The Weibull-Epsilon Distribution: its Properties and Applications

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Abstract:- In this study, a new probability distribution called Weibull-epsilon distribution is introduced. Some of the basic statistical properties have been described. The density and hazard rate function plots exhibit unimodal and bathtub shapes, respectively.. Its varying shapes show it is possible to use in modeling different data generating processes. It is fitted to two real-life datasets and provided better fit when compared with the fit of the Weibull distribution to the same datasets. Particularly for the carbon fiber breaking stress dataset, the new distribution is a better choice over the three parameter Cauchy-Weibull-Logistic distribution, with 1.62 % gain in likelihood per data point. It, therefore, holds a good prospect for practical applications in engineering, social and biological systems, mortality studies in demography and renewable energy modeling.

Keywords: hazard rate function, Weibull-G distributions, Cauchy-Weibull-Logistic distribution, solar radiation model, breaking stress of carbon fibers data

1 Introduction

Of recent, many new probability distribution functions are being introduced in the literature with applications to real life datasets. This is motivated partly by the several methods of generating probability distributions that are compatible with the requirements of the axioms of probability; and partly by the need for new distributions that are flexible enough to provide better fit to a wide variety of real-life data generation processes

Thus, many lifetime probability distributions are constructed from standard distributions and are applicable in renewable energy modeling, reliability study in biological and mechanical systems, and in many other real-life applications. Some relevant studies can be found in [1, 2, 3, 4, 5], for examples. The newly constructed distributions are found to be more flexible than the parent distributions from which they were created [3]. For instance, the family of distributions generated from the parent epsilon distribution [6] are found to assume more flexible shapes [1, 2, 7] typical of the characteristics of, and provide better fit to, many real-life random processes.

Many methods of generating probability distribution functions using continuous life time distributions are found in the literature. For instance, gamma-G [8], beta-G [3], Weibull-G [9], and Kumaraswamy-G [10]. These have been used to generate probability distributions both in the discrete and continuous variable cases. Addition of parameter(s) through exponentiation and frailty process is another method of generating new probability distributions [11, 12]. Many of these distributions have been applied to real life datasets in areas such as engineering, actuarial, environmental and medical sciences, biological system studies, demography, economics, finance and insurance; see for example [13, 14, 15, 16]. A comprehensive survey of some new distributions and their applications can be found in [17, 18, 19].

The Weibull distribution is a very important distribution in practice. Its application in generating probability distributions originated with the work of Gurvich et al [20]. In this study, the focus is on generating the Weibull-epsilon (WE) distribution with the view to applying it to real life datasets. This is new.

2 Family of Weibull-G distributions

The family of Weibull-G distributions proposed by Gurvich et al [20] has a general cumulative distribution function given by

$$G(x;\ \alpha,\xi) = 1 - exp\left(-\alpha \mathrm{H}(x,\xi)\right) \tag{1}$$

where $\alpha > 0, x \in \mathbb{R}^{+}$ and ξ is the parameter vector of the parent distribution. $\mathrm{H}(x,\xi)$ is a monotonically increasing function. The density function is given by

$$g(x;\ \alpha,\xi) = \alpha h(x,\xi) exp\left(-\alpha \mathrm{H}(x,\xi)\right) \tag{2}$$

where $\mathrm{h}(x,\xi)$ is the derivative of $\mathrm{H}(x,\xi)$.

Using the expression (1), Bourguignon et al [9] studied the properties of a Weibull generator of probability density functions, given by

$$F(x) = \alpha \int_{0}^{H[G(x;\boldsymbol{\gamma})]} t^{\alpha-1} e^{-t^{\alpha}} dt$$

$$= 1 - e^{-\{H[G(x;\boldsymbol{\gamma})]\}^{\alpha}} \tag{3}$$

The corresponding probability density function is given by

$$f(x) = \alpha h[G(x;\boldsymbol{\gamma})]\{H[G(x;\boldsymbol{\gamma})]\}^{\alpha-1} e^{-\{H[G(x;\boldsymbol{\gamma})]\}^{\alpha}} \tag{4}$$

The authors used $H[G(x;\boldsymbol{\gamma})] = \frac{G(x;\boldsymbol{\gamma})}{1-G(x;\boldsymbol{\gamma})}$ to transform the cumulative distribution function of the parent distribution, $G(x;\boldsymbol{\gamma})$, and $h[G(x;\boldsymbol{\gamma})]$ in (4) is its derivative.

Many distributions were generated and studied, for example, Weibull-uniform, Weibull-Weibull, Weibull-Burr II and Weibull-normal distributions [9]. There are other forms of the transformation functions that were used to generate many probability distributions, for example, the exponentiated T-X family [21]; the Weibull-gamma distribution [22]; a new Weibull-Pareto distribution [23], Weibull-inverted exponential [24], Weibull-Lindley [25], Weibull-Logistic [26], Weibull-Pareto [27] and Weibull-Dagum distributions [28].

3 The Weibull-Epsilon Distribution

The transformation $H[G(x;\boldsymbol{\gamma})] = \frac{G(x;\boldsymbol{\gamma})}{1-G(x;\boldsymbol{\gamma})}$ is used here with the parent distribution as the epsilon distribution with $\boldsymbol{\gamma}' = (\lambda,\delta)$, given by

$$G(x;\lambda,\delta) = 1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta} \tag{5}$$

In which case,

$$H[G(x;\boldsymbol{\gamma})] = \frac{1 - \left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}}{\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}}$$

$$= \left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta} - 1 \tag{6}$$

and

$$h[G(x;\boldsymbol{\gamma})] = \lambda\left(\frac{\delta^2}{\delta^2-x^2}\right)\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta} \tag{7}$$

Substituting (6) and (7) into (3) and (4) gives the three parameter Weibull-epsilon cumulative distribution and probability density functions, respectively, as

$$F(x;\alpha,\delta,\lambda) = 1 - exp\{-[z-1]^{\alpha}\} \tag{8}$$

and

$$f(x;\alpha,\delta,\lambda) = \alpha\lambda\frac{\delta^2}{\delta^2-x^2} z[z-1]^{\alpha-1} exp\{-[z-1]^{\alpha}\} \tag{9}$$

where $z = \left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}$, $0 < x < \delta, \alpha > 0, \delta > 0$ and $\lambda > 0$.

Plots of the Weibull-epsilon probability density function at various parameter values are given in Figure 1 of Appendix 1.

Lemma

The density function $f(x)$ (equation 9) is a true probability density function.

Proof

It suffices to show that, $0 \le F(x) \le 1$. Note that

$$F(x) = P(X \le x)$$

$$= \int_0^x f(t)\, dt$$

$$= 1 - exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{\alpha}\right\}$$

So that,

$$F(0) = 1 - exp\left\{-\left[\left(\frac{0+\delta}{\delta-0}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{\alpha}\right\}$$

$$= 1 - exp\{-[1-1]^{\alpha}\}$$

$$= 1 - exp\{-0\}$$

$$= 1 - 1 = 0$$

and

$$F(\delta) = 1 - exp\left\{-\left[\left(\frac{\delta+\delta}{\delta-\delta}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{\alpha}\right\}$$

$$= 1 - exp\{-[\infty - 1]^{\alpha}\}$$

$$= 1 - exp\{-\infty\}$$

$$= 1 - 0 = 1$$

Since $F(x)$ is a strictly increasing function and $0 < x < \delta$, the proof follows. ■

4 Hazard and mean residual life functions

4.1 The hazard rate function

For the Weibull-epsilon distribution, the hazard rate function can be obtained as

$$hrf(x) = \frac{f(x)}{S(x)}$$

$$= \frac{f(x)}{1 - F(x)}$$

$$= \frac{\alpha\lambda \frac{\delta^2}{\delta^2 - x^2} z[z-1]^{\alpha-1} exp\{-[z-1]^{\alpha}\}}{exp\{-[z-1]^{\alpha}\}}$$

$$= \alpha\lambda \frac{\delta^2}{\delta^2 - x^2} z[z-1]^{\alpha-1} \quad (10)$$

where $z = \left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}$, $0 < x < \delta$, $\alpha > 0$, $\delta > 0$ and $\lambda > 0$.

Plots of the Weibull-epsilon hazard rate function for varying parameter values are presented in Figure 2 of Appendix 1.

The corresponding cumulative hazard rate function is given by

$$H(x) = -\log exp\{-[z-1]^{\alpha}\}$$

$$= \left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{\alpha} \quad (11)$$

4.2 Mean residual life (MRL) function

The mean residual life (MRL) function is the second most important function used to represent life time distributions. It determines the remaining life of a component or unit of a system that has survived up to a particular point in time. That is, it measures the life expectancy of a component or unit that has survived up to time x. It is given, for a random variable X distributed according to the Weibull-epsilon distribution, by

$$\mu(x) = E(X - x | X > x)$$

$$= \frac{1}{S(x)} \int_x^{\infty} S(u)\, du$$

$$= \frac{\delta^2 - x^2}{\alpha\lambda\delta^2} \left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta} \left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{1-\alpha} \quad (12)$$

The full evaluation of the integral is given in Appendix 2. Plots of the mean residual life function of the Weibull-epsilon distribution at varying parameter values are presented in Figure 3 of Appendix 1.

5 Quantile function

The quantile function is defined by the relation

$$X_p = {F_X}^{-1}(p) \quad (13)$$

where $0 < p < 1$

For a random variable, X, distributed according to the Weibull-epsilon distribution, the quantile function is given by

$$Q(p) = \delta \frac{\gamma - 1}{\gamma + 1} \quad (14)$$

where $\gamma = \left\{1 + [-\log(1-p)]^{\frac{1}{\alpha}}\right\}^{\frac{2}{\lambda\delta}}$, $0 < p < 1$

6 Distribution of order statistics

Let $X_{(r)}, r = 1, 2, \dots, n$ be order statistics of a random variable X characterized according to the Weibull-epsilon distribution.

The distribution of the r^{th} order statistic, $1 \leq r \leq n$, is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!\,(n-r)!} g(x) G(x)^{r-1} [1 - G(x)]^{n-r} \quad (15)$$

where $G(x)$ and $g(x)$ are Weibull-epsilon cumulative distribution and probability density functions, respectively. In most statistical applications interest is centered on the distribution of the extreme values. Thus, from (15) the distributions of the first and n^{th} order statistics are obtained by substituting $r = 1$ and $r = n$, respectively. That is,

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{n!}{(1-1)!\,(n-1)!} g(x) G(x)^{1-1} [1 - G(x)]^{n-1} \\ &= ng(x)[1 - G(x)]^{n-1} \\ &= n\alpha\lambda \frac{\delta^2}{\delta^2 - x^2} z[z-1]^{\alpha-1} exp\{-[z-1]^\alpha\} \\ &\qquad [exp\{-[z-1]^\alpha\}]^{n-1} \\ &= n\alpha\lambda \frac{\delta^2}{\delta^2 - x^2} z[z-1]^{\alpha-1} exp\{-[z-1]^{\alpha n}\} \qquad (16) \end{aligned}$$

$$\begin{aligned} f\left(x_{(n)}\right) &= \frac{n!}{(n-1)!\,(n-n)!} g(x) G(x)^{n-1} [1 - G(x)]^{n-n} \\ &= ng(x)G(x)^{n-1} \\ &= n\alpha\lambda \frac{\delta^2}{\delta^2 - x^2} z[z-1]^{\alpha-1} exp\{-[z-1]^\alpha\}[1 \\ &\qquad - exp\{-[z-1]^\alpha\}]^{n-1} \qquad (17) \end{aligned}$$

where $z = \left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}$

7 Estimation of parameters

One of the problems of the construction of new probability distributions is that parameter estimates are not explicit, that is, there may not exist explicit expressions for the estimate of the parameters of the model. The normal distributions, for instance, has expressions for the estimates of its parameters in terms of the sample values of the random variables; it is not so with the distribution in this study. However, modern computers and packages enable easy estimation of the parameters of any true probability distribution. For example, the ***fitdistrplus*** package of R statistical programming language enables easy estimation of the parameters of any probability distribution when the density, distribution and quantile functions can be specified.

For this study, the density function (9), distribution function (8) and the quantile function (14) are specified, respectively, below.

$$\begin{aligned} dwepsilon &= function(x, a, k, d)\ a * k \\ &\quad * (d\hat{}2/(d\hat{}2 - x\hat{}2)) * (((x + d)/(d \\ &\quad - x))\hat{}(k * d/2)) * ((((x + d)/(d - x))\hat{}(k \\ &\quad * d/2) - 1)\hat{}(a - 1)) * exp(-(((x \\ &\quad + d)/(d - x))\hat{}(k * d/2) - 1)\hat{}a) \end{aligned}$$

$$\begin{aligned} pwepsilon &= function(q, a, k.d)\ 1 - exp(-(((q \\ &\quad + d)/(d - q))\hat{}(k * d/2) - 1)\hat{}a) \end{aligned}$$

$$\begin{aligned} qwepsilon &= function(p, a, k, d)\ d * ((1 \\ &\quad + (-log(1 - p))\hat{}(1/a))\hat{}(2/(k \\ &\quad * d)) - 1)/((1 \\ &\quad + (-log(1 - p))\hat{}(1/a))\hat{}(2/(k * d)) - 1) \end{aligned}$$

The syntax for fitting Weibull-epsilon distribution to a dataset, say y, in ***fitdistrplus*** of R after the specification above is given by

$$\begin{aligned} fwy &= fitdist(y, \text{“}wepsilon\text{”}, start = list\,(a = a0, k \\ &\quad = k0, d = d0), method = \text{“}mle\text{”}) \end{aligned}$$

The default for method of estimation is maximum likelihood (*mle*) and does not need to be specified. Other methods include matching moments (*mme*), matching quantiles (*qme*) and maximum goodness of fit (*mge*). Any of these can replace *mle* as above.

8 Application

Here, the applicability of the Weibull-epsilon distribution is illustrated by fitting to two real life datasets. It is compared with the fit of the Weibull distribution for the first dataset and to the Weibull and Cauchy-Weibull-Logistic distribution.

8.1 Data

The Weibull-epsilon distribution is applied to two datasets to illustrate its flexibility in modeling diverse data generation processes. These are the 2015 daily solar radiation (MJ/m^2) record at Yola [16, 29] and the breaking stress of carbon fibers of length 50 mm, obtained from Almheidat et al [30].

8.2 Parameter Estimation

8.2.1 Solar Radiation Data

The result of fitting the Weibull and Weibull-epsilon distributions to the solar radiation dataset is presented in Table 1. The fitted densities superimposed over the histogram of the data are presented in Figure 4.

Table 1 Weibull and Weibull-epsilon fit result of solar radiation dataset

Dist.	Par.	Est.	Std. E	$-LL$ (AIC)	KS ($p\ val$)	Remark
Weibull	Shape	6.083	0.274	590.2	0.134	Poor fit
	Scale	6.380	0.057	(1184)	(0.000)	
Weibull -Epsilon	α	2.766	0.168	544.9	0.071	Good fit
	λ	0.080	0.002	(1096)	(0.063)	
	δ	8.515	0.157			

Dist. = Distribution, Par. = Parameter, Est. = Estimate, Std. E = Standard Error, $-LL$ = negative log-likelihood function value, KS = Kolmogorov-Smirnov statistic value, $p\ val$ = p-value

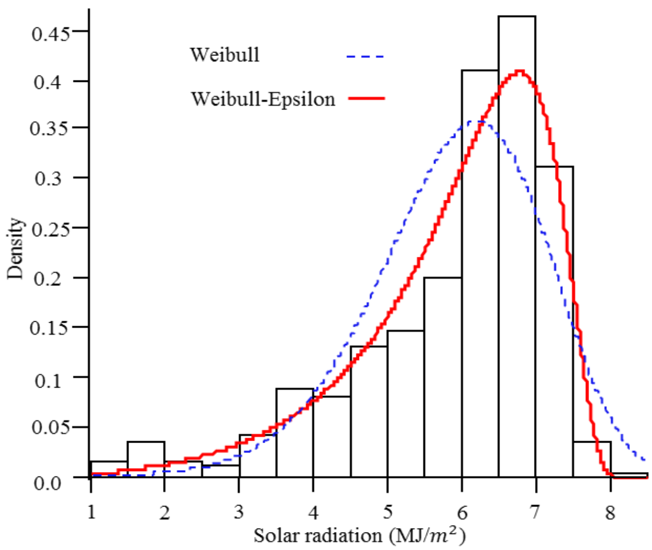

Figure 4 Weibull and Weibull-epsilon fit of solar radiation data

8.2.2 *Breaking Stress Data*

The results of fitting the Weibull and Weibull-epsilon distributions to the breaking stress of carbon fibers dataset is presented in Table 2 below, along with the fit result of the Cauchy-Weibull-Logistic distribution [30]. The fitted density curves superimposed over the histogram of the data are presented in Figure 5.

Table 2 3-P CW{L}*, Weibull and Weibull-epsilon fit result of breaking stress dataset

Dist.	Par.	Est.	Std. E	$-LL$ (AIC)	KS ($p\ val$)	Remark
3-P CW{L}	β	2.144	0.722	86.99	0.057	Good fit
	k	7.932	1.889	(180.0)	(0.983)	
	λ	2.953	0.108			
Weibull	Shape	3.441	0.331	86.08	0.082	Good fit
	Scale	3.062	0.115	(178.2)	(0.761)	
Weibull-Epsilon	α	2.514	0.245	85.93	0.088	Good fit
	λ	0.223	0.008	(177.8)	(0.655)	
	δ	3866	1532			

Dist. = Distribution, Par. = Parameter, Est. = Estimate, Std. E = Standard Error, $-LL$ = negative log-likelihood function value, KS = Kolmogorov-Smirnov statistic value, $p\ val$ = p-value, 3-P CW{L} = 3 parameter Cauchy-Weibull-Logistic distribution adopted from [30]

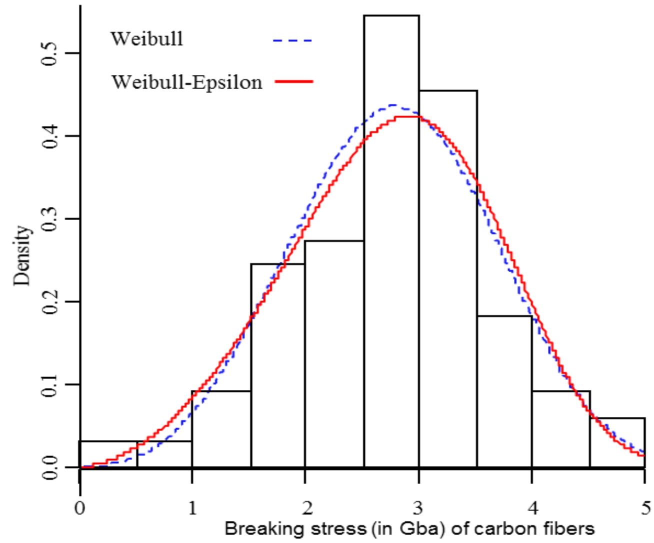

Figure 5 Weibull and Weibull-epsilon fit of breaking stress (in Gba) of carbon fibers

9 Discussion

The plots of the density function of the Weibull-epsilon distribution introduced in this study shows flexibility in taking any form of shape – J-shape, reversed J-shape, positively and negatively skewed shapes. This suggests that it can be used in modeling datasets from different data generating processes. The varying shapes of the plots of its hazard rate function also show it has a potential for modeling in lifetime and reliability studies.

The results in Table 1 show that the Weibull-epsilon distribution provides a good fit to the solar radiation dataset while the Weibull distribution does not. This is depicted in Figure 4. This shows that the distribution can be applicable in modeling solar power for informed deployment of solar panels that will attain maximum energy generating efficiency. The likelihood gain per data point of the Weibull-epsilon distribution is 13.2 % over the Weibull distribution.

The results in Table 2 for the breaking stress of carbon fibers (Gba) data show that the parameter estimates of both the Weibull and Weibull-epsilon distributions are very small; that is, they are precise estimates. This implies that they are very close to their true parameter values. The overall goodness-of-fit results also show that both distributions are compatible with the data. The fit of the distribution to the data shown in Figure 5 gives a pictorial evidence that the models are compatible with the dataset.

Table 2 also shows that the Weibull-epsilon distribution fit performs better than the Weibull and Cauchy-Weibull-Logistic distributions with a percent gain in likelihood per data point of 0.23 % and 1.62 %, respectively.

10 Conclusion

A new probability distribution, called the Weibull-epsilon distribution, is constructed in this study. It is shown to be compatible with solar radiation and breaking stress of carbon fibers datasets. It brings about improvement in fit when compared with the Weibull distribution for the two datasets considered. It is also better than the Cauchy-Weibull-Logistic distribution in fit to the breaking stress of carbon fibers datasets.

The distribution can assume different shapes within its parameter range indicating it holds a good potential in modeling different lifetime data generating processes. The bathtub shapes of its hazard rate function also show the model can be used in reliability studies. This flexibility in shape creates avenue for further research interest in applications, particularly in reliability studies in engineering, social and biological systems, mortality studies in demography and renewable energy modeling.

This study also provide scope for further research on the properties of the distribution, particularly the derivation of moments, and using the distribution as a base for generating other distributions.

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Appendix 1

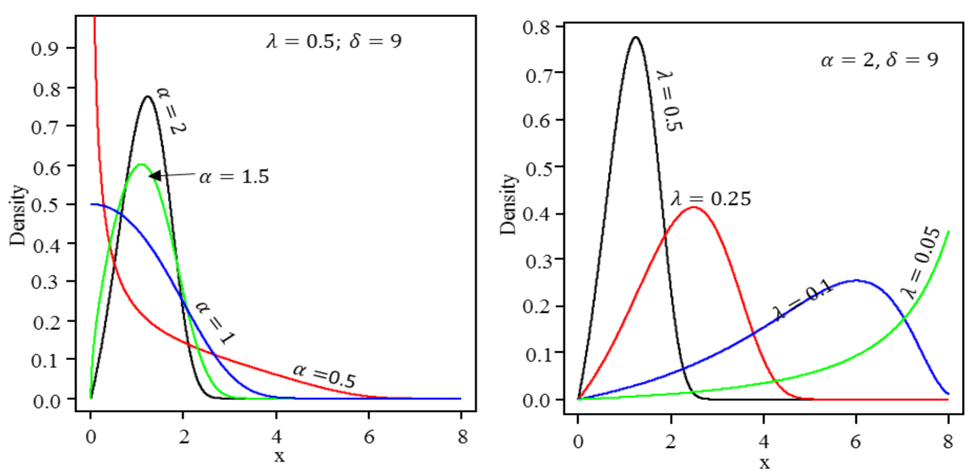

Figure 1 Weibull-Epsilon Density Plots at various parameter values

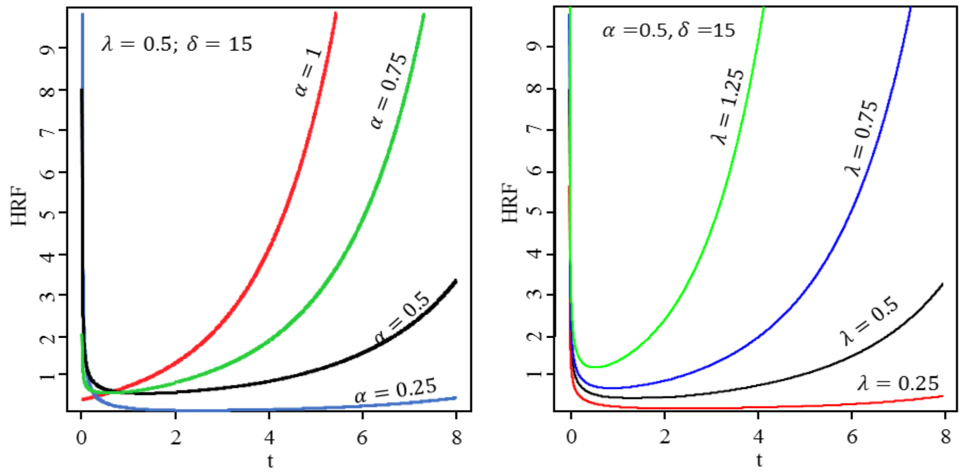

Figure 2 Weibull-epsilon hazard rate function plots

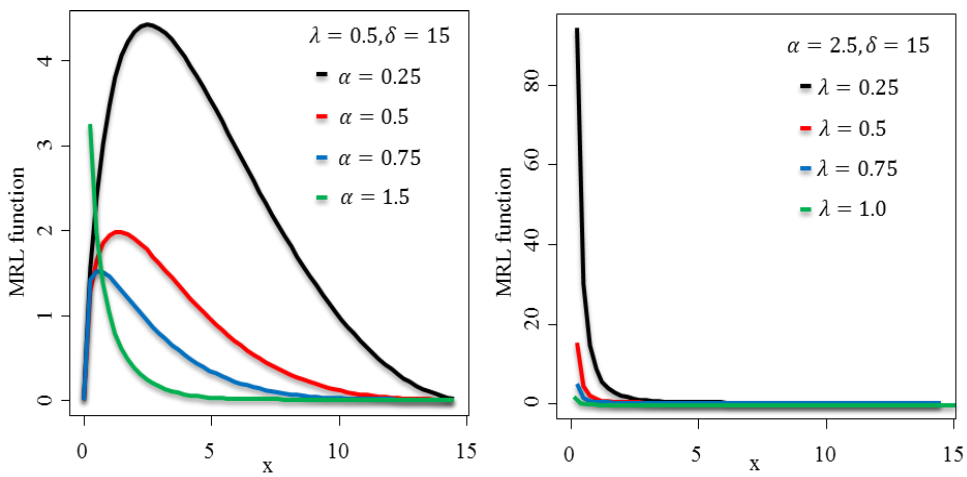

Figure 3 Mean Residual Life function of the Weibull-epsilon distribution at parameter values

Appendix 2

$$E(X - x|X > x) = \frac{1}{S(x)} \int_x^{\infty} S(u)\, du$$

$$= \frac{1}{exp\left\{-\left[\left(\frac{x+\delta}{\delta - x}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{\alpha}\right\}} \int_x^{\delta} exp\left\{-\left[\left(\frac{u+\delta}{\delta - u}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{\alpha}\right\} du$$

But

$$\int_x^{\delta} exp\left\{-\left[\left(\frac{u+\delta}{\delta - u}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{\alpha}\right\} du = -exp\left\{-\left[\left(\frac{u+\delta}{\delta - u}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{\alpha}\right\} \cdot \frac{1}{\frac{d}{du}\left[\left(\frac{u+\delta}{\delta - u}\right)^{\frac{\lambda}{2}\delta} - 1\right]^{\alpha}} \Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\cdot\frac{1}{\alpha\cdot\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha-1}\cdot\frac{\lambda}{2}\delta\cdot\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta-1}\cdot\frac{1\cdot(\delta-u)--1\cdot(u+\delta)}{(\delta-u)^2}}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\cdot\frac{1}{\alpha\lambda\frac{\delta}{2}\frac{2\delta}{(\delta-u)^2}\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta-1}\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha-1}}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\cdot\frac{1}{\alpha\lambda\frac{\delta}{2}\frac{2\delta}{(\delta-u)^2}\frac{\delta-u}{u+\delta}\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha-1}}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\cdot\frac{1}{\alpha\lambda\frac{\delta^2}{\delta^2-u^2}\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha-1}}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-u^2}{\alpha\lambda\delta^2}\left(\frac{u+\delta}{\delta-u}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{u+\delta}{\delta-u}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}\Bigg|_x^{\delta}$$

$$= -exp\left\{-\left[\left(\frac{\delta+\delta}{\delta-\delta}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-\delta^2}{\alpha\lambda\delta^2}\left(\frac{\delta+\delta}{\delta-\delta}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{\delta+\delta}{\delta-\delta}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$
$$+ exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$

$$= -exp\{-[\infty-1]^{\alpha}\}\cdot 0\cdot 0\cdot[\infty-1]^{1-\alpha} + exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$

$$= exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$

Therefore,

$$E(X-x|X>x) = \frac{1}{exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}}exp\left\{-\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{\alpha}\right\}\frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$

$$= \frac{\delta^2-x^2}{\alpha\lambda\delta^2}\left(\frac{x+\delta}{\delta-x}\right)^{-\frac{\lambda}{2}\delta}\left[\left(\frac{x+\delta}{\delta-x}\right)^{\frac{\lambda}{2}\delta}-1\right]^{1-\alpha}$$