Solving Fractional Relaxation Oscillation Differential Equations Using Rishi Transform

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Abstract: - In this article, a new integral transform is proposed. Properties of Rishi transform are suggested based on the duality between Rishi transform and the general integral transform. We apply Rishi transform to the general form of the fractional relaxation oscillation differential equation. Analytic solutions of some specific applications are found to demonstrate the effectiveness of this novel integral transform.

Key-Words: - Rishi transform, general integral transform, fractional relaxation oscillation differential equation.

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1 Introduction

Fractional calculus has gained widespread adoption and is now recognized as an indispensable instrument in understanding and assessing phenomena across numerous fields, making it a thriving hub for mathematicians and scientists to explore their research [1-3]. Nowadays, researchers are paying attention to fractional differential equations (FDEs) due to their ability to provide profound insights into real-world phenomena across different fields [4-6]. The fractional relaxation–oscillation equation is considered one of the most important differential equations according to its applications in many fields [7,8]. Diverse methodologies such as collocation scheme [9], residual power series method [10], Generalized Taylor matrix method [11], optimal homotopy asymptotic method [12], Elzaki decomposition method [13] have been employed to solve the fractional relaxation-oscillation equations.

Integral transform has the power to solve many applications in Engineering, Science, and many other fields. Employing integral transformation can simplify a complex problem into a more manageable one. The value of integral transforms is by transforming differential equations into algebraic ones. Numerous integral transforms in diverse scientific domains have been suggested, examined, and demonstrated to be effective in resolving fractional differential equations such as Laplace transform [14], Sumudu transform [15], Elazaki transform [13,16], Shehu transform [17], ARA transform [18]. The General integral transform [19], is a new integral transform where many integral transforms can be written as a special case of this new integral transform. More properties and applications of this integral transform can be found in[20]. A newly developed integral transform called Rishi transform has been presented and proposed by Kumar in [21] where principal properties and inverse were presented. Rishi transform was effectively used to solve the first-kind and second-kind linear Volterra integral equations [21-23], nonlinear first-kind Volterra integral equations in [24], nonlinear second-kind Volterra integral equations in [25], fractional order differential equations [26]. the model of the bacteria growth [27]. This paper aims to find the solution of some fractional relaxation oscillation differential equations. The duality between the general integral transform and Rishi transform is established. Using this duality, several features of Rishi transform were discovered.

This paper is arranged as follows. In section 2 some basic definitions of fractional Calculus with definitions and properties of the general integral transform were introduced. In section 3 the definition and properties of Rishi transform. In section 4, The fractional relaxation oscillation differential equations were described and we applied them generally by Rishi transform. In section 5, Some numerical applications were successfully proved using Rishi transform.

2 preliminaries

Definition 2.1

The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u(t), t > 0$ is defined by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau \quad (1)$$

Definition 2.2

The Caputo fractional derivative of order $\alpha > 0$ of a function $u(t), t > 0$ is defined by

$$\boldsymbol{D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}u^{(n)}(\tau)d\tau} \quad (2)$$

Definition 2.3

The Mittag-Leffler function $E_{\alpha,\beta}(t)$, where $\alpha, \beta \in \mathbb{R}^+$ is defined by

$$\boldsymbol{E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}} \quad (3)$$

The Mittag-Leffler function with one parameter is

$$\boldsymbol{E_{\alpha}(t) = E_{\alpha,1}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+1)}} \quad (4)$$

It follows that $\boldsymbol{E_{1,1}(t) = e^t}$.

A new general integral transform has been introduced in [19] where many integral transforms can be written as a special case of this general integral transform.

Definition 2.4

The General integral Transform of the integrable function $u(t), t \geq 0$, with $p(s) \neq 0, q(s)$ are positive real functions is defined by

$$T\{u(t)\} = \int_0^{\infty} p(s)u(t)e^{-q(s)t}dt = \mathcal{T}_u(s) \quad (5)$$

The general integral transform of some functions is listed in Table 1.

Table 1: General integral transform of some functions

	$u(t)$	$\mathcal{T}_u(s) = T\{u(t)\}$
1)	1	$\frac{p(s)}{q(s)}$
2)	t^{α}	$\frac{\Gamma(\alpha+1)p(s)}{q^{\alpha+1}(s)}, \ \alpha > 0$
3)	$sinct$	$\frac{cp(s)}{q^2(s)+c^2}$
4)	$cosct$	$\frac{p(s)q(s)}{q^2(s)+c^2}$
5)	$sinhct$	$\frac{cp(s)}{q^2(s)-c^2}$
6)	$coshct$	$\frac{p(s)q(s)}{q^2(s)-c^2}$
7)	$t^{\beta-1}E_{\alpha,\beta}(ct^{\alpha})$, $\alpha,\beta \in \mathbb{R}^+$	$\frac{p(s)q^{\alpha}(s)}{q^{\alpha+\beta}(s)-cq^{\beta}(s)}$
8)	$u_1(t) * u_2(t)$	$\frac{1}{p(s)}\mathcal{T}_{u_1}(s).\mathcal{T}_{u_2}(s)$
9)	$D^{\alpha}u(t)$, $n-1 < \alpha \leq n$	$q^{\alpha}(s)\mathcal{T}_u(s) - p(s)\sum_{k-=0}^{n-1}q^{\alpha-1-k}u^{(k)}(0)$
10)	$I^{\alpha}u(t)$, $n-1 < \alpha \leq n$	$\frac{1}{q^{\alpha}(s)}\mathcal{T}_u(s)$

3 Rishi Transform

Rishi transform is a new integral transform that provides exact solutions to problems without requiring time-consuming computations.

Definition 3.1

If $u(t), t \geq 0$ is a piecewise continuous function with exponential order, then the Rishi transform of $u(t)$ is defined by

$$R\{u(t)\} = \mathfrak{R}_u(s,\sigma) = \frac{\sigma}{s}\int_0^{\infty} u(t)e^{\frac{-s}{\sigma}t}dt, \ s > 0, \sigma > 0 \quad (6)$$

The duality between Rishi transform and Laplace transform is as follows

If $L\{u(t)\} = \int_0^{\infty} u(t)e^{-\omega t}dt = U(s)$ is the Laplace transform of a piecewise continuous function $u(t), t \geq 0$, then we have the relation between Rishi transform and Laplace transform is $\mathfrak{R}_u(s,\sigma) = \frac{\sigma}{s}U(\frac{s}{\sigma})$.

The duality between Rishi transform and the General integral transform is as follows

Depending on the definition of the general integral transform in the equation (5). We can find the relationship between Rishi transform and the general integral transform by letting $p(s) = \frac{\sigma}{s}$ and $q(s) = \frac{s}{\sigma}$. Using Table 1 we can conclude Rishi transform for some fundamental functions as shown in Table 2.

Table 2: Rishi transform for some basic functions

	$u(t)$	$R\{u(t)\} = \mathfrak{R}_u(s,\sigma)$
1)	1	$\frac{\sigma^2}{s^2}$
2)	t^{α}	$\frac{\sigma^{\alpha+2}}{s^{\alpha+2}}\Gamma(\alpha+1), \ \alpha > 0$
3)	$sinct$	$\frac{c\sigma^3}{s(s^2+c^2\sigma^2)}$
4)	$cosct$	$\frac{\sigma^2}{(s^2+c^2\sigma^2)}$
5)	$sinhct$	$\frac{c\sigma^3}{s(s^2-c^2\sigma^2)}$
6)	$coshct$	$\frac{\sigma^2}{(s^2-c^2\sigma^2)}$

The following are the characteristics of Rishi transform for linearity and convolution.

Property 1. (The linearity property)

If $R\{u_1(t)\} = \mathfrak{R}_{u_1}(s,\sigma)$, $R\{u_2(t)\} = \mathfrak{R}_{u_2}(s,\sigma)$, where $c_1, c_2 \in \mathbb{R}$, then

$$R\{c_1u_1(t)+c_2u_2(t)\} = c_1\mathfrak{R}_{u_1}(s,\sigma) + c_2\mathfrak{R}_{u_2}(s,\sigma) \quad (7)$$

Property 2. (The convolution property)

If $R\{u_1(t)\} = \mathfrak{R}_{u_1}(s,\sigma)$ and $R\{u_2(t)\} = \mathfrak{R}_{u_2}(s,\sigma)$, then

$$R\{u_1(t)*u_2(t)\} = \frac{s}{\sigma}\mathfrak{R}_{u_1}(s,\sigma).\mathfrak{R}_{u_2}(s,\sigma). \quad (8)$$

The convolution property in Table 1 and the relationship between the general integral transform and the Rishi transform can be used to illustrate Property 2.

The Rishi transform for Mittag-Leffler function, fractional integral, and fractional derivative is provided by the following theorems.

Theorem 3.2

If $\alpha,\beta \in \mathbb{R}^+$, then $R\{t^{\beta-1}E_{\alpha,\beta}(ct^\alpha)\} = \frac{s^{\alpha-1}\sigma^{\beta+1}}{s^{\alpha+\beta}-c\sigma^\alpha s^\beta}$.

Proof:

The general integral transform for the Mittag-Leffler function is shown in table 1 as follows

$$T\{t^{\beta-1}E_{\alpha,\beta}(ct^\alpha)\} = \frac{p(s)q^\alpha(s)}{q^{\alpha+\beta}(s)-cq^\beta(s)}$$

Put $p(s) = \frac{\sigma}{s}$ and $q(s) = \frac{s}{\sigma}$, we get the Rishi transform of the Mittag-Leffler function will be

$$R\{t^{\beta-1}E_{\alpha,\beta}(ct^\alpha)\} = \frac{\frac{\sigma}{s}\cdot\frac{s^\alpha}{\sigma^\alpha}}{\left(\frac{s}{\sigma}\right)^{\alpha+\beta}-c\frac{s^\beta}{\sigma^\beta}}$$

$$= \frac{\sigma}{s}\cdot\frac{s^\alpha}{\sigma^\alpha}\cdot\frac{\sigma^{\alpha+\beta}}{s^{\alpha+\beta}-c\sigma^\alpha s^\beta} = s^{\alpha-1}.\frac{\sigma^{\beta+1}}{s^{\alpha+\beta}-c\sigma^\alpha s^\beta}.$$

Using the definition of Mittag-Leffler function in equation (4), we conclude the following Lemma.

Lemma 3.3

If $\alpha \in \mathbb{R}^+$, then $R\{E_\alpha(ct^\alpha)\} = \frac{\sigma^2 s^{\alpha-2}}{s^\alpha - c\sigma^\alpha}$.

Theorem 3.4

If $\alpha > 0$, then the Rishi transform of the fractional Riemann-Liouville integral is

$$R\{I^\alpha u(t)\} = \frac{\sigma^\alpha}{s^\alpha}\mathfrak{R}_u(s,\sigma).$$

Proof:

The definition of the general integral transform of the fractional Riemann-Liouville integral in Table 1 is

$$T(I^\alpha u(t)) = \frac{1}{q^\alpha(s)}\mathcal{T}_u(s),$$

Our conclusion is derived easily from the relationship between the general integral transform and Rishi transform by inserting $q(s) = \frac{s}{\sigma}$.

Theorem 3.5

If $n-1 < \alpha \le n$, $n \in \mathrm{Z}^+$, then the Rishi transform of the Caputo fractional derivative is

$$R\{D^\alpha u(t)\} = \frac{s^\alpha}{\sigma^\alpha}\mathfrak{R}_u(s,\sigma) - \sum_{k-=0}^{n-1}\frac{s^{\alpha-k-2}}{\sigma^{\alpha-k-2}}u^{(k)} \quad (9)$$

Proof:

The general integral transform for the Caputo fractional derivative is displayed in table 1 as follows

$$T(D^\alpha u(t)) = q^\alpha(s)\mathcal{T}_u(s) - p(s)\sum_{k-=0}^{n-1}q^{\alpha-1-k}u^{(k)}(0)$$

Once we solve $q(s) = \frac{s}{\sigma}$, we have our answer.

4 The fractional Relaxation Oscillation Differential Equation with Rishi Transform.

The model for fractional relaxation-oscillation can be described as follows

$$D^\alpha u(t) - cu(t) = g(t),\ t > 0 \quad (10)$$

where c is a positive constant.

$When$ $0 < \alpha \le 1$, equation (10) with the condition $u(0) = u_0$ represents the relaxation with the power law attenuation. For $1 < \alpha \le 2$, equation (10) with the conditions $u(0) = u_0$, $u'(0) = u_1$ describe the damped oscillation with the viscoelastic intrinsic damping of the oscillator. $When$ $0 < \alpha \le 2$, the model is called the fractional relaxation-oscillation equation.

Consider the fractional relaxation differential equation

$$D^\alpha u - cu = g(t)\ ,\quad 0 < \alpha \le 1. \quad (11)$$

With initial condition$u(0) = u_0$.

By applying Rishi transform to equation (11), we get

$$\mathfrak{R}(D^\alpha u - cu) = \mathfrak{R}(g(t))$$

Using theorem 3.5 and applying the inverse of Rishi transform, we have

$$\mathfrak{R}(s,\sigma) = u_0\frac{\sigma^2 s^{\alpha-2}}{s^\alpha-c\sigma^\alpha} + \mathfrak{R}(g(t))\frac{\sigma^\alpha}{s^\alpha-c\sigma^\alpha}$$

$$= u_0\frac{\sigma^2 s^{\alpha-2}}{s^\alpha-c\sigma^\alpha} + \frac{s}{\sigma}\mathfrak{R}(g(t))R(t^{\alpha-1}E_{\alpha,\alpha}(ct^\alpha)) \quad (12)$$

Taking the inverse Rishi transform, we have the solution

$$u(t) = u_0E_\alpha(ct^\alpha)) + g(t)*t^{\alpha-1}E_{\alpha,\alpha}(ct^\alpha)$$

If $g(t) = 0$, then the solution of equation (11) is

$$u(t) = u_0E_\alpha(ct^\alpha). \quad (13)$$

Consider the fractional oscillation differential equation

$$D^\alpha u - cu = g(t)\ ,\quad 1< \alpha \le 2. \quad (14)$$

With initial conditions $u(0) = u_0$, $u'(0) = u_1$.

Apply Rishi transform to equation (14) we get

$$\mathfrak{R}(s,\sigma) = u_0\frac{\sigma^2 s^{\alpha-2}}{s^\alpha-c\sigma^\alpha} + u_1\frac{\sigma^3 s^{\alpha-3}}{s^\alpha-c\sigma^\alpha} + \mathfrak{R}(g(t))\frac{\sigma^\alpha}{s^\alpha-c\sigma^\alpha}. \quad (15)$$

By taking the inverse Rishi transform, we get

$$u(t) = u_0E_{\alpha,1}(ct^\alpha) + u_1tE_{\alpha,2}(ct^\alpha) + g(t)*t^{\alpha-1}E_{\alpha,\alpha}(ct^\alpha)$$

If $g(t) = 0$, then the solution of equation (14) is

$$u(t) = u_0E_{\alpha,1}(ct^\alpha) + u_1tE_{\alpha,2}(ct^\alpha) \quad (16)$$

5 Analytical applicatons

In this section, analytical applications are applied to explain the utility of Rishi transform and find the exact analytical solution on some fractional relaxation-oscillation differential equations.

Application 1:

If we have the initial value problem,

$$D^{0.5}u + u = 0 \ , u(0) = 1. \quad (17)$$

Then according to equation (13), the solution of equation (17) is $u(t) = E_{0.5}(-t^{0.5})$

An identical solution is notified in [9-12, 26].

Application 2:

Consider the initial value problem

$$D^{0.5}u + u = t \ , u(0) = 1 \ . \quad (18)$$

According to equation (12), the Rishi transform of equation (18) is

$$\Re(s,\sigma) = \sigma^2 \frac{s^{-1.5}}{s^{0.5}+\sigma^{0.5}} + \frac{s^{-0.5}\sigma^{3.5}}{s^3+\sigma^{0.5}s^{2.5}}$$

Using theorem 3.2 we get the solution of equation (18) will be

$$u(t) = E_{0.5}(-t^{0.5}) + t^{1.5}E_{0.5,2.5}(-t^{0.5})$$

A similar solution is considered in [29].

Application 3:

Consider the initial value problem

$$D^{0.5}u - u = 1 \ , u(0) = 0 \ . \quad (19)$$

Apply Rishi transform we have

$$\Re(s,\sigma) = \frac{s^{-0.5}\sigma^{2.5}}{s^2-\sigma^{0.5}s^{1.5}}$$

Take the Rishi transform and apply Theorem 3.2 we have the solution

$$u(t) = t^{0.5}E_{0.5,1.5}(t^{0.5})$$

An identical solution is considered in [9, 31].

Application 4:

Consider the initial value problem

$$D^{1.5}u + u = 0 \ , u(0) = 1 \ , u'(0) = 1. \quad (20)$$

By applying equation (16), the solution will be

$$u(t) = E_{1.5,1}(-t^{1.5}) + tE_{1.5,2}(-t^{1,5})$$

The same solution is found in [19, 26, 30].

Application 5:

For the initial value problem

$$D^{1.5}u + 3u = 3t^3 + \frac{4}{\Gamma(1.5)}t^{1.5} \ ,$$

$$u(0) = u'(0) = 0 \quad (21)$$

Apply Rishi transform on equation (21) we get

$$\Re(s,\sigma) = \left(18\frac{\sigma^5}{s^5} + 6\frac{\sigma^{3.5}}{s^{3.5}}\right)\frac{\sigma^{1.5}}{s^{1.5}+3\sigma^{1.5}}$$

This is equivalent to

$$\Re(s,\sigma) = 6\frac{\sigma^5}{s^5}$$

By taking the inverse of the Rishi transform, we get the exact solution $u(t) = t^3$, which is identical to the solution found in [29].

Declaration of Competing Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

6 Conclusion

This work has successfully proposed a new integral transform called the "Rishi Transform" and has found the precise answer for some applications of the fractional relaxation oscillation differential equation. Fundamental properties of Rishi transform are presented, along with the duality between Rishi transform and general integral transform.

References:

[1] Diethelm, K. and Ford, N.J., 2010. The analysis of fractional differential equations. Lecture notes in mathematics, 2004.

[2] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J., 2006. Theory and applications of fractional differential equations (Vol. 204). elsevier.

[3] Podlubny, I., 1998. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. elsevier.

[4] Abuteen, E., 2024. Solving Fractional Riccati Differential equation with Caputo-Fabrizio fractional derivative. European Journal of Pure and Applied Mathematics, 17(1), pp.372-384.

[5] Kumar, S., Shaw, P.K., Abdel-Aty, A.H. and Mahmoud, E.E., 2024. A numerical study on fractional differential equation with population growth model. Numerical Methods for Partial Differential Equations, 40(1), p.e22684.

[6] Jassim, H.K. and Abdulshareef Hussein, M., 2023. A new approach for solving nonlinear fractional ordinary differential equations. Mathematics, 11(7), p.1565.

[7] Rasmussen, A., Wyller, J. and Vik, J.O., 2011. Relaxation oscillations in spruce–budworm interactions. Nonlinear Analysis: Real World Applications, 12(1), pp.304-319.

[8] Chen, W., Zhang, X.D. and Korošak, D., 2010. Investigation on fractional and fractal derivative relaxation-oscillation models. International Journal of Nonlinear Sciences and Numerical Simulation, 11(1), pp.3-10.

[9] Izadi, M., 2022. A computational algorithm for simulating fractional order relaxation–oscillation equation. SeMA Journal, 79(4), pp.647-661.

[10] Arora, G., Pant, R., Emadifar, H. and Khademi, M., 2023. Numerical solution of fractional

relaxation–oscillation equation by using residual power series method. Alexandria Engineering Journal, 73, pp.249-257.

[11] Gülsu, M., Öztürk, Y. and Anapalı, A., 2013. Numerical approach for solving fractional relaxation–oscillation equation. Applied Mathematical Modelling, 37(8), pp.5927-5937.

[12] Hamarsheh, M., Ismail, A. and Odibat, Z., 2015. Optimal homotopy asymptotic method for solving fractional relaxation-oscillation equation. Journal of Interpolation and Approximation in Scientific Computing, 2, pp.98-111.

[13] Chanchlani, L., Agrawal, M., Pandey, R.M., Purohit, S.D. and Suthar, D.L., 2023. Applications of Elzaki decomposition method to fractional relaxation-oscillation and fractional biological population equations. Applied Mathematics in Science and Engineering, 31(1), p.2154766.

[14] Liang, S., Wu, R. and Chen, L., 2015. Laplace transform of fractional order differential equations. Electron. J. Differ. Equ, 139, p.2015.

[15] AKGÜL, A. and ÖZTÜRK, G., 2023. Application of the sumudu transform to some equations with fractional derivatives. Sigma Journal of Engineering and Natural Sciences, 41(6), pp.1132-1143.

[16] Pant, R., Arora, G. and Emadifar, H., 2024. Elzaki residual power series method to solve fractional diffusion equation. Plos one, 19(3), p.e0298064.

[17] Shah, R., Saad Alshehry, A. and Weera, W., 2022. A semi-analytical method to investigate fractional-order gas dynamics equations by Shehu transform. Symmetry, 14(7), p.1458.

[18] Qazza, A., Burqan, A. and Saadeh, R., 2021. A new attractive method in solving families of fractional differential equations by a new transform. Mathematics, 9(23), p.3039.

[19] Khalouta, A., 2023. A new general integral transform for solving Caputo fractional-order differential equations. International Journal of Nonlinear Analysis and Applications, 14(1), pp.67-78.

[20] Jafari, H., 2021. A new general integral transform for solving integral equations. Journal of Advanced Research, 32, pp.133-138.

[21] Kumar, R., Chandel, J. and Aggarwal, S., 2022. A new integral transform “Rishi Transform” with application. Journal of Scientific Research, 14(2), pp.521-532.

[22] Aggarwal, S., Kumar, R. and Chandel, J., 2023. Solution of Linear Volterra Integral Equation of Second Kind via Rishi Transform. Journal of Scientific Research, 15(1).

[23] Thakur, D. and Thakur, P.C., Rishi Transform for Solving Second Kind Linear Volterra Integral Equations.

[24] Aggarwal, S., Kumar, R. and Chandel, J., 2022. Exact solution of non-linear Volterra integral equation of first kind using Rishi transform. Bulletin of Pure & Applied Sciences-Mathematics and Statistics, 41(2), pp.159-166.

[25] Aggarwal, S., Kumar, R. and Chandel, J., 2023. Application of Rishi transform to the solution of nonlinear Volterra integral equation of the second kind. Bulletin of Pure & Applied Sciences-Mathematics, (2).

[26] Turab, A., Hilmi, H., Guirao, J.L., Jalil, S., Chorfi, N. and Mohammed, P.O., 2024. The Rishi Transform method for solving multi-high order fractional differential equations with constant coefficients. AIMS Mathematics, 9(2), pp.3798-3809.

[27] Aggarwal, S., 2022. Solution of the model of the bacteria growth via Rishi transform. Journal of Advanced Research in Applied Mathematics and Statistics, 7(1&2), 5-11.

[28] Dimitrov, Y., 2016. Higher-order numerical solutions of the fractional relaxation-oscillation equation using fractional integration. arXiv preprint arXiv:1603.08733.

[29] Rani, D., Mishra, V. and Cattani, C., 2019. Numerical inverse Laplace transform for solving a class of fractional differential equations. Symmetry, 11(4), p.530.

[30] Qazza, A., Burqan, A. and Saadeh, R., 2021. A new attractive method in solving families of fractional differential equations by a new transform. Mathematics, 9(23), p.3039.

[31] Kazem, S., 2013. Exact solution of some linear fractional differential equations by Laplace transform. International Journal of nonlinear science, 16(1), pp.3-11.