Sixth-Order Hybrid Boundary Value Method for Systems of Boundary Value Problems

Grace O. Akinlabi and Raphael B. Adeniyi

***Abstract*—** Hybrid Boundary Value Methods (HyBVMs) are a new class of Boundary Value Methods (BVMs) proposed recently for the approximation of Ordinary Differential Equations (ODEs). Just like the BVMs, the HyBVMs are also based on the Linear Multistep Methods (LMMs) while utilizing data at both step and off-step points. Numerical tests on both linear and nonlinear Boundary Value Problems (BVPs) were presented using HyBVMs of order 6. The results were compared with two symmetric schemes: Extended Trapezoidal Rule (ETR) and Top Order Method (TOM).

***Keywords*—** Boundary value methods, hybrid BVMs, boundary value problems, linear multistep method, numerical methods for ODEs.

I. Introduction

Boundary value Problems (BVPs) are applicable in Sciences and Engineering as they result from the modelisation of real world phenomena.

This class of problem is more difficult to handle, since it is a broader class of continuous problems unlike the Initial Value Problems (IVPs). They are usually solved by the Shooting Method (SHM). The SHM works by reducing the BVP to its equivalent IVP but it suffers from numerical instability as a result of this [1].

The Boundary Value Methods (BVMs) were proposed to remove this instability. The process of developing and applying the BVMs makes them suitable for solving BVPs without necessarily converting the BVPs to their equivalent Initial Value Problems (IVPs). For instance, in the derivation of BVMs, the same continuous scheme used to generate the main methods is also used in generating the additional methods; these are then applied at the end points thereby avoiding some of the stability problems encountered in the application of the SHM.

Lots of BVMs have been proposed by different authors and used for the approximation of different types of differential problems. Their convergence and stability properties have also been fully discussed [3] – [9].

Our focus in this work is to develop new BVMs that utilize data at off-step points and which will be called Hybrid Boundary Value Methods (HyBVMS). In deriving these methods, we will be adopting the Adams Moulton methods, which is a LMM of the form:

$$y_{n+k} - y_{n+k-1} = h\sum_{i=0}^{k}\beta_i f_{n+i} \tag{1.1}$$

This is done by using the Adam Moulton Methods at both step and off-step points. These methods are then applied as BVMs and used to solve the BVP of the form:

$$y'(x) = f\left(x, y(x)\right) \tag{1.2}$$

$$a_0 y(0) - b_0 y(0) = \alpha_0\ ,\quad a_1 y(1) - b_1 y(1) = \alpha_1$$

where all $f:\mathbb{R}^2 \to \mathbb{R}^2$ are continuous functions that satisfy the existence and uniqueness conditions, guaranteed by Henrici in [10].

Several authors have proposed different hybrid methods for the approximation of ODEs [11] – [16].

The application of BVMs for the numerical approximation of BVPs was first proposed by Brugnano and Trigiante in [17] with the two symmetric schemes: Extended Trapezoidal Rule (ETR) of order 4 and Top Order Method (TOM) of order 6.

II. Boundary Value Methods [18]

In this section, we present some of the important definitions on BVMs.

Definition 1: A polynomial $p(z)$ of degree $k = k_1 + k_2$ is called an S_{k_1,k_2} - polynomial if its roots are such that

$$|z_1| \le |z_2| \le \ldots \le |z_{k_1}| < 1 < |z_{k_1+1}| \le \ldots \le |z_k|\ .$$

Definition 2: A polynomial $p(z)$ of degree $k = k_1 + k_2$ is called an N_{k_1,k_2} - polynomial if its roots are such that

$$|z_1| \le |z_2| \le \ldots \le |z_{k_1}| \le 1 < |z_{k_1+1}| \le \ldots \le |z_k|$$

Definition 3: A BVM with (k_1, k_2) - boundary conditions is O_{k_1,k_2} - stable if its polynomial $p(z)$ is an N_{k_1,k_2} - polynomial.

Definition 4: A BVM with (k_1, k_2) - boundary conditions is (k_1, k_2) - Absolutely stable for a given complex q if the stability polynomial, $\pi(z,q)$ is an S_{k_1,k_2} - polynomial.

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Definition 5: The region D_{k_1,k_2} of the complex plane defined by

$$D_{k_1,k_2} = \{ q \in \mathbb{C} : \pi(z,q) \text{ is of type } (k_1, 0, k_2) \}$$

is called the region of (k_1, k_2) - Absolute stability.

III. Hybrid Boundary Value Methods

The HyBVMs are generalizations of the hybrid Adams-Moulton (AM) Methods. The hybrid AM can be written as:

$$y_{n+k} - y_{n+k-1} = h \sum_{i=0\left(\frac{1}{2}\right)}^{k} \beta_i f_{n+i} \tag{3.1}$$

These methods are used as IVMs but not BVMs and have been used in the past for the approximation of ODEs.
However, if we choose $k = v$ in (3.1)

$$\text{where } \; v = \begin{cases} \dfrac{k}{2}, & \text{for even } k \\ \dfrac{k+1}{2}, & \text{for odd } k \end{cases}$$

we obtain the HyBVMs.

For instance, the HyBVMs with an odd number of steps have the form:

$$y_{n+v} - y_{n+v-1} = h \sum_{r=0\left(\frac{1}{2}\right)}^{2v-1} \beta_r f_{n+r} \tag{3.2}$$

with the polynomial of the form: $p(z) = z^{v-1}(z-1)$. They are to be used with *(v-1,v)* boundary conditions and of order *2k+2*.
In this work, we apply one of the even HyBVMs (*k=2*) in the solution of some BVPs.
The sixth order HyBVM is given as:

$$y_{n+2} - y_n = \frac{h}{45}\left[7 f_n + 32 f_{n+\frac{1}{2}} + 12 f_{n+1} + 32 f_{n+\frac{3}{2}} + 7 f_{n+2} \right] \tag{3.3}$$

which is to be used together with the following initial methods:

$$y_{\frac{1}{2}} - y_0 = \frac{h}{1440}\left[251 f_0 + 646 f - 264 f_1 + 106 f_{\frac{3}{2}} - 19 f_2 \right]$$

and the final methods

$$y_{N-1} - y_N = -\frac{h}{180}\left[29 f_N + 124 f_{N-\frac{1}{2}} + 24 f_{N-1} + 4 f_{N-\frac{3}{2}} - f_{N-2} \right]$$

$$y_{N-\frac{3}{2}} - y_N = -\frac{h}{160}\left[27 f_N + 102 f_{N-\frac{1}{2}} + 72 f_{N-1} + 42 f_{N-\frac{3}{2}} - 3 f_{N-2} \right]$$

IV. Numerical Examples

In this section, we apply the HyBVM stated above to two (2) systems of BVPs. The maximum errors and Rate of Convergence (ROC) are compared with ETRs and TOMs.

Example 1: Consider the nonlinear second order BVP [19]:

$$y'' = \frac{(y')^2 + y^2}{2e^x} \quad , \qquad x \in (0,1)$$

with boundary conditions:

$$y(0) - y'(0) = 0\,, \quad y(1) + y'(1) = 2e$$

with exact solution: $\quad y(x) = e^x$

To solve, we first recast to its equivalent first order system:

$$y_1' = y_2$$

$$y_2' = \frac{(y_2)^2 + (y_1)^2}{2e^x}$$

for $x \in (0,1)$

with boundary conditions:

$$y_1(0) - y_2(0) = 0\,, \quad y_1(1) + y_2(1) = 2e$$

with exact solutions: $\quad y_1(x) = e^x, \; y_2(x) = e^x$

Example 2: Consider the linear second order BVP [17]:

$$y'' - 4y = 16x + 12x^2 - 4x^4 \quad , \qquad x \in (0,1)$$

with boundary conditions:

$$y(0) = y'(1) = 0$$

with exact solution: $\quad y(x) = x^4 - 4x$

To solve, we first recast to its equivalent first order system:

$$y_1' = y_2$$

$$y_2' = 4y_1 + 16x + 12x^2 - 4x^4$$

for $x \in (0,1)$

with boundary conditions:

$$y_1(0) = 0\,, \quad y_2(1) = 0$$

with exact solutions: $\quad y_1(x) = e^x, \; y_2(x) = e^x$

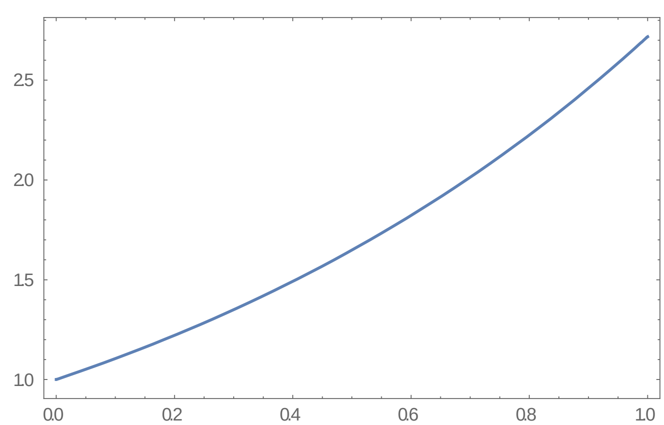

Fig. 1: Exact Solution of Example 1

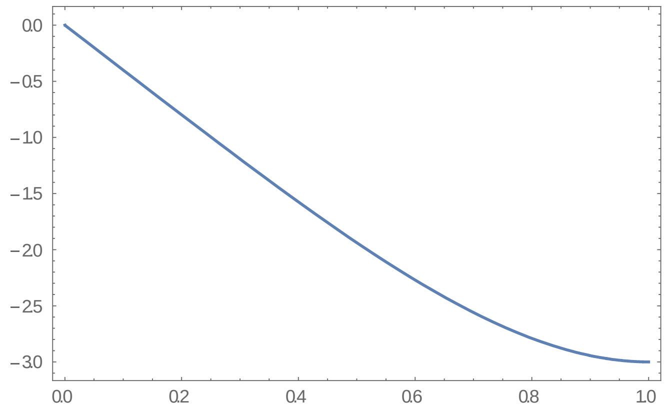

Fig. 2: Exact Solution of Example 2

V. Conclusion

In this work, we have applied a sixth-order HyBVM to two systems of BVPs and compared the maximum error and rate of convergence of the solutions with other two BVMs: ETR and TOM called symmetric schemes. In constructing these methods, we have adopted the Adams Moulton methods derived through interpolation and collocation procedure by utilizing data at both step and off-step points and implemented them as BVMs.

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Table I: Maximum errors for HBVM with ETR and TOM for Example 1

N	HBVM of order 4		ETR		TOM
	$\lVert e \rVert_\infty$	Rate	$\lVert e \rVert_\infty$	Rate	$\lVert e \rVert_\infty$
20	1.592e-11	-	7.448e-08	-	3.189e-11
40	1.765e-13	6.50	1.480e-09	5.65	1.972e-14
80	4.082e-15	5.43	2.576e-11	5.84	5.722e-16
160	6.280e-16	2.70	4.234e-13	5.93	6.732e-14

Table II: Maximum errors for HBVM with ETR and TOM for Example 2

h	HBVM of order 4		ETR of order 4		TOM of order 6
	$\lVert e \rVert_\infty$	ROC	$\lVert e \rVert_\infty$	ROC	$\lVert e \rVert_\infty$
0.25	1.332e-15	-	2.628e-3	-	1.776e-15
0.125	6.280e-16	1.08	1.955e-4	3.75	1.776e-15
0.0625	4.965e-16	0.34	1.359e-5	3.85	1.332e-15
0.03125	4.578e-16	0.12	8.989e-7	3.92	2.664e-15
0.015625	1.601e-16	1.52	5.785e-8	3.96	6.661e-15