On the Reciprocal Sums of Multiples-of-p-indexed Fibonacci Numbers

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Abstract: In this paper we derive some identities related to the reciprocal sums of multiples-of-p-indexed Fibonacci numbers.

Key–Words: Fibonacci numbers, reciprocal, floor function.

1 Introduction

The classical Fibonacci numbers, denoted by F_n, are generated from the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2),$$

with initial condition $F_0 = 0$ and $F_1 = 1$. Over the decades, numerous results on the properties and applications of the Fibonacci numbers have been reported [4].

Recently Ohtsuka and Nakamura [6] found interesting properties of the Fibonacci numbers and proved Theorem 1 below, where $\lfloor\cdot\rfloor$ indicates the floor function and $\mathbb{N}_e$ ($\mathbb{N}_o$, respectively) denotes the set of positive even (odd, respectively) integers.

Theorem 1 *Let $n \geq 1$. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, & \text{if } n \in \mathbb{N}_o, \end{cases} \tag{1}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \in \mathbb{N}_e; \\ F_{n-1}F_n, & \text{if } n \in \mathbb{N}_o. \end{cases} \tag{2}$$

After the work of Ohtsuka and Nakamura [6], diverse results in the same direction have appeared in the literature [1–3], [5], [7–9]. In particular, Wang and Zhang [8], [9] considered the even/odd-indexed Fibonacci numbers and the Fibonacci 3-subsequences. According to the results of [8], [9], Theorem 2 and Theorem 3 below hold.

Theorem 2 *Let $n \geq 1$. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{2k}} \right)^{-1} \right\rfloor = F_{2n} - F_{2n-2} - 1, \tag{3}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{2k}^2} \right)^{-1} \right\rfloor = F_{4n-2} - 1. \tag{4}$$

Theorem 3 *For $n \geq 1$, we have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{3k}} \right)^{-1} \right\rfloor = \begin{cases} 2F_{3n-2}, & \text{if } n \in \mathbb{N}_e; \\ 2F_{3n-2} - 1, & \text{if } n \in \mathbb{N}_o, \end{cases} \tag{5}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{3k}^2} \right)^{-1} \right\rfloor = \begin{cases} F_{3n}^2 - F_{3n-3}^2, & \text{if } n \in \mathbb{N}_e; \\ F_{3n}^2 - F_{3n-3}^2 - 1, & \text{if } n \in \mathbb{N}_o. \end{cases} \tag{6}$$

Before going further, we note that the following identities can be easily proved:

$$\begin{aligned} F_{4n-2} &= F_{2n}^2 - F_{2n-2}^2, \\ 2F_{3n-2} &= F_{3n} - F_{3n-3}. \end{aligned}$$

The purpose of this paper is to generalize Theorem 1–Theorem 3. More precisely, we obtain identities related to the numbers

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{pk}} \right)^{-1} \right\rfloor, \ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} \right)^{-1} \right\rfloor, \quad p = 1, 2, 3, \cdots.$$

2 Main Results

First, we present two lemmas which will be used to prove our main results.

Lemma 4 *[4]*

$$F_m F_n - F_{m+k} F_{n-k} = (-1)^{n-k} F_{m+k-n} F_k.$$

Lemma 5 below is obtained by letting $n = k + 1$ and interchanging the roles of k, m in Lemma 4.

Lemma 5

$$F_{m+k} = F_k F_{m+1} + F_{k-1} F_m.$$

Proposition 6

$$\frac{1}{F_{pn}-F_{pn-p}} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}}, \quad \textit{if } p\in\mathbb{N}_e \textit{ or } p,n\in\mathbb{N}_o; \quad (7)$$

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}} < \frac{1}{F_{pn}-F_{pn-p}}, \quad \textit{if } p\in\mathbb{N}_o \textit{ and } n\in\mathbb{N}_e. \quad (8)$$

Proof: Consider

$$\begin{aligned} X_1 &= \frac{1}{F_{pn} - F_{pn-p}} - \frac{1}{F_{pn+p} - F_{pn}} - \frac{1}{F_{pn}} \\ &= \frac{\hat{X}_1}{(F_{pn} - F_{pn-p})(F_{pn+p} - F_{pn})F_{pn}}, \end{aligned}$$

where, by Lemma 4

$$\begin{aligned} \hat{X}_1 &= F_{pn-p}F_{pn+p} - F_{pn}^2 \\ &= (-1)^{pn-p-1}F_p^2. \end{aligned}$$

If $p \in \mathbb{N}_e$ or $p, n \in \mathbb{N}_o$, then $X_1 < 0$ and

$$\frac{1}{F_{pn} - F_{pn-p}} - \frac{1}{F_{pn+p} - F_{pn}} < \frac{1}{F_{pn}}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{F_{pn} - F_{pn-p}} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}}, \quad \text{if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o.$$

Similarly, if $p \in \mathbb{N}_o$ and $n \in \mathbb{N}_e$, then $X_1 > 0$ and we obtain

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}} < \frac{1}{F_{pn} - F_{pn-p}}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e.$$

Hence the proof is completed. □

Proposition 7

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}} < \frac{1}{F_{pn}-F_{pn-p}-1}, \quad \textit{if } p\in\mathbb{N}_e \textit{ or } p,n\in\mathbb{N}_o. \quad (9)$$

Proof: Consider

$$\begin{aligned} X_2 &= \frac{1}{F_{pn} - F_{pn-p} - 1} - \frac{1}{F_{pn+2p} - F_{pn+p} - 1} \\ &\quad - \frac{1}{F_{pn}} - \frac{1}{F_{pn+p}} \\ &= \frac{\hat{X}_2}{(F_{pn} - F_{pn-p} - 1)(F_{pn+2p} - F_{pn+p} - 1)} \\ &\quad \times \frac{1}{F_{pn}F_{pn+p}}, \end{aligned}$$

where, by Lemma 4

$$\begin{aligned} \hat{X}_2 &= (F_{pn+2p} - 1)(F_{pn-p}F_{pn+p} - F_{pn}^2) \\ &\quad + (F_{pn-p} + 1)(F_{pn}F_{pn+2p} - F_{pn+p}^2) \\ &\quad - F_{pn-p}F_{pn} - F_{pn} - F_{pn+p} + F_{pn+p}F_{pn+2p} \\ &= (-1)^{pn-p-1}F_p^2(F_{pn+2p} - 1) \\ &\quad + (-1)^{pn-1}F_p^2(F_{pn-p} + 1) \\ &\quad - F_{pn-p}F_{pn} - F_{pn} - F_{pn+p} + F_{pn+p}F_{pn+2p}. \end{aligned}$$

Now assume that $p \in \mathbb{N}_e$. We can easily show that $\hat{X}_2 > 0$ for $n = 1$. Hence let $n \geq 2$. By Lemma 5, we have

$$\begin{aligned} \hat{X}_2 &= -F_p^2(F_{pn+2p} + F_{pn-p}) - F_{pn-p}F_{pn} - F_{pn} \\ &\quad - F_{pn+p} + F_{pn+p}F_{pn+2p} \\ &= (F_{pn+p} - F_p^2)F_{pn+2p} - F_p^2F_{pn-p} - F_{pn-p}F_{pn} \\ &\quad - F_{pn} - F_{pn+p} \\ &= (F_pF_{pn+1} + F_{p-1}F_{pn} - F_p^2)F_{pn+2p} \\ &\quad - F_p^2F_{pn-p} - F_{pn-p}F_{pn} - F_{pn} - F_{pn+p}. \end{aligned}$$

Since, for $n \geq 2$

$$F_pF_{pn+1} - F_p^2 \geq F_pF_{pn},$$

then

$$\begin{aligned} \hat{X}_2 &\geq (F_p + F_{p-1})F_{pn}F_{pn+2p} - F_p^2F_{pn-p} \\ &\quad - F_{pn-p}F_{pn} - F_{pn} - F_{pn+p} \\ &= (F_p + F_{p-1})F_{pn}(F_{2p}F_{pn+1} + F_{2p-1}F_{pn}) \\ &\quad - F_p^2F_{pn-p} - F_{pn-p}F_{pn} - F_{pn} - F_pF_{pn+1} \\ &\quad - F_{p-1}F_{pn} \\ &= (F_{2p}F_{pn} - 1)F_pF_{pn+1} + (F_pF_{2p-1}F_{pn} - 1)F_{pn} \\ &\quad + (F_{p-1}F_{2p}F_{pn+1} - F_{pn-1})F_{pn} \\ &\quad + F_{p-1}F_{2p-1}F_{pn} - F_p^2F_{pn-p} - F_{pn} \\ &> 0. \end{aligned}$$

If $p, n \in \mathbb{N}_o$, then

$$\hat{X}_2 = -F_p^2(F_{pn+2p} - F_{pn-p} - 2) - F_{pn-p}F_{pn}$$

$$\begin{aligned}
&-F_{pn} - F_{pn+p} + F_{pn+p}F_{pn+2p} \\
> \;& -F_p^2(F_{pn+2p} + F_{pn-p}) - F_{pn-p}F_{pn} - F_{pn} \\
&-F_{pn+p} + F_{pn+p}F_{pn+2p} \\
> \;& 0.
\end{aligned}$$

Consequently, we have

$$\frac{1}{F_{pn}} + \frac{1}{F_{pn+p}} < \frac{1}{F_{pn} - F_{pn-p} - 1} - \frac{1}{F_{pn+2p} - F_{pn+p} - 1},$$

from which we can obtain the inequality

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}} < \frac{1}{F_{pn} - F_{pn-p} - 1}, \quad \text{if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o,$$

and the proof is completed. □

Theorem 8 below follows from Proposition 6 and Proposition 7.

Theorem 8

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{pk}} \right)^{-1} \right\rfloor = F_{pn} - F_{pn-p} - 1, \quad \textit{if } p \in \mathbb{N}_e \textit{ and } n \geq 1. \tag{10}$$

Proposition 9

$$\frac{1}{F_{pn} - F_{pn-p} + 1} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}}, \quad \textit{if } p \in \mathbb{N}_o \textit{ and } n \in \mathbb{N}_e. \tag{11}$$

Proof: Consider

$$\begin{aligned}
X_3 &= \frac{1}{F_{pn} - F_{pn-p} + 1} - \frac{1}{F_{pn+2p} - F_{pn+p} + 1} \\
&\quad - \frac{1}{F_{pn}} - \frac{1}{F_{pn+p}} \\
&= \frac{\hat{X}_3}{(F_{pn} - F_{pn-p} + 1)(F_{pn+2p} - F_{pn+p} + 1)} \\
&\quad \times \frac{1}{F_{pn}F_{pn+p}},
\end{aligned}$$

where

$$\begin{aligned}
\hat{X}_3 &= \hat{X}_2 + 2(F_{pn} + F_{pn+p})(F_{pn-p} - F_{pn} \\
&\quad + F_{pn+p} - F_{pn+2p}) \\
&= (-1)^{pn-p-1}F_p^2(F_{pn+2p} - 1) \\
&\quad + (-1)^{pn-1}F_p^2(F_{pn-p} + 1) \\
&\quad - F_{pn-p}F_{pn} - F_{pn} - F_{pn+p} + F_{pn+p}F_{pn+2p} \\
&\quad + 2(F_{pn} + F_{pn+p})(F_{pn-p} - F_{pn} + F_{pn+p} \\
&\quad - F_{pn+2p}).
\end{aligned}$$

Here, $\hat{X}_2$ is as defined in the proof Proposition 7.

If $p \in \mathbb{N}_o$ and $n \in \mathbb{N}_e$, then, by Lemma 4 and Lemma 5

$$\begin{aligned}
\hat{X}_3 &= F_p^2F_{pn+2p} + F_{pn-p}F_{pn} - 2(F_{pn}^2 - F_{pn-p}F_{pn+p}) \\
&\quad + 2(F_{pn+p}^2 - F_{pn}F_{pn+2p}) - F_p^2F_{pn-p} \\
&\quad - F_{pn} - F_{pn+p} - F_{pn+p}F_{pn+2p} - 2F_p^2 \\
&= F_p^2F_{pn+2p} + F_{pn-p}F_{pn} + 2F_p^2 - F_p^2F_{pn-p} \\
&\quad - F_{pn} - F_{pn+p} - F_{pn+p}F_{pn+2p} \\
&= F_p^2(F_{2p}F_{pn+1} + F_{2p-1}F_{pn}) + F_{pn-p}F_{pn} \\
&\quad + 2F_p^2 - F_p^2F_{pn-p} - F_{pn} \\
&\quad - (F_pF_{pn+1} + F_{p-1}F_{pn}) \\
&\quad - (F_pF_{pn+1} + F_{p-1}F_{pn})(F_{2p}F_{pn+1} + F_{2p-1}F_{pn})
\end{aligned}$$

For the case where $p = 1$ and $n \in \mathbb{N}_e$, it is easily seen that $\hat{X}_3 < 0$. If $p \geq 3$ and $n \in \mathbb{N}_e$, then

$$\begin{aligned}
\hat{X}_3 &< (F_p^2F_{2p}F_{pn+1} - F_pF_{2p}F_{pn+1}^2) \\
&\quad + (F_p^2F_{2p-1}F_{pn} - F_{2p-1}F_{2p}F_{pn}F_{pn+1}) \\
&\quad + (F_{pn-p}F_{pn} - F_{p-1}F_{2p}F_{pn}F_{pn+p}) \\
&\quad + (2F_p^2 - F_p^2F_{pn-p}) \\
&< 0.
\end{aligned}$$

Hence we have

$$\frac{1}{F_{pn} - F_{pn-p} + 1} - \frac{1}{F_{pn+2p} - F_{pn+p} + 1} < \frac{1}{F_{pn}} + \frac{1}{F_{pn+p}}.$$

Repeatedly applying the above inequality, we obtain

$$\frac{1}{F_{pn} - F_{pn-p} + 1} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e,$$

and the proof is completed. □

From Proposition 6, Proposition 7 and Proposition 9, we obtain the following result.

Theorem 10 *Let $p \in \mathbb{N}_o$. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{pk}} \right)^{-1} \right\rfloor = \begin{cases} F_{pn} - F_{pn-p}, & \textit{if } n \in \mathbb{N}_e\,; \\ F_{pn} - F_{pn-p} - 1, & \textit{if } n \in \mathbb{N}_o. \end{cases} \tag{12}$$

Proposition 11

$$\frac{1}{F_{pn}^2 - F_{pn-p}^2} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2}, \quad \textit{if } p \in \mathbb{N}_e \textit{ or } p, n \in \mathbb{N}_o; \tag{13}$$

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} < \frac{1}{F_{pn}^2 - F_{pn-p}^2}, \quad \textit{if } p \in \mathbb{N}_o \textit{ and } n \in \mathbb{N}_e. \tag{14}$$

Proof: Consider

$$\begin{aligned} Y_1 &= \frac{1}{F_{pn}^2 - F_{pn-p}^2} - \frac{1}{F_{pn+p}^2 - F_{pn}^2} - \frac{1}{F_{pn}^2} \\ &= \frac{\hat{Y}_1}{(F_{pn}^2 - F_{pn-p}^2)(F_{pn+p}^2 - F_{pn}^2)F_{pn}^2}, \end{aligned}$$

where, by Lemma 4

$$\begin{aligned} \hat{Y}_1 &= F_{pn-p}^2 F_{pn+p}^2 - F_{pn}^4 \\ &= (-1)^{pn-p-1}(F_{pn-p}F_{pn+p} + F_{pn}^2). \end{aligned}$$

If $p \in \mathbb{N}_e$ or $p, n \in \mathbb{N}_o$, then $Y_1 < 0$ and

$$\frac{1}{F_{pn}^2 - F_{pn-p}^2} - \frac{1}{F_{pn+p}^2 - F_{pn}^2} < \frac{1}{F_{pn}^2}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{F_{pn}^2 - F_{pn-p}^2} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2}, \quad \text{if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o.$$

Similarly, if $p \in \mathbb{N}_o$ and $n \in \mathbb{N}_e$, then $Y_1 > 0$ and we obtain

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}} < \frac{1}{F_{pn} - F_{pn-p}}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e.$$

Hence the proof is completed. □

Proposition 12

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} < \frac{1}{F_{pn}^2 - F_{pn-p}^2 - 1}, \quad \text{if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o. \tag{15}$$

Proof: Consider

$$\begin{aligned} Y_2 &= \frac{1}{F_{pn}^2 - F_{pn-p}^2 - 1} - \frac{1}{F_{pn+2p}^2 - F_{pn+p}^2 - 1} \\ &\quad - \frac{1}{F_{pn}^2} - \frac{1}{F_{pn+p}^2} \\ &= \frac{\hat{Y}_2}{(F_{pn}^2 - F_{pn-p}^2 - 1)(F_{pn+2p}^2 - F_{pn+p}^2 - 1)} \\ &\quad \times \frac{1}{F_{pn}^2 F_{pn+p}^2}, \end{aligned}$$

where, by Lemma 4

$$\begin{aligned} \hat{Y}_2 &= (F_{pn}^2 + F_{pn-p}F_{pn+p})(F_{pn}^2 - F_{pn-p}F_{pn+p}) \\ &\quad - F_{pn-p}^2(F_{pn+p}^2 + F_{pn}F_{pn+2p}) \\ &\quad \times (F_{pn+p}^2 - F_{pn}F_{pn+2p}) \\ &\quad - F_{pn+2p}^2(F_{pn}^2 + F_{pn-p}F_{pn+p}) \\ &\quad \times (F_{pn}^2 - F_{pn-p}F_{pn+p}) \\ &\quad - (F_{pn+p}^2 + F_{pn}F_{pn+2p})(F_{pn+p}^2 - F_{pn}F_{pn+2p}) \\ &\quad + F_{pn+p}^2 F_{pn+2p}^2 - F_{pn-p}^2 F_{pn}^2 - F_{pn}^2 - F_{pn+p}^2 \\ &= (-1)^{pn-p} F_p^2 (F_{pn+2p}^2 + 1)(F_{pn}^2 + F_{pn-p}F_{pn+p}) \\ &\quad + (-1)^{pn-1} F_p^2 (F_{pn-p}^2 + 1)(F_{pn+p}^2 + F_{pn}F_{pn+2p}) \\ &\quad + F_{pn+p}^2 F_{pn+2p}^2 - F_{pn-p}^2 F_{pn}^2 - F_{pn}^2 - F_{pn+p}^2. \end{aligned}$$

Assume that $p \in \mathbb{N}_e$. Since

$$\begin{aligned} F_{pn+2p}^2 &> F_{pn}^2 + F_{pn-p}F_{pn+p}, \\ F_{pn+p}^2 F_{pn+2p}^2 &> F_{pn-p}^2 F_{pn}^2 + F_{pn}^2 + F_{pn+p}^2, \end{aligned}$$

then

$$\begin{aligned} \hat{Y}_2 &= F_p^2 (F_{pn+2p}^2 + 1)(F_{pn}^2 + F_{pn-p}F_{pn+p}) \\ &\quad - F_p^2 (F_{pn-p}^2 + 1)(F_{pn+p}^2 + F_{pn}F_{pn+2p}) \\ &\quad + F_{pn+p}^2 F_{pn+2p}^2 - F_{pn-p}^2 F_{pn}^2 - F_{pn}^2 - F_{pn+p}^2 \\ &> 0, \end{aligned}$$

and so $Y_2 > 0$ for $p \in \mathbb{N}_e$.

If $p, n \in \mathbb{N}_o$, then we also have $Y_2 > 0$.

Consequently, if $p \in \mathbb{N}_e$ or $p, n \in \mathbb{N}_o$, we have

$$\frac{1}{F_{pn}^2} + \frac{1}{F_{pn+p}} < \frac{1}{F_{pn} - F_{pn-p} - 1} - \frac{1}{F_{pn+2p} - F_{pn+p} - 1},$$

from which we obtain the inequality

$$\sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} < \frac{1}{F_{pn}^2 - F_{pn-p}^2 - 1}, \quad \text{if } p \in \mathbb{N}_e \text{ or } p, n \in \mathbb{N}_o,$$

and the proof is completed. □

From Proposition 11 and Proposition 12, we obtain the following the result.

Theorem 13

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} \right)^{-1} \right\rfloor = F_{pn}^2 - F_{pn-p}^2 - 1, \quad \text{if } p \in \mathbb{N}_e \text{ and } n \geq 1. \tag{16}$$

Proposition 14

$$\frac{1}{F_{pn}^2 - F_{pn-p}^2 + 1} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e. \tag{17}$$

Proof: Consider

$$\begin{aligned} Y_3 &= \frac{1}{F_{pn}^2 - F_{pn-p}^2 + 1} - \frac{1}{F_{pn+2p}^2 - F_{pn+p}^2 + 1} \\ &\quad -\frac{1}{F_{pn}^2} - \frac{1}{F_{pn+p}^2} \\ &= \frac{\hat{Y}_3}{(F_{pn}^2 - F_{pn-p}^2 + 1)(F_{pn+2p}^2 - F_{pn+p}^2 + 1)} \\ &\quad \times \frac{1}{F_{pn}^2 F_{pn+p}^2}, \end{aligned}$$

where

$$\begin{aligned} \hat{Y}_3 &= \hat{Y}_2 + 2(F_{pn}^2 + F_{pn+p}^2)(F_{pn-p}^2 - F_{pn}^2 + F_{pn+p}^2 \\ &\quad -F_{pn+2p}^2) \\ &= (-1)^{pn-p} F_p^2 (F_{pn+2p}^2 + 1)(F_{pn}^2 + F_{pn-p} F_{pn+p}) \\ &\quad +(-1)^{pn-1} F_p^2 (F_{pn-p}^2 + 1)(F_{pn+p}^2 + F_{pn} F_{pn+2p}) \\ &\quad +F_{pn+p}^2 F_{pn+2p}^2 - F_{pn-p}^2 F_{pn}^2 - F_{pn}^2 - F_{pn+p}^2 \\ &\quad +2(F_{pn}^2 + F_{pn+p}^2)(F_{pn-p}^2 - F_{pn}^2 + F_{pn+p}^2 \\ &\quad -F_{pn+2p}^2). \end{aligned}$$

Here, $\hat{Y}_2$ is as defined in the proof Proposition 12.

If $p \in \mathbb{N}_o$ and $n \in \mathbb{N}_e$, then $\hat{Y}_3 < 0$ and we have

$$\frac{1}{F_{pn}^2 - F_{pn-p}^2 + 1} - \frac{1}{F_{pn+2p}^2 - F_{pn+p}^2 + 1} < \frac{1}{F_{pn}^2} + \frac{1}{F_{pn+p}^2}.$$

Repeatedly applying the above inequality, we obtain

$$\frac{1}{F_{pn}^2 - F_{pn-p}^2 + 1} < \sum_{k=n}^{\infty} \frac{1}{F_{pk}^2}, \quad \text{if } p \in \mathbb{N}_o \text{ and } n \in \mathbb{N}_e,$$

and the proof is completed. □

From Proposition 11, Proposition 12 and Proposition 14, we obtain the following result.

Theorem 15 *Let $p \in \mathbb{N}_o$. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{pk}^2} \right)^{-1} \right\rfloor = \begin{cases} F_{pn}^2 - F_{pn-p}^2, & \text{if } n \in \mathbb{N}_e\,; \\ F_{pn}^2 - F_{pn-p}^2 - 1, & \text{if } n \in \mathbb{N}_o. \end{cases} \tag{18}$$

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