2825(0.0798)	0.4950(0.2480)	0.6443(0.4172)	0.8477(0.7195)
				2.0	0.2364(0.0559)	0.9407(0.8862)	0.9106(0.8301)	0.9510(0.9051)
	30	15	0.75	1.5	0.1115(0.0124)	0.1263(0.0163)	0.0725(0.0082)	0.2501(0.0645)
				2.5	0.1008(0.0102)	0.2163(0.0483)	0.2594(0.0693)	0.3168(0.1017)
			1.5	1.5	0.2831(0.0802)	0.4955(0.2484)	0.6451(0.4182)	0.8482(0.7203)
				2.0	0.2361(0.0558)	0.9396(0.8841)	0.9098(0.8285)	0.9502(0.9036)
		23	0.75	1.5	0.1003(0.0101)	0.0750(0.0067)	0.1185(0.0215)	0.1378(0.0236)
				2.5	0.0911(0.0083)	0.1450(0.0221)	0.1443(0.0258)	0.2144(0.0494)
			1.5	1.5	0.2324(0.0540)	0.2847(0.0820)	0.3232(0.1116)	0.6179(0.3845)
				2.0	0.2070(0.0429)	0.6146(0.3843)	0.6831(0.4694)	0.7517(0.5669)
60	30	15	0.75	1.5	0.1224(0.0150)	0.1782(0.0321)	0.2175(0.0487)	0.3691(0.1368)
				2.5	0.1047(0.0110)	0.3639(0.1336)	0.3813(0.1459)	0.4250(0.1810)
			1.5	1.5	0.2945(0.0868)	0.8773(0.7710)	0.8895(0.7917)	0.3645(0.10452)
				2.0	0.2606(0.0680)	0.7956(0.5735)	0.6758(0.5735)	0.3524(0.1352)
		23	0.75	1.5	0.1030(0.0106)	0.1057(0.0114)	0.0593(0.0055)	0.2410(0.0596)
				2.5	0.0948(0.0090)	0.2056(0.0437)	0.2527(0.0654)	0.3148(0.1002)
			1.5	1.5	0.2507(0.0629)	0.4636(0.2172)	0.6195(0.3853)	0.8390(0.7046)
				2.0	0.2198(0.0483)	0.9233(0.8532)	0.8950(0.8016)	0.9474(0.8981)
	45	23	0.75	1.5	0.1029(0.0106)	0.1059(0.0115)	0.0591(0.0056)	0.2414(0.0599)
				2.5	0.0948(0.0090)	0.2033(0.0426)	0.2507(0.0643)	0.3132(0.0991)
			1.5	1.5	0.2508(0.0629)	0.4685(0.2221)	0.6232(0.3899)	0.8415(0.7088)
				2.0	0.2201(0.0484)	0.9231(0.8529)	0.8949(0.8014)	0.9473(0.8979)
		34	0.75	1.5	0.0952(0.0091)	0.0575(0.0043)	0.1116(0.0180)	0.1321(0.0211)
				2.5	0.0877(0.0077)	0.1329(0.0183)	0.1414(0.0234)	0.2138(0.0480)
			1.5	1.5	0.2198(0.0483)	0.2746(0.0761)	0.3254(0.1109)	0.6211(0.3878)
				2.0	0.1950(0.0380)	0.6230(0.3927)	0.6810(0.4656)	0.7560(0.5728)

Table 5: The Average bias (ABS) and Mean Squared Errors (MSEs) in parentheses for the parameter β using Picard and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=0.75.

n	m	k	α	β	Picard Estimations	Bayes Estimations		
						CH-Prior	Gamma Prior	Kernel Prior
20	10	5	0.75	1.5	0.1602(0.0257)	0.3200(0.1029)	0.6963(0.4850)	0.4691(0.2203)
				2.5	0.2604(0.0678)	0.2670(0.0714)	0.2839(0.0808)	0.1374(0.0190)
			1.5	1.5	0.1557(0.0242)	0.2602(0.0687)	0.7033(0.4946)	0.5552(0.3083)
				2.0	0.2559(0.0655)	0.3070(0.0943)	0.2748(0.0756)	0.1958(0.0384)
		8	0.75	1.5	0.1559(0.0243)	0.3512(0.1281)	0.7963(0.6346)	0.4847(0.2355)
				2.5	0.2561(0.0656)	0.2384(0.0569)	0.1725(0.0302)	0.1800(0.0328)
			1.5	1.5	0.1533(0.0235)	0.5136(0.2658)	0.8091(0.6548)	0.5865(0.3441)
				2.0	0.2534(0.0642)	0.2593(0.0673)	0.1570(0.0247)	0.2557(0.0655)
	15	8	0.75	1.5	0.1557(0.0243)	0.3520(0.1287)	0.7963(0.6347)	0.4849(0.2357)
				2.5	0.2560(0.0655)	0.2388(0.0571)	0.1726(0.0303)	0.1798(0.0327)
			1.5	1.5	0.1533(0.0235)	0.5133(0.2656)	0.8096(0.6556)	0.5871(0.3447)
				2.0	0.2534(0.0642)	0.2589(0.0671)	0.1570(0.0247)	0.2557(0.0655)
		11	0.75	1.5	0.1547(0.0239)	0.3514(0.1330)	0.8674(0.7536)	0.5022(0.2536)
				2.5	0.2548(0.0649)	0.2277(0.0521)	0.0955(0.0100)	0.2104(0.0452)
			1.5	1.5	0.1526(0.0233)	0.6785(0.4618)	0.8907(0.7936)	0.6113(0.3740)
				2.0	0.2528(0.0639)	0.2074(0.0433)	0.0706(0.0051)	0.2991(0.0896)
	20	10	0.75	1.5	0.1562(0.0244)	0.6080(0.3710)	0.8586(0.7374)	0.5326(0.2839)
				2.5	0.2564(0.0658)	0.2284(0.0523)	0.1031(0.0107)	0.2252(0.0508)
			1.5	1.5	0.1534(0.0235)	0.3774(0.1427)	0.8648(0.7478)	0.6137(0.3766)
				2.0	0.2535(0.0643)	0.2610(0.0681)	0.0950(0.0091)	0.2814(0.0792)
		15	0.75	1.5	0.1551(0.0241)	0.7525(0.5684)	0.9588(0.9197)	0.5800(0.3369)
				2.5	0.2553(0.0652)	0.1525(0.0236)	0.0121(0.0002)	0.2854(0.0818)
			1.5	1.5	0.1527(0.0233)	0.7013(0.4937)	0.9681(0.9373)	0.6760(0.4571)

40				2.0	0.2528(0.0639)	0.2086(0.0435)	0.0140(0.0002)	0.3575(0.1279)
	30	15	0.75	1.5	0.1551(0.0241)	0.7537(0.5702)	0.9590(0.9199)	0.5802(0.3371)
				2.5	0.2553(0.0652)	0.1526(0.0236)	0.0120(0.0002)	0.2857(0.0820)
			1.5	1.5	0.1527(0.0233)	0.7014(0.4938)	0.9686(0.9382)	0.6767(0.4580)
				2.0	0.2528(0.0639)	0.2086(0.0435)	0.0139(0.0002)	0.3574(0.1278)
		23	0.75	1.5	0.1547(0.0239)	0.7956(0.6433)	0.0154(0.0354)	0.6361(0.4074)
				2.5	0.2549(0.0650)	0.0897(0.0086)	0.1035(0.0113)	0.3532(0.1264)
			1.5	1.5	0.1525(0.0232)	0.0784(0.0153)	0.1353(0.0352)	0.7560(0.5719)
				2.0	0.2525(0.0638)	0.1090(0.0121)	0.1273(0.0163)	0.4517(0.2043)
60	30	15	0.75	1.5	0.1541(0.0237)	0.7949(0.6337)	0.9604(0.9224)	0.5909(0.3494)
				2.5	0.2542(0.0646)	0.1967(0.0388)	0.0085(0.0001)	0.2916(0.0851)
			1.5	1.5	0.1522(0.0232)	0.5586(0.3122)	0.9660(0.9332)	0.6690(0.4476)
				2.0	0.2523(0.0636)	0.2170(0.0471)	0.0128(0.0002)	0.3467(0.1202)
		23	0.75	1.5	0.1535(0.0236)	0.9838(0.9691)	0.0253(0.0256)	0.6609(0.4373)
				2.5	0.2536(0.0643)	0.0802(0.0072)	0.1100(0.0123)	0.3699(0.1372)
			1.5	1.5	0.1518(0.0231)	0.9149(0.8378)	0.2546(0.0352)	0.7560(0.5716)
				2.0	0.2519(0.0635)	0.1374(0.0189)	0.1208(0.0146)	0.4433(0.1966)
	45	23	0.75	1.5	0.1535(0.0236)	0.9833(0.9681)	0.1573(0.0365)	0.6605(0.4368)
				2.5	0.2536(0.0643)	0.0781(0.0068)	0.1113(0.0125)	0.3720(0.1387)
			1.5	1.5	0.1519(0.0231)	0.9129(0.8343)	0.1324(0.0684)	0.7559(0.5715)
				2.0	0.2519(0.0635)	0.1374(0.0189)	0.1209(0.0146)	0.4434(0.1966)
		34	0.75	1.5	0.1533(0.0235)	0.1534(0.0153)	0.1546(0.6583)	0.7288(0.5336)
				2.5	0.2534(0.0642)	0.0400(0.0020)	0.1934(0.0377)	0.4473(0.2016)
			1.5	1.5	0.1517(0.0230)	0.0356(0.0352)	0.1375(0.0145)	0.8484(0.7201)
				2.0	0.2518(0.0634)	0.0118(0.0002)	0.2135(0.0456)	0.5490(0.3016)

Table 6: The Average bias (ABS) and Mean Squared Errors (MSEs) in parentheses for the parameter β using Picard and Bayes methods with m = (n/2 and 3n/4) and k=(m/2 and 3m/4) at T=1.5.

n	m	k	α	β	Picard Estimations	Bayes Estimations		
						CH-Prior	Gamma Prior	Kernel Prior
20	10	5	0.75	1.5	0.1596(0.0255)	0.1424(0.0208)	0.6966(0.4854)	0.3873(0.1501)
				2.5	0.2598(0.0675)	0.1734(0.0302)	0.2839(0.0808)	0.1844(0.0341)
			1.5	1.5	0.1554(0.0241)	0.0737(0.0065)	0.7033(0.4946)	0.4550(0.2071)
				2.0	0.2556(0.0653)	0.2135(0.0456)	0.2748(0.0756)	0.1518(0.0231)
		8	0.75	1.5	0.1555(0.0242)	0.0724(0.0061)	0.7963(0.6346)	0.4262(0.1820)
				2.5	0.2557(0.0654)	0.1853(0.0344)	0.1725(0.0302)	0.1205(0.0147)
			1.5	1.5	0.1531(0.0234)	0.3502(0.1281)	0.8091(0.6548)	0.5107(0.2609)
				2.0	0.2532(0.0641)	0.1849(0.0345)	0.1570(0.0247)	0.0785(0.0062)
	15	8	0.75	1.5	0.1554(0.0242)	0.0730(0.0062)	0.7963(0.6347)	0.4263(0.1822)
				2.5	0.2556(0.0654)	0.1857(0.0346)	0.1726(0.0303)	0.1206(0.0147)
			1.5	1.5	0.1531(0.0235)	0.3488(0.1273)	0.8096(0.6556)	0.5112(0.2614)
				2.0	0.2532(0.0641)	0.1842(0.0342)	0.1570(0.0247)	0.0785(0.0062)
		11	0.75	1.5	0.1545(0.0239)	0.0277(0.0012)	0.8674(0.7536)	0.4559(0.2089)
				2.5	0.2545(0.0648)	0.2047(0.0419)	0.0955(0.0100)	0.0774(0.0063)
			1.5	1.5	0.1525(0.0232)	0.5811(0.3428)	0.8907(0.7936)	0.5498(0.3024)
				2.0	0.2526(0.0638)	0.1390(0.0202)	0.0706(0.0051)	0.0282(0.0008)
40	20	10	0.75	1.5	0.1559(0.0243)	0.5778(0.3384)	0.8586(0.7374)	0.4762(0.2270)
				2.5	0.2561(0.0656)	0.1220(0.0155)	0.1031(0.0107)	0.0775(0.0060)
			1.5	1.5	0.1532(0.0235)	0.1703(0.0303)	0.8648(0.7478)	0.5419(0.2937)
				2.0	0.2533(0.0642)	0.1971(0.0389)	0.0950(0.0091)	0.0483(0.0023)
		15	0.75	1.5	0.1548(0.0240)	0.6815(0.4709)	0.9588(0.9197)	0.5352(0.2868)
				2.5	0.2550(0.0650)	0.0781(0.0069)	0.0121(0.0383)	0.0124(0.0003)
			1.5	1.5	0.1525(0.0233)	0.7874(0.6389)	0.9681(0.9373)	0.6152(0.3786)
				2.0	0.2526(0.0638)	0.1722(0.0297)	0.0140(0.0002)	0.0239(0.0006)
			0.75	1.5	0.1548(0.0240)	0.6827(0.4723)	0.9590(0.9199)	0.5354(0.2870)
				2.5	0.2550(0.0650)	0.0785(0.0070)	0.0120(0.0002)	0.0125(0.0003)

	30	15	1.5	1.5	0.1526(0.0233)	0.7869(0.6375)	0.9686(0.9382)	0.6158(0.3792)
				2.0	0.2526(0.0638)	0.1720(0.0297)	0.0139(0.0002)	0.0239(0.0006)
		23	0.75	1.5	0.1544(0.0239)	0.6107(0.4163)	0.7694(0.3265)	0.6003(0.3626)
				2.5	0.2546(0.0648)	0.0949(0.0097)	0.1035(0.0113)	0.0560(0.0035)
			1.5	1.5	0.1523(0.0232)	0.5163(0.2859)	0.2362(0.1546)	0.7030(0.4945)
				2.0	0.2524(0.0637)	0.2369(0.0561)	0.1273(0.0163)	0.1027(0.0106)
60	30	15	0.75	1.5	0.1539(0.0237)	0.2314(0.0576)	0.9604(0.9224)	0.5439(0.2960)
				2.5	0.2540(0.0645)	0.2277(0.0518)	0.0085(0.0001)	0.0092(0.0001)
			1.5	1.5	0.1521(0.0231)	0.0061(0.0352)	0.9660(0.9332)	0.6078(0.3694)
				2.0	0.2521(0.0636)	0.2375(0.0564)	0.0128(0.0002)	0.0178(0.0003)
		23	0.75	1.5	0.1533(0.0235)	0.5195(0.2742)	0.8586(0.7374)	0.6215(0.3866)
				2.5	0.2534(0.0642)	0.2186(0.0478)	0.1031(0.0107)	0.0635(0.0041)
			1.5	1.5	0.1517(0.0230)	0.2064(0.0439)	0.8648(0.7478)	0.7005(0.4908)
				2.0	0.2518(0.0634)	0.2333(0.0544)	0.0950(0.0191)	0.0973(0.0095)
	45	23	0.75	1.5	0.1533(0.0235)	0.5183(0.2735)	0.9588(0.9197)	0.6211(0.3862)
				2.5	0.2534(0.0642)	0.2183(0.0477)	0.0121(0.0002)	0.0645(0.0043)
			1.5	1.5	0.1518(0.0230)	0.2036(0.0429)	0.9681(0.9373)	0.7005(0.4907)
				2.0	0.2518(0.0634)	0.2333(0.0544)	0.0140(0.0002)	0.0973(0.0095)
		34	0.75	1.5	0.1531(0.0235)	0.6268(0.4051)	0.9590(0.9199)	0.6941(0.4836)
				2.5	0.2532(0.0641)	0.2162(0.0468)	0.0120(0.0002)	0.1286(0.0168)
			1.5	1.5	0.1516(0.0230)	0.9104(0.8461)	0.9686(0.9382)	0.7946(0.6317)
				2.0	0.2517(0.0633)	0.2283(0.0521)	0.0139(0.0356)	0.1717(0.0295)