An extension result of the Mixed Convection Boundary Layer flow over a vertical permeable surface embedded in a Porous Medium

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Abstract: In this paper we are concerned with the solution of the third-order non-linear differential equation $f''' + ff'' + \beta f'(f' - 1) = 0$, satisfying the boundary conditions $f(0) = a \in \mathbb{R}$, $f'(0) = b < 0$ and $f'(t) \longrightarrow \lambda$, as $t \to +\infty$ where $\lambda \in \{0, 1\}$ and $\beta < 0$. The problematic arises in the study of the Mixed Convection Boundary Layer flow over a permeable vertical surface embedded in a Porous Medium. We prove the non-existence and the sign of convex and convex-concave solutions of the above problem according to the mixed convection parameter $b < 0$, the permeable parameter $a \in \mathbb{R}$ and the temperature parameter $\beta < 0$.

Key–Words: Opposing mixed convection, Boundary layer problem, Existence and nonexistence, Convex solution, Convex-Concave solution.

1 Introduction

Owing to their numerous applications in geophysical and industrial manufacturing processes, the problem of boundary layers related to heating and cooling surfaces embedded in fluid-saturated porous media have attracted considerable attention of researchers during the last few decades. Areas of applications as geothermal energy extraction, oil reservoir modelling, magnetohydrodynamic, casting and welding in manufacturing processes, (see [8], [11] and [12]) or in boundary layer flows (see [4] and [10]) etc. In this paper, our interest focuses on the analysis of the boundary value problems $\mathcal{P}_{\lambda(a,b)}$

$$\begin{cases} f''' + ff'' + \beta f'(f' - 1) = 0 \\ f(0) = a, \ a \in \mathbb{R} \\ f'(0) = b < 0 \\ f'(t) \longrightarrow \lambda \text{ as } t \longrightarrow +\infty \end{cases} \qquad \mathcal{P}_{\lambda(a,b)}$$

where $\lambda \in \{0, 1\}$ has already been examined in [3], [7] and [14] with $a = 0$. This problem comes from the study of the mixed convection boundary layer flow along a semi-infinite vertical permeable plate embedded in a saturated porous medium, with a prescribed power law of the distance from the leading edge for the temperature. The parameter β is a temperature power-law profile and b is the mixed convection parameter, namely $b = \frac{R_a}{Pe} - 1$, with R_a the Rayleigh number and P_e the Péclet number. For more details on the physical derivation and the numerical results, the interested reader can consult references [3] and [13].

Mathematical results about the problem $\mathcal{P}_{\lambda(a,b)}$ with $\lambda = 1$ can be found in [1], [6], [7], [9] and [14]. The case where $a \geq 0$, $b \geq 0$, $\beta > 0$ and $\lambda \in \{0, 1\}$ was treated by Aïboudi and al. in [1], and the results obtained generalize the ones of [9]. In [6], Brighi and Hoernel established some results about the existence and uniqueness of convex and concave solution of $\mathcal{P}_{1(a,b)}$ where $-2 < \beta < 0$ and $b > 0$. These results can be recovered from [4], where the general equation $f''' + ff'' + \mathbf{g}(f') = 0$ is studied.

Recently, in [7], the authors prove some theoretical results about the problem $\mathcal{P}_{1(0,b)}$ with $-2 < \beta < 0$, $b = 1 + \varepsilon$ and $\varepsilon < -1$. In particular, the authors prove that there exist $\varepsilon_* \in (-1.807, -1.806)$ and $\varepsilon^* \in (-1.193, -1.192)$, such that:

(i) $\mathcal{P}_{1(0,b)}$ has no convex solution for any $-2 < \beta < 0$ and each $\varepsilon \leq \varepsilon_*$.

(ii) $\mathcal{P}_{1(0,b)}$ has a convex solution for each $-2 < \beta < 0$ and each $\varepsilon \in [\varepsilon^*, -1)$.

In [14] one can found interesting new result about the existence of convex solution of $\mathcal{P}_{1(0,b)}$ where $0 < \beta < 1$ under some conditions. In [2] the results obtained by Aïboudi and al generalize the ones of [14]. In [7] and [14], the method used by the authors to prove the existence of a convex solution for the case $a = 0$ seems difficult to generalize for $a \neq 0$.

The problem $\mathcal{P}_{\lambda(a,b)}$ with $\beta = 0$ is the well known Blasius problem. For a broad view, see [5]. See also [15].

The aim of this paper is to extend the study of existence and nonexistence of the solutions of $\mathcal{P}_{\lambda(a,b)}$ with $\beta < 0$ and $\lambda \in \{0, 1\}$. We will focus our attention on convex and convex-concave solutions of the equation

$$f''' + ff'' + \beta f'(f' - 1) = 0. \qquad (1)$$

As usually, to get a convex or convex-concave solution of $\mathcal{P}_{\lambda(a,b)}$, we will use the shooting technique which consists of finding the values of a parameter $c \geq 0$ for which the solution of (1) satisfying the initial conditions $f(0) = a$, $f'(0) = b$ and $f''(0) = c$, exists on $[0, +\infty)$, and is such that $f'(t) \to \lambda$ as $t \to +\infty$. We denote by f_c the solution of the following initial value problem and by $[0, T_c)$ its right maximal interval of existence:

$$\begin{cases} f''' + ff'' + \beta f'(f' - 1) = 0 \\ f(0) = a \\ f'(0) = b < 0 \\ f''(0) = c \geq 0 \end{cases} \qquad \mathcal{P}_{(a,b,c)}$$

2 On Blasius Equation

In this section, we recall some basic properties of the supersolutions of the Blasius equation. Let $I \subset \mathbb{R}$ be an interval and $f : I \longrightarrow \mathbb{R}$ be a function.

Definition 1. *We say that f is a supersolution of the Blasius equation $f''' + ff'' = 0$ if f is of class C^3 and if $f''' + ff'' \geq 0$ on I.*

Proposition 2. *Let $t_0 \in \mathbb{R}$. There does not exist non-positive convex supersolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [4], Proposition 2.5. □

3 Preliminary Results

Proposition 3. *Let f be a solution of the equation (1) on some maximal interval $I = (T_-, T_+)$.*

1. *If F is any anti-derivative of f on I, then $(f''e^F)' = -\beta f'(f' - 1)e^F$.*
2. *Assume that $T_+ = +\infty$ and that $f'(t) \longrightarrow \lambda \in \mathbb{R}$ as $t \to +\infty$. If moreover f is of constant sign at infinity, then $f''(t) \longrightarrow 0$ as $t \to +\infty$.*
3. *If $T_+ = +\infty$ and if $f'(t) \longrightarrow \lambda \in \mathbb{R}$ as $t \to +\infty$, then $\lambda = 0$ or $\lambda = 1$.*
4. *If $T_+ < +\infty$, then f'' and f' are unbounded near T_+.*
5. *If there exists a point $t_0 \in I$ satisfying $f''(t_0) = 0$ and $f'(t_0) = \mu$, where $\mu = 0$ or 1 then for all $t \in I$, we have $f(t) = \mu(t - t_0) + f(t_0)$.*

Proof. The first item follows immediately from equation (1). For the proof of items 2-5, see [4], Proposition 3.1 with $g(x) = \beta x(x - 1)$. □

4 The Boundary Value Problem in the Convex and Convex-Concave Case with $\beta < 0$

In the following we take $a, b \in \mathbb{R}$ and $\lambda \in \{0, 1\}$ with $b < 0$ and $\beta < 0$. We are interested here in convex and convex-concave solutions of the boundary value problem $\mathcal{P}_{\lambda(a,b)}$. As mentioned in the introduction, we will use the shooting method to find these solutions. Define the following sets:

$$C_1 = \{c \geq 0 : f_c' \leq 0 \text{ and } f_c'' \geq 0 \text{ on } [0, T_c)\},$$

$$C_2 = \{c \geq 0 : \exists t_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c' < 0 \text{ on } (0, t_c),$$
$$f_c' > 0 \text{ on } (t_c, t_c + \varepsilon_c) \text{ and } f_c'' > 0 \text{ on } (0, t_c + \varepsilon_c)\},$$

$$C_3 = \{c \geq 0 : \exists s_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c'' > 0 \text{ on } (0, s_c),$$
$$f_c'' < 0 \text{ on } (s_c, s_c + \varepsilon_c) \text{ and } f_c' < 0 \text{ on } (0, s_c + \varepsilon_c)\}.$$

Lemma 4. *f_c is a convex solution of the boundary value problem $\mathcal{P}_{0(a,b)}$ if and only if $c \in C_1$.*

Proof. See Appendix A of [4] with $g(x) = \beta x(x - 1)$. □

Lemma 5. *The set C_3 is empty.*

Proof. See Lemma A.5 of [4] with $g(x) = \beta x(x - 1)$ and $\beta < 0$. □

From the previous Lemma, we have $C_1 \cup C_2 = [0, +\infty)$ and $C_1 \cap C_2 = \emptyset$.

5 The $a \le 0$ case

Lemma 6. *The set C_1 is empty.*

Proof. For contradiction, assume that $C_1 \neq \emptyset$ and let $c \in C_1$. From Lemma 4, f_c is a convex solution of the boundary value problem $\mathcal{P}_{0(a,b)}$. Hence f_c and f'_c are negative on $[0, +\infty)$. This implies that $f'''_c + f_c f''_c = -\beta f'_c(f'_c - 1) > 0$ on $(0, +\infty)$. Hence, f_c is a nonpositive convex supersolution of the Blasius equation on $(0, +\infty)$. This contradicts Proposition 2. □

Remark 7. *From the previous lemma and Lemma 5, $C_2 = [0, +\infty)$.*

Remark 8. *From Proposition 3.1, items 1, 3 and 5, if $c \in C_2$, then there are only three possibilities for the solution of the initial value problem $\mathcal{P}_{(a,b,c)}$:*

1. *f_c is convex on its right maximal interval of existence $[0, T_c)$ and $f'_c(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$);*
2. *there exists a point $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$ and $0 < f'_c(t_0) < 1$;*
3. *f_c is a convex solution of $\mathcal{P}_{1(a,b)}$.*

Lemma 9. *Let $\beta < 0$, $a \le 0$ and $b \le -1$. If $c \ge 0$ and if there exists $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$ and $0 < f'_c(t_0) < 1$, then $f_c(t_0) > 0$.*

Proof. Let $c \ge 0$ and assume that there exists $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$ and $0 < f'_c(t_0) = \theta < 1$. Suppose that $f_c(t_0) \le 0$. Let us consider the function $L_c = 3f''^2_c + 2\beta f'^3_c - 3\beta f'^2_c$. Then, from (1), we have $L'_c = -6 f_c f''^2_c > 0$ on $[0, t_0)$ and hence:

$$L_c(0) = 3c^2 + 2\beta b^3 - 3\beta b^2 < L_c(t_0) = 2\beta\theta^3 - 3\beta\theta^2.$$

It follows that $\theta^2 - b^2 > 0$ which implies that $\theta > 1$. This is a contradiction. □

For the rest of this section, if it is defined, we will set $a_* = -\sqrt{\dfrac{1 - b^2}{\beta - 2b}}$.

Lemma 10. *Let $b \le -1$ and $c \ge 0$. Let $t_* > 0$ be the first point such that $f_c(t_*) = 0$. If, either $2b \le \beta < 0$, or $\beta < 2b$ and $a \ge a_*$, then $f'_c(t_*) > 1$.*

Proof. From Remark 7, Remark 8 and Lemma 9, we know that the point t_* exists. Let $K_c = 2f_c f''_c - f'^2_c + f^2_c(2f'_c - \beta)$. From (1), we obtain $K'_c = 2(2 - \beta) f_c f'^2_c$ on $(0, t_*)$. Therefore, K_c is decreasing on $(0, t_*)$ and hence $K_c(0) > K_c(t_*)$. It follows that if $2b \le \beta < 0$ then

$$f'^2_c(t_*) > -2ac + b^2 + a^2(\beta - 2b) \ge b^2,$$

which implies that $f'_c(t_*) > 1$. The same result is obtained where $b \le -1$, $\beta < 2b$ and $a \ge a_*$. □

Theorem 11. *Let $\beta < 0$ and $a, b \in \mathbb{R}$ with $b < 0$ and $a \le 0$.*

1) *The boundary value problem $\mathcal{P}_{0(a,b)}$ has no convex solution.*
2) *If $b \le -1$ and if either $2b \le \beta < 0$, or $\beta < 2b$ and $a \ge a_*$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has no convex and no convex-concave solution.*
3) *If $b \le -1$ and if, either $2b \le \beta < 0$, or $\beta < 2b$ and $a \ge a_*$, then, for any $c \ge 0$, the solution f_c of the initial value problem $\mathcal{P}_{(a,b,c)}$ is convex on its right maximal interval of existence $[0, T_c)$ and $f'_c(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$).*

Proof. The first result follows from Lemma 4 and Lemma 6. The second result follows from Proposition 3,item1, Lemma 9, and Lemma 10. The third result follows from Remark 7, Remark 8, and Lemma 10. □

6 The $a > 0$ case

Let $a, b \in \mathbb{R}$ with $\beta < 2b < 0$ and $a > 0$. We consider the solution f_c of the initial value problem $P_{(a,b,c)}$ on the right maximal interval of existence $[0, T_c)$.

Let us set $a^* = -\dfrac{b}{\sqrt{2b - \beta}}$.

Lemma 12. *Let $a \ge a^*$, $c \ge 0$ and $\beta < 2b < 0$. If f_c is a solution of the initial value problem $P_{(a,b,c)}$, then f_c is positive on the right maximal interval of existence $[0, T_c)$.*

Proof. Assume that there exists $t_* \in (0, T_c)$ such that $f_c > 0$ on $[0, t_*)$ and $f_c(t_*) = 0$. Let $K_c = 2f_c f''_c - f'^2_c + f^2_c(2f'_c - \beta)$. From (1), we obtain $K'_c = 2(2 - \beta) f_c f'^2_c > 0$ on $(0, t_*)$. Therefore, K_c is increasing on $(0, t_*)$ and hence $K_c(0) < K_c(t_*)$. It follows that

$$0 > -f'^2_c(t_*) > a^2(2b - \beta) - b^2.$$

This is a contradiction. □

Remark 13. *From the previous Lemma and Lemma 5.16 of [4], if there exists $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$, then $f_c(t_0) > 0$ and f_c is a convex-concave solution of $\mathcal{P}_{0(a,b)}$.*

Lemma 14. *The set C_2 is not empty.*

Proof. Assume C_2 is empty, then from Lemma 5, $C_1 = [0, +\infty)$ and f_c is a convex solution of $\mathcal{P}_{0(a,b)}$ for all $c \in [0, +\infty)$. Let $A_c = f_c'' + f_c(f_c' - 1)$. From (1), we obtain $A_c' = (1-\beta)f_c'(f_c' - 1)$. Since f_c is a convex solution of $\mathcal{P}_{0(a,b)}$, then $f_c' < 0$. Therefore, A_c is increasing on $[0, +\infty)$ and hence $A_c(0) < A_c(t)$ as $t \to +\infty$. It follows that $c < -a(b-1)$. This is a contradiction. □

Remark 15. *From the Remark 13, Proposition 3.1, items 1, 3 and 5, if $c \in C_2$, then there are only three possibilities for the solution of the initial value problem $\mathcal{P}_{(a,b,c)}$:*

1. *f_c is convex on its right maximal interval of existence $[0, T_c)$ and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$);*
2. *f_c is convex-concave solution of $\mathcal{P}_{0(a,b)}$.*
3. *f_c is a convex solution of $\mathcal{P}_{1(a,b)}$.*

Lemma 16. *If $\beta < 2b < 0$ and $a \geq a^*$ then there exists $c_0 \in C_2$ such that if $c \geq c_0$ then f_c is a convex solution of $\mathcal{P}_{(a,b,c)}$ on its right maximal interval of existence $[0, T_c)$ and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$).*

Proof. From Remark 15 and Lemma 14, we know that, if $c \in C_2$, then f_c is a convex solution of $\mathcal{P}_{1(a,b)}$, a convex-concave solution of $\mathcal{P}_{0(a,b)}$ or f_c is convex on its right maximal interval of existence $[0, T_c)$ and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$).

Let $c \in C_2$, be such that f_c is a convex solution of $\mathcal{P}_{1(a,b)}$ or a convex-concave solution of $\mathcal{P}_{0(a,b)}$. Therefore, we have $b < f_c' < 1$ on $[0, +\infty)$ and, from Lemma 12, we have $f_c > 0$. It follows that

$$(f_c'' + f_c(f_c' - 1))' = (1-\beta)f_c'(f_c' - 1) \geq -\frac{1}{4}(1-\beta)$$

on $[0, +\infty)$. Integrating between 0 and $t \geq 0$, and using the fact that $f_c > 0$, we obtain

$$\begin{aligned} f_c''(t) &\geq -\frac{1}{4}(1-\beta)t + a(b-1) + c - f_c(t)(f_c'(t) - 1) \\ &\geq -\frac{1}{4}(1-\beta)t + a(b-1) + c. \end{aligned}$$

Integrating once again we get

$$\forall t \geq 0, \qquad 1 > f_c'(t) \geq -\frac{1}{8}(1-\beta)t^2 + (a(b-1)+c)t + b. \tag{2}$$

Let us set $P_c(t) = -\frac{1}{8}(1-\beta)t^2 + (a(b-1)+c)t + b - 1$. We have $P_c(t) < 0$ for all $t \geq 0$. It means that P_c has no positive roots. Thus c cannot be too large, because, on the contrary, its discriminant $\Delta = \big(a(b-1)+c\big)^2 + \frac{1}{2}(1-\beta)(b-1)$ and $a(b-1)+c$ would be positive, and hence the polynomial P_c would have two positive roots, a contradiction.

Therefore, there exists $c_0 > 0$ such that for any $c > c_0$, f_c is convex on its right maximal interval of existence $[0, T_c)$ and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$). This completes the proof. □

Theorem 17. *Let $\beta < 2b < 0$, $a \geq a^* > 0$ and f_c be a solution of the initial value problem $\mathcal{P}_{(a,b,c)}$.*

1) *For all $c \geq 0$, f_c is positive.*
2) *there exists $c_0 > 0$ such that for any $c > c_0$, f_c is convex on its right maximal interval of existence $[0, T_c)$ and $f_c'(t) \to +\infty$ as $t \to T_c$ (with $T_c < +\infty$).*

Proof. The first result follows from Lemma 12. The second result follows from the first result, Remark 13, Remark 15, Lemma 14 and Lemma 16. □

7 Conclusion

In this work we have presented a set of new and important results for $\beta < 0$ and $b < 0$, we summarize as follows:

1. If $a \leq 0$.
   (a) The boundary value problem $\mathcal{P}_{0(a,b)}$ has no convex solution on $[0, +\infty)$.
   (b) If $b \leq -1$ and if either $2b \leq \beta < 0$ or $\beta < 2b$ and $a \geq a_*$ with $a_* = -\sqrt{\dfrac{1-b^2}{\beta - 2b}}$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has no convex and no convex-concave solution.
   (c) If $b \leq -1$ and either $2b \leq \beta < 0$ or $\beta < 2b$ and $a \geq a_*$, and if f_c is a solution of the initial problem $\mathcal{P}_{(a,b,c)}$ with $c \geq 0$ then f_c is a convex solution of the boundary value problem $\mathcal{P}_{+\infty(a,b)}$.
2. for $a > 0$
   (a) if $a \geq a^* > 0$ where $a^* = -\dfrac{b}{\sqrt{2b-\beta}}$, all solution of the initial value problem $\mathcal{P}_{(a,b,c)}$ is a positive.
   (b) The boundary value problem $\mathcal{P}_{+\infty(a,b)}$ has infinitely many positive convex solutions.

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