Approximate solution of Singular Integro-Differential Equations for displaced Fejér points

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Abstract: The main proposal of this article is the investigation and theoretical background of the direct-approximate methods for the numerical solution of singular integro-differential(SIDE) equations(Cauchy type kernel) with unknown function defined on the smooth contours of the Lypunov type. The equations are studied in the Lebesgue spaces. The SIDE are defined on the displaced Fejér points of complex plane. The numerical schemes of collocation and mechanical quadrature methods for the SIDE defined on an arbitrary smooth closed contour of complex plane are elaborated. The theorems of convergence of these methods have been proved in Lebesgue spaces.

Key–Words: Displaced Fejér points, Singular Integro-Differential Equations, Collocation Method, Mechanical Quadrature Methods

1 Introduction

Many theoretical and practical problems of mathematics, elasticity, aerodynamics, mechanics, thermoelasticity, queuing analysis, and mathematical biology lead to various classes of singular integral equations(SIE) and singular integro- differential equations(SIDE) (one- dimensional and multidimensional) (see [1]-[6] and the literature cited therein). The general theory of SIE and SIDE has been widely investigated in the last decades [7]-[19]. The general theory SIDE has been widely investigated in the monographes Muskhelishvili and Vekua [20]-[22]. On these monographs have been proved that the exact solution for systems of SIDE can be found in some particular cases. Even in these cases the analytical solution of SIE is rarely available. It should be mentioned that using a con- formal mapping, one can transform an arbitrary smooth closed contour to the unit circle. However, this approach may not simplify the problem. The transition to another contour, different from the standard one, implies many difficulties:

- The coefficients, kernel, and right part of transforming equation lose their smoothness;
- The power of smoothness appears in convergence speed of collocation method. So that the evaluations of convergence speed will depend from particular contour;
- The numerical schemes of researched methods become more difficult. The singularity appears in new kernel and we are not able to use the numerical schemes of mechanical quadrature method because of a singularity of the new kernel.

That is why there is a necessity to elaborate the approximate methods for solving SIDE and proving the convergence. As usual the direct-approximate methods for the SIDE have been studied in two cases: when the contour was a unit circle and when the contour Γ was an interval on the real line. However, the case where the contour Γ is assumed to be a smooth closed Jordan boundary of a simply connected domain around the origin has not been studied enough. We note that the theoretical background of collocation methods and mechanical quadratic methods for SIDE was studied in [23]-[29].

In this research, we will study theoretical foundation for SIDE defined on an arbitrary smooth closed contour in the complex plane with displaced Fejér points on contour Γ. The paper is organized as follows. In Section 2 we introduce some definitions and notations. We present the evaluation of the norm for Lagrange Interpolation Polynomial in Section 3. In Section 4 we describe a Singular Integro-Differential Equations with Regular Integral Operator Equal Zero. We Formulate the Problem. In Section 4 we present the numerical schemes of collocation methods for case when $h_r(t,\tau) = 0$. We formulate the auxiliary results. We formulate the convergence theorems for collocation methods. We present the numerical schemes of collocation methods for case when $h_r(t,\tau) \neq 0$ and convergence theorem in Section 5 . In Section 6 we elaborated the numerical schemes of mechanical quadrature methods.

2 Definitions of Function Spaces and Notations

Let Γ be an arbitrary smooth closed contour bounding a simply-connected region F^+ of the complex plane, let $t = 0 \in F^+$, $F^- = C \setminus \{F^+ \cup \Gamma\}$, where C is the complex plane. Let $z = \psi(w)$ be a function, mapping conformably the outside of unit circle $\Gamma_0 = \{|w| = 1\}$ on the domain F^- so that

$$\psi(\infty) = \infty,\ \psi^{(\prime)}(\infty) = 1. \tag{1}$$

We assume that the function $z = \psi(w)$ has the second derivative, satisfying on Γ_0 the Hölder condition with some parameter μ $(0 < \mu < 1)$; the class of such contours is denoted by $C(2;\mu)$ [29]. Define $\overset{\circ}{W}{}_p^{(q)}$ as

$$\overset{\circ}{W}{}_p^{(\nu)} = \Big\{ g \in L_p(\Gamma) : g^{(q)} \in L_p(\Gamma),$$

$$\left. \frac{1}{2\pi i}\int\limits_\Gamma g(\tau)\tau^{-k-1}d\tau = 0, \quad k = 0,\ldots,q-1 \right\}.$$

The norm in $\overset{\circ}{W}{}_p^{(q)}$ is determined by the equality

$$||g||_{p,q} = ||g^{(q)}||_{L_p}.$$

Let U_n be the Lagrange interpolating polynomial

$$(U_n g)(t) = \sum_{s=0}^{2n} g(t_s)\cdot l_s(t), \tag{2}$$

$$l_j(t) = \prod_{k=0,k\neq j}^{2n} \frac{t - t_k}{t_j - t_k}\left(\frac{t_j}{t}\right)^n \equiv$$

$$\sum_{k=-n}^{n} \Lambda_k^{(j)} t^k,\ t \in \Gamma, \quad j = 0,\ldots,2n.$$

In complex space $L_p(\Gamma)$ $1 < p < \infty$ of the functions $g(t) \in L_p(\Gamma)$ with norm

$$||g| = \left(\frac{1}{l}\int\limits_\Gamma |g|^p(\tau)d\tau\right),$$

where l is the length of Γ. By $H_\beta(\Gamma)$ we denote the classical Hölder spaces which satisfies the Hölder condition with exponent β $(0 < \beta < 1)$ and with norm

$$\|g\|_\beta = \|g\|_C + H(g;\beta),$$

$$H(g,\beta) = \sup_{t'\neq t''} \frac{\left|g(t'') - g(t')\right|}{|t' - t''|^\beta}, t', t'' \in \Gamma.$$

By $H_\beta^{(q)}(\Gamma)$ $q = 0,1,\ldots$, $H^{(0)}(\Gamma) = H(\Gamma)$ we denote the space of q times continuously-differentiable functions. The derivatives of the qth order for these functions are elements of space $H_\beta(\Gamma)$. The norm in $H_\beta^{(q)}(\Gamma)$ is given by the formula

$$||g||_{\beta,q} = \sum_{k=0}^{q} ||g^{(k)}||_c + H(g^{(q)};\beta). \tag{3}$$

Definition 1 *By index of the function[30] $G(t)$ with respect to the contour Γ we understand the increment of its argument, in traversing the curve in the positive direction, divided by 2π.*

If the increment of a quality ω in traversing the contour Γ be denoted by $[\omega]_\Gamma$, the index of $G(t)$ can be written in the form

$$\chi = IndG(t) = \frac{1}{2\pi}[argG(t)]_\Gamma.$$

3 Function Approximation in Lebesgue spaces

In this section we present the evaluation of norm for Lagrange Interpolation Polynomial (2) $U_n : C(\Gamma) \to L_p(\Gamma)$, $1 < p < \infty$.

Theorem 2 *Let $F(t) \in C(2;\mu)$ and*

$$t_j(\theta) = \psi\left[\exp\left(\frac{2\pi i}{2n+1}(j-n) + \theta i\right)\right], \tag{4}$$

$$j = 0,\ldots,2n, \theta \in (0;2\pi) \quad i^2 = -1,$$

are displaced Fejér points. Then $||U_n||_{C\to L_p} \leq B_1$ $(< \infty)$, $B_1 = B_1(p)$ is the constant which depends from p.

Remark 3 *On the Theorem 2 we proved that the operator U_n is bounded as operator reflecting from $C(\Gamma)$ to $L_p(\Gamma)$.*

As it was proved in [23] the operator $U_n : L_p(\Gamma) \to L_p(\Gamma)$ is unbounded even for simplest case:when Γ is an unit circle. This means $||U_n||_{L_p} = \infty$.

Theorem 4 *Let $\Gamma \in C(2;\mu)$ and the points t_j $j = 0,...,2n$ form the displaced Fejér points [31],[32] (4) on Γ. Then for every continuous function $g(t)$ from Γ the following relation holds*

$$||g - U_n g||_p \leq (1 + B_1(p))E_n(g;\Gamma) \qquad (5)$$

$E_n(g;\Gamma$ is the best uniform approximation of the function $g(t)$ by polynomials from the class $\Upsilon = \{\sum_{k=-n}^{n} r_k t^k.\}$

Remark 5 *If $g(t) \in H_\alpha^r(\Gamma)$, $r = 0,1,2,\ldots$, then from the relation (5) the following inequality takes place:*

$$||g - U_n g||_p \leq (1 + B_1(p))C\frac{H(g^{(r);\alpha})}{n^{r+\alpha}},$$

where C is a constant.

4 Problem Formulation. Singular Integro-Differential Equations with Regular Integral Operator Equal Zero ($h_r(t,\tau) = 0$)

We consider the SIDE

$$Lx \equiv \sum_{r=0}^{q} \left\{ \tilde{A}_r(t)x^{(r)} + \frac{\tilde{B}_r(t)}{\pi i}\int_\Gamma \frac{x^{(r)}(\tau)}{\tau - t}d\tau \right\} = y(t), \quad t \in \Gamma, \qquad (6)$$

$\tilde{A}_r(t), \tilde{B}_r(t)$, and $y(t)$ are known functions from $C(\Gamma)$; $x^{(0)}(t) = x(t)$ is the unknown function $x^{(r)}(t) = \dfrac{d^r x(t)}{dt^r}$, $(r = 1,\ldots,q)$, (q is a positive integer). We search for a solution of (6) in class of the functions, satisfying the condition

$$\frac{1}{2\pi i}\int_\Gamma x(\tau)\tau^{-k-1}d\tau = 0,$$

$$k = 0,\ldots,q-1. \qquad (7)$$

The SIDE (6 with conditions (7) we denote as "problem (6- (7)." Using the Riesz operators $P = \frac{1}{2}(I+S)$, $Q = I - P$, (where I is the identity operator, and $(Sx)(t) = \dfrac{1}{\pi i}\int_\Gamma \dfrac{x(\tau)}{\tau - t}d\tau$ is the singular operator (with Cauchy kernel)), we rewrite the Eq. (6) in the form convenient for consideration:

$$(Mx \equiv) \sum_{r=0}^{q}[A_r(t)(Px^{(r)})(t) + B_r(t)(Qx^{(r)})(t)] = f(t), \quad t \in \Gamma, \qquad (8)$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $r = 0,\ldots,q$.

4.1 Auxiliary Results

We denote by $L_{p,q}$ the image of the space L_p with respect to the map $P + t^{-q}Q$ equipped with the norm of L_p, $1 < p < \infty$. We formulate Lemma 6 and Lemma 7 from [33]. We use these lemmas to prove the convergence theorems.

Lemma 6 *The differential operator $D^q : \overset{\circ}{W}{}_p^{(q)} \to L_{p,q}$, $(D^q g)(t) = g^{(q)}(t)$ is continuously invertible and its inverse operator $D^{-q} : L_{p,q} \to \overset{\circ}{W}{}_p^{(q)}$ is determined by the equality*

$$(D^{-q}g)(t) = (N^+g)(t) + (N^-g)(t),$$

$$(N^+g)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\times \int_\Gamma (Pg)(\tau)(\tau - t)^{q-1}\log(1 - \frac{t}{\tau})d\tau,$$

$$(N^-g)(t) = \frac{(-1)^{q-1}}{2\pi i(q-1)!}\times \int_\Gamma (Qg)(\tau)(\tau - t)^{q-1}\log(1 - \frac{\tau}{t})d\tau.$$

From Lemma 6 it follows

Lemma 7 *The operator $B : \overset{\circ}{W}{}_p^{(q)} \to L_p, B = (P + t^q Q)D^q$ is invertible and*

$$B^{-1} = D^{-q}(P + t^{-q}Q).$$

The proofs of Lemma 6 and Lemma 7 can be found in [33].

We search for the approximate solution of problem (6)-(7) in the form

$$x_n(t) = \sum_{k=0}^{n} \xi_k^{(n)} t^{k+q} + \sum_{k=-n}^{-1} \xi_k^{(n)} t^k, \quad t \in \Gamma, \qquad (9)$$

with the unknown coefficients $\xi_k^{(n)} = \xi_k$ $(k = -n,\ldots,n)$; we note that the function $x_n(t)$, constructed by formula (9), satisfies condition (7).

4.2 Collocation Methods. Convergence Theorem

Let $\Theta_n(t) = Mx_n(t) - f(t)$ be the residual of SIDE. The collocation method consists in setting it equal to zero at some distinct points $t_j,\ j = 0,\dots,2n$ on Γ. Thus we obtain a system of linear algebraic equations (SLAE) for the unknown complex numbers $\xi_k,\quad (k = -n,\dots,n)$ which can be determined by solving

$$\Theta_n(t_j) = 0,\ j = 0,\dots,2n. \tag{10}$$

Using the formulae from [23]

$$(Px)^{(r)}(t) = (Px^{(r)})(t),$$
$$(Qx)^{(r)}(t) = (Qx^{(r)})(t), \tag{11}$$

and the relations

$$(t^{k+q})^{(r)} = \frac{(k+q)!}{(k+q-r)!}t^{k+q-r},\ k = 0,\dots,n,$$

$$(t^{-k})^{(r)} = (-1)^r\frac{(k+r-1)!}{(k-1)!}t^{-k-r},\ k = 1,\dots,n,$$

$$S(\tau^k) = \begin{cases} t^k, & \text{when } k \geq 0; \\ -t^k, & \text{when } k < 0; \end{cases} \tag{12}$$

from (8) and (10) we obtain the system of linear algebraic equations (SLAE) for collocation method:

$$\sum_{r=0}^{q}\left\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t_j^{k+q-r}\xi_k + B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}t_j^{-k-r}\xi_{-k}\right\} = f(t_j),\ j = 0,\dots,2n. \tag{13}$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$,$r = 0,\dots,q$.
The SLAE (13) contains $2n+1$ equations and $2n+1$ unknowns ξ_k, $k = -n,\dots,n$.

Theorem 8 *Let the following conditions be satisfied:*

1) $\Gamma \in C(2,\mu),\quad 0 < \mu < 1$;
2) *the functions* $A_r(t)$ *and* $B_r(t)$ *belong to the space* $H_\alpha(\Gamma)$ $0 < \alpha < 1$;
3) $A_q(t)B_q(t) \neq 0,\quad t \in \Gamma$;
4) *the index of the function* $t^qB_q^{-1}(t)A_q(t)$ *is equal to zero;*
5) *function* $f(t) \in C(\Gamma)$;
6) *the operator* $M :\overset{\circ}{W}{}_p^{(q)} \to L_p(\Gamma)$ *is linear and invertible;*
7) *the points* t_j $(j = 0,\dots 2n)$ *form a system of displaced Fejér knots on* Γ*:*

$$t_j(\theta) = \psi\left[\exp\left(\frac{2\pi i}{2n+1}(j-n) + \theta i\right)\right],$$
$$j = 0,\dots,2n, \theta \in (0;2\pi) \quad i^2 = -1.$$

Then, the SLAE (13) of collocation method has the unique solution ξ_k $(k = -n,\dots,n)$, *for the numbers* $n \geq n_1$, *large enough numbers. The approximate solutions* $x_n(t)$, *constructed by formula (9), converge when* $n \to \infty$ *in the norm of space* $\overset{\circ}{W}{}_p^{(q)}$ *to the exact solution* $x(t)$ *of the problem (6)-(7) and the following estimation for convergence holds:*

$$||x - x_n||_{p,q} = O\left(\frac{1}{n^\alpha}\right) + O(\omega(f;\frac{1}{n})) \overset{def}{=} \delta_n, \tag{14}$$

To prove the Theorem 9 we can apply the scheme from [23].

5 Collocation methods for Singular Integro-Differential Equations with Regular Integral Operator Different from Zero ($h_r(t,\tau) \neq 0$)

In the complex space $L_p(\Gamma)$ we consider the SIDE

$$(Mx \equiv)\sum_{r=0}^{q}[\tilde{A}_r(t)x^{(r)}(t) + \tilde{B}_r(t)\frac{1}{\pi i}\int_\Gamma\frac{x^{(r)}(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int_\Gamma h_r(t,\tau)x^{(r)}(\tau)d\tau] = f(t),\quad t \in \Gamma, \tag{15}$$

where $\tilde{A}_r(t), \tilde{B}_r(t)$, $h_r(t,\tau)$ $(r = 0,\dots,q)$ and $f(t)$ are known functions; $x^{(0)}(t) = x(t)$ is the unknown function $x^{(r)}(t) = \dfrac{d^rx(t)}{dt^r}$, $(r = 1,\dots,q)$, (q is a positive integer). The equation (15) with conditions (7) we denote as problem "(15)-(7)". Applying the similar approach as in 4 for collocation methods to the SIDE(15) we obtain the following SIDE for the problem (15)-(7)

$$\sum_{r=0}^{q}A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t_j^{k+q-r}\xi_{k,\rho}$$

$$+B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}t_j^{-k-r}\times\xi_{-k,\rho}$$

$$+\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}\sum_{s=0}^{2n}h_r(t_j,t_s)t_s^{1+k-r}\Lambda_{-k}^{(s)}\xi_{k,\rho}+$$

$$+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\sum_{s=0}^{2n}h_r(t_j,t_s)t_s^{-k-r}\Lambda_{k}^{(s)}\xi_{-k}$$

$$=f(t_j), \qquad (16)$$

Theorem 9 *Let the following conditions be satisfied:*

1) $\Gamma\in C(2,\mu),\quad 0<\mu<1;$
2) *the functions* $A_r(t)$ *and* $B_r(t)$ *belong to the space* $H_\alpha(\Gamma)$ $0<\alpha<1;$
3) $A_q(t)B_q(t)\neq 0,\quad t\in\Gamma;$
4) *the index of the function* $t^qB_q^{-1}(t)A_q(t)$ *is equal to zero;*
5) $h_r(t,\tau)$ $(r=0,\dots,q)\in C(\Gamma\times\Gamma)$, $0<\beta\le 1$, *function* $f(t)\in C(\Gamma)$;
6) *the operator* $M:\overset{\circ}{W}{}_p^{(q)}\to L_p(\Gamma)$ *is linear and invertible;*
7) *the points* t_j $(j=0,\dots 2n)$ *form a system of displaced Fejér knots on* Γ*:*

$$t_j(\theta)=\psi\left[\exp\left(\frac{2\pi i}{2n+1}(j-n)+\theta i\right)\right],$$

$$j=0,\dots,2n,\theta\in(0;2\pi)\quad i^2=-1.$$

Then, the SLAE (16) of collocation method has the unique solution ξ_k $(k=-n,\dots,n)$, *for the numbers* $n\ge n_1$, *large enough. The approximate solutions* $x_n(t)$, *converge when* $n\to\infty$ *in the norm of space* $\overset{\circ}{W}{}_p^{(q)}$ *to the exact solution* $x(t)$ *of the problem (15)-(7) and the following estimation for convergence holds:*

$$||x-x_n||_{p,q}=O\left(\frac{1}{n^\alpha}\right)+O(\omega(f;\frac{1}{n}))+$$

$$O(\omega^t(h;\frac{1}{n}))\overset{def}{=}\delta_n, \qquad (17)$$

Here $\omega(g;\delta)=\sup\limits_{|t'-t''|\le\delta}|g(t')-g(t'')|$, $(t',t'')\in\Gamma$ is the continue module for function $g(t)$.

6 Mechanical Quadrature Methods for Singular Integro-Differential Equations

We approximate the integrals in SLAE (16) by quadrature formula:

$$\frac{1}{2\pi i}\int\limits_\Gamma g(\tau)\tau^{l+k}d\tau\cong\frac{1}{2\pi i}\int\limits_\Gamma U_n(\tau^{l+1}\cdot g(\tau))\tau^{k-1}d\tau,$$

where $k=0,\dots,n$, at $l=0,1,2,\dots$ and $k=-1,\dots,-n$, for $l=-1,-2,\dots$, and U_n is the Lagrange interpolation operator defined by formula (2).

Thus, we obtain the following SLAE from (16):

$$\sum_{r=0}^{q}A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t_j^{k+q-r}\xi_{k,\rho}$$

$$+B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}t_j^{-k-r}\times\xi_{-k,\rho}$$

$$+\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}\sum_{s=0}^{2n}h_r(t_j,t_s)t_s^{1+k-r}\Lambda_{-k}^{(s)}\xi_{k,\rho}+$$

$$+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\sum_{s=0}^{2n}h_r(t_j,t_s)t_s^{-k-r}\Lambda_{k}^{(s)}\xi_{-k,\rho}$$

$$=f(t_j),j=0,\dots,2n. \qquad (18)$$

Theorem 10 *Let all conditions of Theorem 9 be satisfied. Then the SLAE () has a unique solution* ξ_k, $k=-n,\dots,n$ *for numbers* $n\ge n_2(\ge n_1)$ *large enough. The approximate solutions* $x_n(t)$ *converge when* $n\to\infty$ *in the norm* $\overset{\circ}{W}{}_p^{(q)}$ *to exact solution* $x(t)$ *of the problem (15)-(7) and the following estimation for the convergence is true:*

$$||x-x_n||_{p,q}=\delta_n+O(\omega^\tau(h;\frac{1}{n})). \qquad (19)$$

To prove the Theorem 10 we can use the scheme from [23].

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