An Iterative method for solving the Partial integro- differential Equations

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Abstract: In this paper, we purpose and apply the parametric iteration method (PIM) for the numerical solution of partial integro- differential equations (PIDEs).The approximate solutions arise according to the auxiliary parameter h. The solution process is illustrated by some of examples. The results of PIM have compared with the exact solution for illustration of simplicity and efficiency of PIM. The convergence of proposed method has been provided.

Keywords: parametric iteration method, approximation solution, partial integro-differential equation,

1. Introduction

Partial integro differential equations appear in the many various fields. For example, they arise in modeling of heat conduction in materials with memory; population dynamics, viscoelasticity and theory of nuclear reactors [1-6].Also some of finance mathematics problems are modeled as a Partial integro differential equation and some of linear system arise of partial differential equation and integral equation [7]. So far various numerical schemes are proposed for solving PIDE [1,8-11]. In this paper, we purpose a parametric iteration method for solving partial integro differential equations. Now, we consider partial integro-differential equation as following:

$$a(x)\frac{\partial^n u}{\partial t^n}+b(x)\frac{\partial^n u}{\partial x^n}=g(x,t)+c(x)u+\lambda\int_0^t k(x,t,s)u(x,s)ds$$

(1)

(with prescribed conditions) , where $a(x),b(x),c(x)$ and $g(x,t)$ are known functions and $u(x,t)$ is an unknown function. In this work the numerical solution of (1) provides by PIM when $g,k,u,$ are continuous. PIM provides a sequence of iterations for the solution of PIDE. Some examples are approximated by PIM and the results are compared with the exact solution.

2. The basic idea of the PIM

In this section, the basic concepts the PIM are described for the PIDE. Then the convergence analysis is discussed.

2.1. PIM

The PIM provides the solution of Eq. (1) as a sequence of approximations. For this purpose we assume L is the continuous operator on $C[a,b]$ respectively. To describe the basic idea of the PIM, we consider Eq. (1) as follows:

$$L(u) = g(x,t),$$

(2)

where L has the property $L(g) \equiv 0$ when $g \equiv 0$. That L is an auxiliary linear operator with respect to u, and $g(x,t)$ is the known continuous function.

Now, we construct a family of iterative processes for Eq. (1) as follows:

$$L[u_{k+1}(x)-u_k(x)] = hH(x,t)A[u_k(x,t)],$$

(3)

where,

$$A[u_k(x,t)] = a\frac{\partial^n u}{\partial t^n}+b\frac{\partial^n u}{\partial x^n}-cu-\lambda\int_0^t k(t,s)u(x,s)ds-g(x,t)$$

$(k = 0,1,\dots)$,   (4)

(with prescribed conditions) ,and $u_0(x,t)$ is the initial guess which can be chosen arbitrary but

the suitable selection is effective for rate of convergence [12],or it can also be solved from its corresponding linear homogeneous equation $L[u_0(x,t)]=0$ or linear non homogeneous equation $L[u_0(x)]=g(x)$ and using of prescribed conditions. The parameter $h\neq 0$ and the function $H(x,t)\neq 0$ denote the so-called auxiliary parameter and auxiliary function. Also we have the great freedom to choose the auxiliary linear operator L, the auxiliary parameter h, the auxiliary function $H(x,t)$ and the initial approximation $u_0(x,t)$. Therefore, if the successive approximations $u_k(x,t)$, $k\geq 0$ obtained by PIM in terms of the auxiliary parameter h then exact solution may be given by $u(x,t)=\lim_{k\to\infty}u_k(x,t)$.

Now we let V be the solution space of Eq. (1) and $\{\phi_{ij}(x,t):\phi_{ij}(x,t)\in V, i,j=0,1,...\}$ denote the set of basis functions

that $\phi_{ij}(x,t)=\varphi_i(x)\varphi_j(t), (i,j=0,1,...)$. Hence we can represent the solution in the

series $u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}\varphi_i(x)\varphi_j(t)$, where c_{ij} are

real coefficient. The auxiliary linear operator L, the initial approximation $u_0(x,t)$ and the auxiliary function $H(x,t)$ should be chosen in such a way that all solutions of the corresponding PIM equation (4) exist and can be expressed by this set of base functions. Now we choose use the following the base functions as basis for solution space i.e.
$\{(x-a)^m\mid m=0,1,...\}$, therefore,

$\phi_{ij}(x,t)=\phi_i(x)\phi_j(t)=(x-a)^i(t-a)^j, (i,j=0,1,...)$

(5)

Therefore we consider

$$u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}\,\phi_{ij}(x,t)$$

(6)

where c_{ij} be determined. Now, for solving Eq. (1) we set the auxiliary operator L as the follows:

$$L[u(x,t)]=\frac{\partial^n u}{\partial t^n}$$

(7)

Now we truncate the infinite series (6) as below:

$$u(x,t)\approx\sum_{i=0}^{N}\sum_{j=0}^{M}c_{ij}\,\phi_{ij}(x)$$

(8)

If we set $N=2, M=2$ then we can consider the initial guess as following:

$$u_0(x,t)=\sum_{i=0}^{2}\sum_{j=0}^{2}c_{ij}\,\varphi_i(x)\varphi_j(t)$$

(9)

That with due attention to $L[u_0(x,t)]=g(x,t)$ and prescribed conditions

$c_{00},c_{01},c_{02},c_{10},c_{11},c_{12},c_{20},c_{21},c_{22}$ will be determined. Also we set $H(x,t)=1$, and we use the PIM for the computation approximation solutions of (1).

2.2 The h- curve and the valid region of h

Assume that we obtain a family of solution series in the auxiliary parameter h by means of PIM. We consider this solution as a function in terms of h, x ,t then we derive (once or more) this function respect to x,t in $x=\alpha, t=\beta$ that $\alpha,\beta\in[a,b]$ i.e. we let $y=G(x,t,h)$ that y is the solution of (1) then we set

$$\Gamma=\frac{\partial^{n+m}G}{\partial x^n\partial t^m}\big|_{x=\alpha,t=\beta},\quad \alpha,\beta\in[a,b],\quad (n,m=1,2,..)$$

(10)

therefore Γ will be in terms of h, now we plot curve of Γ, according to these h curves, it is easy to discover the valid region of h, which correspond to the line segments nearly parallel to the horizontal axis. This region is called valid region of h that we note it with R_h. We ensure the solution series converge for the any $h\in R_h$.

2.3 The convergence of PIM
Definition2.3.1: for $\alpha,\beta\in Z_+^n$
and $\varphi\in C^{\infty}(R^n)$, $p_{\alpha,\beta}(\varphi)=\sup_{x\in R^n}\left|x^{\alpha}\partial^{\beta}\varphi(x)\right|$ is
defined that we show it as $\|\varphi\|_{\alpha,\beta}$ [13].

Definition2.3.2: the Schwartz space $S(R^n)$ or S, consist of all functions $\varphi \in C^\infty(R^n)$ such that $\|\varphi\|_{\alpha,\beta}$ is finite for pair of multi-indices $\alpha, \beta \in Z_+^n$.[13].

Proposition: Polynomials in R^n are Schwartz functions.[13]

Lemma: for each $\alpha \in Z_+^n$ the partial differential operator $\partial^\alpha : S \to S$ is continuous linear operator in S.
Proof:[13].

Now, we set $K = \int_0^t k(x,t,s)ds$ and we rewrite Eq. (1) as the follows:

$$a\frac{\partial^n u}{\partial t^n} + b\frac{\partial^n u}{\partial x^n} = g(x,t) + cu + \lambda K u(x,s)$$

(11)

Theorem (convergence): Let K, L and ∂^n be continuous operators in S. It is proved that the sequence $\{u_k(x)\}$- where $u_k(x)$ is produced by the parametric iteration formulation of (5) - converges to the exact solution of (1).

Proof. If the sequence $u_k(x,t)$ converges, we can write

$$U(x,t) = \lim_{k \to \infty} u_k(x,t)$$

(12)
Then

$$U(x,t) = \lim_{k \to \infty} u_{k+1}(x,t)$$

(13)
With attention to (9), (11), (12)

$$\lim_{k \to \infty} L[u_{k+1}(x,t) - u_k(x,t)] = L[\lim_{k \to \infty} u_{k+1}(x,t) - \lim_{k \to \infty} u_k(x,t)] = 0$$

(14)

According to (5) we have

$$L\lim_{k \to \infty}[u_{k+1}(x,t) - u_k(x,t)] = \lim_{k \to \infty} hH(x)A[u_k(x,t)] = 0$$

(15)

Since $h \neq 0$ and $H(x,t) \neq 0$, we obtain

$$\lim_{k \to \infty} A[u_k(x,t)] = 0$$

(16)

Thus according to continuity of the operators K and ∂^n, (15) gives

$$\lim_{k \to \infty} A[u_k(x)] = \lim_{k \to \infty}[a\frac{\partial^n}{\partial t^n}u_k(x,t) + b\frac{\partial^n}{\partial x^n}u_k(x,t) - g(x,t) - \lambda K u_k^n(x,s)]$$

$$= a\lim_{k \to \infty}\frac{\partial^n}{\partial t^n}u_k(x,t) + b\lim_{k \to \infty}\frac{\partial^n}{\partial x^n}u_k(x,t) - \lim_{k \to \infty}g(x,t) - \lambda K\lim_{k \to \infty}u_k(x,s)$$

$$= a\frac{\partial^n}{\partial t^n}(\lim_{k \to \infty}u_k(x,t)) + b\frac{\partial^n}{\partial x^n}(\lim_{k \to \infty}u_k(x,t)) - \lim_{k \to \infty}g(x,t) - \lambda K(\lim_{k \to \infty}u_k(x,s))$$

$$= a\frac{\partial^n}{\partial t^n}(U(x,t)) + b\frac{\partial^n}{\partial x^n}(U(x,t)) - g(x,t) - \lambda K(U(x,s))$$

(17)
and (16), (17) the following term is obtained.

$$a\frac{\partial^n}{\partial t^n}(U(x,t)) + b\frac{\partial^n}{\partial x^n}(U(x,t)) - g(x,t) - \lambda K(U(x,s)) = 0$$

Hence, $U(x,t)$ should be the exact solution of the Eq. (1).

3. Illustrative Examples

In this section, we use the PIM for solving the partial integro- differential equations and compare the results with the exact solution to show efficiency of PIM.

Example 3.1: We consider the partial integro- differential equations with the exact solution $u(x,t) = xt$, as the following:

$$xu_x = u_{tt} + x\sin t + \int_0^t \sin(t-s)u(x,s)ds,$$

$0 \leq x \leq t \leq 1$
With the conditions
$u(x,0) = 0,\quad u(1,t) = t,\quad u_t(x,0) = x$
For solving this equations by PIM we define,

$$L[u(x,t)] = \frac{\partial^2 u}{\partial t^2},$$

(18)
Then we set initial approximation from (9) as

$$u_0(x,t) = \sum_{i=0}^{2}\sum_{j=0}^{2}c_{ij}\,\varphi_i(x)\varphi_j(t) \quad \text{Now for}$$

determining coefficients we should use $L[u_0(x,t)] = g(x,t)$ that $g(x,t) = x\sin t$ and conditions (). Therefore the coefficients are obtained as below:

$c_{11} = 1$ and other coefficients are zero, thus the initial guess is $u_0(x,t) = xt$

With using (3), (14), the iterative scheme for Eq. (13) is as follows:

$$u_{k+1}(x,t) = u_k(x,t) + h\left(-u_k(x,t) + x\sin t + \int_0^t\int_0^s (x\frac{\partial}{\partial x}u_k(x,v) - \int_0^v (\sin(v-r)u_k(x,r)dr)dvds\right), k = 0,1,\dots$$

For this example the exact solution is achieved i.e. $u_k(x,t) = xt, (k = 0,1,\dots)$. Also the

valid region of h i.e. R_h was presented in figure 1.

Example 3.2: As a second example, we consider the partial integro- differential of the second kind with exact

solution $u(x,t) = e^x \cos t$, as the following:

$$u_{tt} = u_x + 2 \int_0^t (t-s)u(x,s)ds - 2e^x,$$

$0 \leq x \leq t \leq 1$

Such that,

$u(x,0) = x^2, \quad u_t(x,0) = 1, \quad u(0,t) = t, \quad u_x(0,t) = 0.$

Similar to example (1) we set;

$L[u(x,t)] = \dfrac{\partial^2 u}{\partial t^2}$, and $g(x,t) = -2e^x$, then we

consider initial solution as following:

$$u_0(x,t) = \sum_{i=0}^{2} \sum_{j=0}^{2} c_{ij}\,\varphi_i(x)\varphi_j(t)$$

Therefore by using $L[u_0(x,t)] = g(x,t)$ and conditions ().The coefficients are obtained as below:

$c_{00} = 1, \ , c_{02} = -1, c_{10} = 1, \ c_{12} = -1,$ And other coefficients are zero.

If we let $H(x,t) = 1$ then we obtain iteration formula from (3), (14) as following:

$$u_{k+1}(x,t) = u_k(x,t) + h\left(u_k(x,t) + e^x(-1+t^2) + \int_0^t\int_0^s (-\frac{\partial}{\partial x}u_k(x,v) - 2(\int_0^v (v-r)u_k(x,r)dr)dvds) \right), k=0,1,\ldots$$

The obtained result for 8th iteration was shown in table 2; also the valid region of h i.e. R_h was presented in figure 2.

4. Numerical Results and Figures

In the previous section obtained exact solution for example 1. thus for example 2, we present the numerical method in table 1 and plot the h-curve for two examples. All of computation was done with Maple15.

Table 1: Numerical results of example2,

x_i	t_i	$u_8(x,t)$	$u_{exact}(x,t)$
0	0	1	1
0.125	0.125	1.124307249	1.124307252

0.25	0.25	1.244108176	1.244108176
0.375	0.375	1.353880601	1.353880601
0.5	0.5	1.446889037	1.446889037
0.625	0.625	1.515078567	1.515078569
0.75	0.75	1.548985348	1.548985348
0.875	0.875	1.537671528	1.537671527
1	1	1.468693940	1.468693940

Fig1: valid region of h for example 2: $R_h = [-1.8, 0)$
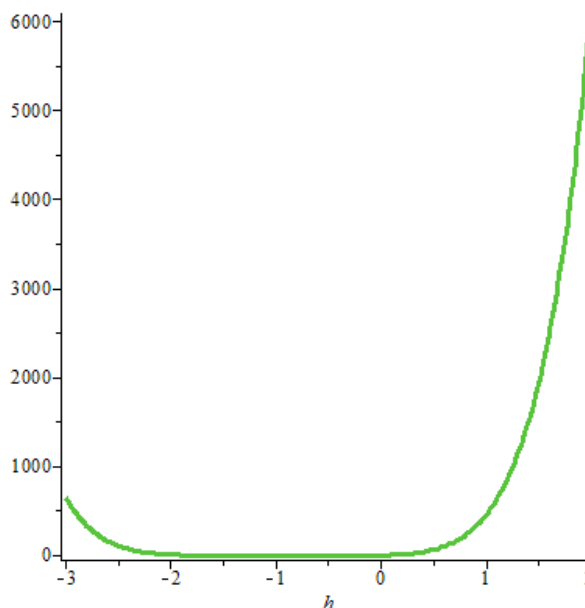
5. Conclusion

In this paper, we applied parametric iteration for solving partial integro- differential equations of the second. The results compared with the corresponding values of exact solutions show that PIM is simplicity in practice and highly accurately for partial integro- differential equations. Also we present the convergence of PIM for solutions of partial integro- differential equations in the valid region of h (R_h). More ever, if we increase iterations by PIM scheme, leads to more accurate solutions.

6. References

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