Elementary Proof of Riemann's Hypothesis by the Modified Chi-square Function

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Abstract: - The present article shows a proof of Riemann's Hypothesis (RH) which is both general (i.e. valid for all the non-trivial zeroes of the zeta function) and elementary (that is not using the theory of the complex functions) in which the real constant $\sigma=+1/2$ arises by itself and automatically. The modified chi-square function in one of its four forms $(\pm 1/\cdot)X_k^2(\Omega,x/\omega)$ is used as an interpolating function of the progressions $\{n^{\pm\alpha}\}$, of their summations $\{\sum n^{\pm\alpha}\}$ and of the progressions $\{N^{\pm\alpha+1}/(\pm\alpha+1)\}$, with $\alpha\in\mathbb{R}$ $n,N\in\mathbb{N}$ so that $k=2\pm2\alpha$ and in the real plane (α,k) two half-lines are set up with $k<2$. The use of the Euler-MacLaurin formula with the one-to-one correspondence between the summation operation $\sum$ and the shift vector operator $\underline{\boldsymbol{\Sigma}}\equiv(\Sigma_\alpha,\Sigma_k)$ in the real 2D plane (α,k) lead to find the zeroes of Euler's function. Finally, the extrusion to the third imaginary axis $i\underline{\mathbf{t}}$ leads to prove Riemann's hypothesis.

Key-Words: - Riemann's hypothesis, modified chi-square function, numeric progressions

1 Introduction

Since 1859, when formulated by G.F.B. Riemann for the first time, Riemann's hypothesis (RH), "*all the non-trivial zeroes of the zeta function ζ(s)=ζ(σ+it) have real part equal to +1/2*", has been a challenge in number theory in that, though accepted and experimentally verified up to values of $t \approx 10^{15}$ and beyond, it has never been proven. Its proof would have important consequences in both mathematics and physics [1-8]. As the experimental confirmation is not enough for mathematics, the present article shows a general (that is valid for any value of $t\in\mathbb{R}$ up to ∞) and elementary (in the sense that it does not use the theory of complex functions) proof of RH.

The proof uses some simple techniques [9-11]:

1) the modified chi-square function as an interpolating function of the progressions $\{n^{\pm\alpha}\}$, of their summations $\{\sum n^{\pm\alpha}\}$ and of the progressions $\{N^{\pm\alpha+1}/(\pm\alpha+1)\}$, with $\alpha\in\mathbb{R}$ $n,N\in\mathbb{N}$ and the relationships $k=2\pm2\alpha$;

2) the Euler-MacLaurin formula valid for high enough N values;

3) the on-to-one correspondence between the summation operation $\sum$ and the shift vector operator $\underline{\boldsymbol{\Sigma}}$ in the plane (a k) so that the Euler equation $\zeta(\alpha)=0$ can be replaced by $\underline{\boldsymbol{\Sigma}}=\underline{\mathbf{0}}=$**null_vector** i.e. $|\underline{\boldsymbol{\Sigma}}|=0$ thus finding all the roots of Euler's equations that is the zeroes of the Euler zeta function;

4) the extrusion to the third imaginary axis $i\underline{\mathbf{t}}$ in both the positive and negative direction thus finding all the zeroes of Riemann's zeta functions, in the same way as above, all of them having real part +1/2.

2 Problem Formulation

The modified chi-square function $(\pm 1/\cdot)X_k^2(\Omega,x/\omega)$ with k degrees of freedom in one of its four forms:

$$(\pm 1/\cdot)X_k^2(\Omega,x/\omega) = \\ = (\pm 1/\cdot)[\Omega/(2\Gamma_{k/2})]\cdot[x/(2\omega)]^{(k/2-1)}\cdot e^{-x/2\omega} \quad (1)$$

with $k<2$ is a general version of the standard chi-square function $X_k^2(x)$ also used in statistics [12-15] with the values $\Omega = \omega = 1$ and is used as a fit function of the finite progressions $\{n^{\pm\alpha}\}$, of their summations $\{\sum_{(n=1\to N)}n^{\pm\alpha}\}$ and of the progressions $\{N^{\pm\alpha+1}/(\pm\alpha+1)\}$ taking advantage of the adjustment of its parameters k, Ω and ω just like the $f(x)=x^{\pm\alpha}$ function and just as the $\Gamma(x)=\Gamma_x$ function is an interpolating function of factorial numbers n! being $x\in\mathbb{R}^+$ and $n\in\mathbb{N}$.

As a matter of fact it is very easy to verify that it is possible to write

$$X_k^2(\Omega,x/\omega)=\{\Omega/[2\Gamma_{k/2}\cdot(2\omega)^{k/2-1}]\}\cdot x^{k/2-1}\cdot e^{-x/(2\omega)}$$

and, being Ω a free parameter that can be selected at one's own choice, being always $x \ll \omega$ as also experimentally verified and shown later on, it is possible to write

$$(\pm 1/\cdot)X_k^2(\Omega,x/\omega) = (\pm 1/\cdot)1\cdot x^{k/2-1}\cdot e^{-x/(2\omega)} \approx$$

$$\approx (\pm 1/\cdot) x^{k/2-1} = (\pm 1/\cdot) x^{\alpha} = \pm\, x^{\pm\alpha}$$

and, as we are not interested in the progressions $\{-n^{\pm\alpha}\}$ fitted by $-x^{\pm\alpha}$ we get

$$x^{k/2-1} = x^{\pm\alpha} \quad \text{i.e.} \quad k = 2 \pm 2\alpha$$

with real domain α and real co-domain $k<2$ for all the progressions $\{n^{\pm\alpha}\}$ and all their interpolating functions $(\pm 1/\cdot)X_k^2(\Omega,x/\omega)$. In such a way, from the geometric standpoint, two half-lines are created in the Euclidean real plane (α k) of equations

$$k = 2 + 2\alpha \ \text{ for } \alpha<0 \ \text{ and } \ k = 2 - 2\alpha \ \text{ for } \alpha>0 \qquad (2)$$

along which all the progressions $\{n^{\pm\alpha}\}$ and all their interpolating functions $(\pm 1/\cdot)X_k^2(\Omega,x/\omega)$ lay.

Simple numerical checks can be performed, as shown in the example of Fig. 1 where the fit of the progression $\{n^{-0.05}\}$ by the $X_k^2(A,x/x_o)$ function is shown with the fit parameters
$A=1\cdot 10^{-6}$ $x_o=9.74002\cdot 10^{125}$ $k=1.90=2+2\alpha$ $\Gamma_{k/2}=\Gamma_{0.95}=1.03145332$ $X^2_{\text{test-value}}=5.592\cdot 10^{-19}$ R=I=1.000000000000 within the precision of the calculations $\delta R=\delta I=1\cdot 10^{-12}$ being R the Bravais-Pearson correlation coefficient and I the non-linear index of correlation, both measuring the goodness of the fit. In addition $\langle n^{-0.05}\rangle = \langle X_k^2\rangle = 0.43246$ and $\sigma_{progr}=\sigma_{fit}=0.019826$

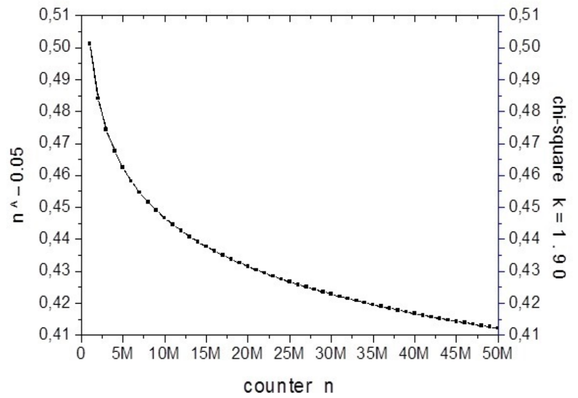

Fig. 1 Fit between the progression $\{n^{-0.05}\}$ and the function $X_k^2(A,x/x_o)$ with k=2+2α

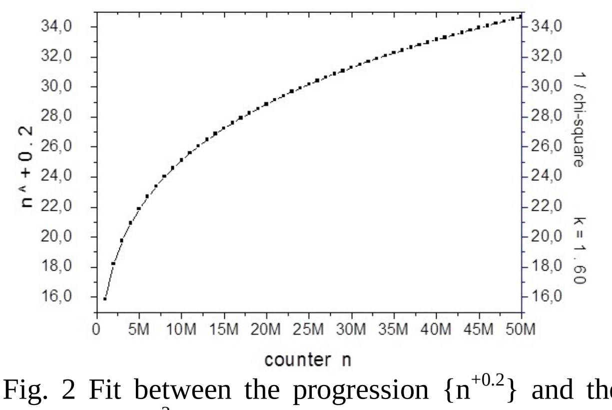

Fig. 2 Fit between the progression $\{n^{+0.2}\}$ and the function $1/X_k^2(A,x/x_o)$ with k=2–2α

Again in Fig. 2 the fit of the progression $\{n^{+0.2}\}$ by the $1/X_k^2(A,x/x_o)$ function is shown with the fit parameters $A=1\cdot 10^{-6}$ $x_o=3.422264\cdot 10^{31}$ $k=1.60= =2-2\alpha$ $\Gamma_{k/2}=\Gamma_{0.8}=1.16422971$ $X^2_{\text{test-val}}=1.84248\cdot 10^{-9}$ R=I=1.000000000000 again within the precision of the calculations $\delta R=\delta I=1\cdot 10^{-12}$ and $\langle n^{+0.2}\rangle=\langle 1/X_k^2\rangle =29.117001$ $\sigma_{progr}=\sigma_{fit}=4.621575$

Of course, in the fitting procedure, all the statistical tools have been used to make the fits at the utmost reliable level that is to match the data points and the fit curve as much as possible.

Both examples show that the decay (or growth according to the case) parameter is much greater than the number of terms $x_o \gg n_{max}$ as already anticipated.

The next step of the proof involves the examination of the summation progressions $\{\sum_{(n=1\to N)} n^{\pm\alpha}\}$ which, owing to the Euler-MacLaurin formula are equal to $\{N^{\pm\alpha+1}/(\pm\alpha+1)\}$ for N high enough, apart from an error term $\varepsilon\to 0$ for $N\to\infty$ as shown in the Fig. 3, Fig. 4 and Fig. 5 where the examples of the progression $\{n^{-0.5}\}\equiv X_k^2(A,x/x_o)$ and of the $\sum$ summation progression $\{\sum_{(n=1\to N)} n^{-0.5}\} \equiv \{N^{+0.5}/0.5\} \approx 1/X_k^2(A,x/x_o)$ are shown.

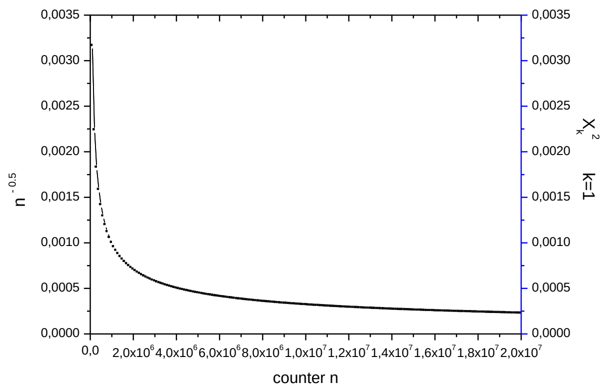

Fig. 3 Fit between the progression $\{n^{-0.5}\}$ and the function $X_k^2(A,x/x_o)$ with k=2+2α=1

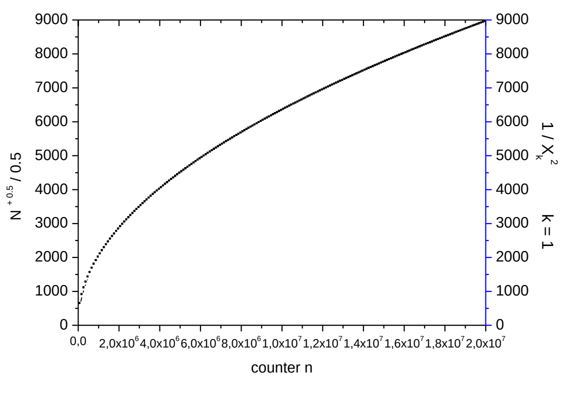

Fig. 4 Fit between the progression $\{N^{+0.5}/0.5\}$ and the function $1/X_k^2(A,x/x_o)$ with k=2–2α=1

Plain to say that this progression with the value of α=–1/2 is of the extreme importance in view of the proof of Riemann's hypothesis. Thus it deserves the utmost attention.

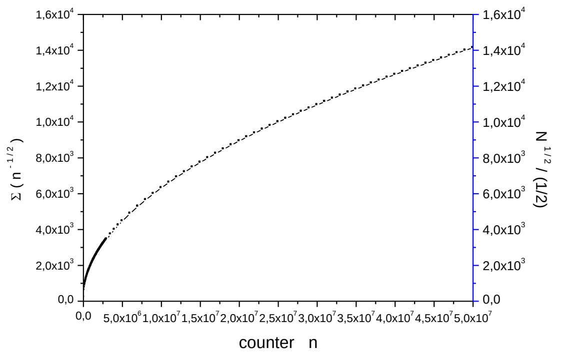

Fig. 5 Fit between the two progressions $\{\sum_{(n=1\to 2E7)} n^{-0.5}\}$ and $\{N^{+0.5}/0.5\}$

The previous Fig. 3 shows the progression $\{n^{-0.5}\}$ for $2\cdot 10^7$ terms as fitted by the $X_k^2(A,x/x_o)$ function with $k=1.0=2+2\alpha$ $x_o=2.00000\cdot 10^{12}$ with $R=I=0.999999999999$ $LSS=5.777417\cdot 10^{-10}$ $X^2_{test\text{-}value}=1.86468\cdot 10^{-13}$ while Fig. 4 shows the fit between the progression $\{N^{+0.5}/0.5\}$ again for $2\cdot 10^7$ terms as fitted by the $1/X_k^2(A,x/x_o)$ function with $k=1.0=2-2\alpha$ $x_o=5.00000\cdot 10^{11}$ $R=I=0.999999999999$ $LSS=2.199636\cdot 10^{-8}$ $X^2_{test\text{-}value}=3.297636\cdot 10^{-5}$

Fig. 5 shows the fit between the progression $\{\sum_{(n=1\to 2E7)} n^{-0.5}\}$ for $2\cdot 10^7$ terms and the progression $\{N^{+0.5}/0.5\}$ again for $2\cdot 10^7$ terms with $R=I=1.000000000000$ and $\delta R=\delta I= 1\cdot 10^{-12}$

Finally, Fig. 6 shows the error term (in percentage) and its linear (on a log-log scale) fit with the asymptotic limit $\lim_{(N\to\infty)}\varepsilon(N)=0$.

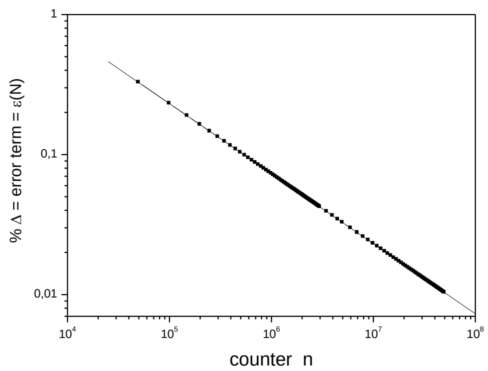

Fig. 6 Percentage error between the two progressions $\{\sum_{(n=1\to 2E7)} n^{-0.5}\}$ and $\{N^{+0.5}/0.5\}$

Thus it is possible to write the following schemes, all of them for $k<2$:

$$\{n^{-\alpha}\} \Rightarrow X_k^2(A,x/x_o) \quad k=2+2\alpha \quad \alpha>0$$
$$\{\sum_{(n=1\to N)} n^{-\alpha}\} \Rightarrow \{N^{-\alpha+1}/(-\alpha+1)\} \Rightarrow 1/X_k^2(A,x/x_o) \quad k=2-2\alpha \quad \alpha>0$$

that is

$$\{n^{-\alpha}\} \approx X_k^2(A,x/x_o) \quad k=2+2\alpha \quad \alpha>0$$
$$\{\sum_{(n=1\to N)} n^{-\alpha}\} \approx \{N^{-\alpha+1}/(-\alpha+1)\} \approx 1/X_k^2(A,x/x_o) \quad k=2-2\alpha \quad \alpha>0$$

where, in addition, being $\Delta\alpha = +1$ always and $k=2\pm 2\alpha$ one gets $\Delta k = \pm 2$ according to the case.

However, it is of the utmost importance to highlight that the latest relation $\Delta k = \pm 2$ holds for $\alpha < -1$ and $\alpha > 0$ in that within the range $\alpha \in (-1,0)$ there is a different situation, as described in the following.

3 PROBLEM SOLUTION

It is possible, now, to move to the geometric representation of the situation on the (α k) Euclidean real plane as in Fig. 7 where the two half-lines $k=2+2\alpha$ for $\alpha<0$ and $k=2-2\alpha$ for $\alpha>0$ both valid for $k<2$ are shown, crossing one each other at the point $(\alpha,k)\equiv(0,2)$, and being the pillars of the whole topic. As a matter of fact, owing to the above relationships from the geometric viewpoint any summation operation $\sum_{(n=1\to N)} \equiv \sum$ can be put in a one-to-one correspondence with a shift vector operator in the Euclidean plane (α k) with components $(\Sigma_\alpha\ \Sigma_k) \equiv (\Delta\alpha\ \Delta k)$ so that

$\sum \leftrightarrow \underline{\mathbf{\Sigma_{1,2}}} \equiv (\Sigma_\alpha\ \Sigma_k) \equiv (\Delta\alpha\ \Delta k) \equiv (+1\ \pm 2)$ and $|\underline{\mathbf{\Sigma_{1,2}}}| = \Sigma_{1,2} = \sqrt{5}$ for $\alpha < -1$ and $\alpha > 0$ respectively.

As for the α range (–1,0) and $k\in(0,2)$ the situation is easily verified again in Fig. 7.

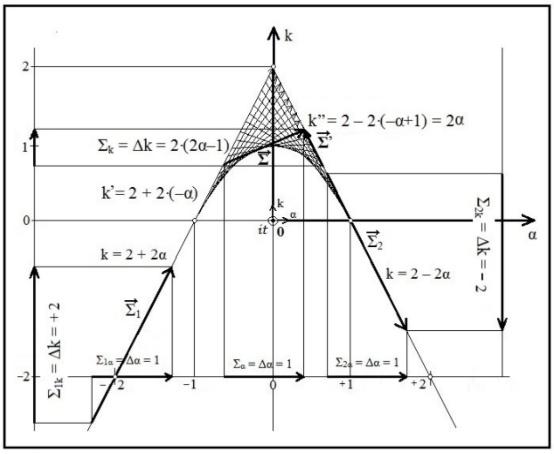

Fig. 7 Examples of the three vectors $\underline{\mathbf{\Sigma_1}}$ $\underline{\mathbf{\Sigma_2}}$ and $\underline{\mathbf{\Sigma}}$ on the plane (α,k) with the two half lines $k=2\pm 2\alpha$

Here it can be seen that, while the $\underline{\mathbf{\Sigma_1}}$ and $\underline{\mathbf{\Sigma_2}}$ shift vectors lay on the two half-lines $k = 2\pm 2\alpha$ with

$k < 2$ and $\alpha < -1$ $\alpha > 0$, the $\underline{\mathbf{\Sigma}}$ vector, valid for $\alpha \in (-1, 0)$, has its application point on the first half-line ($k=2+2\alpha$) and its final point on the second one ($k=2-2\alpha$) spinning around the focus $F \equiv (0, 0.75)$ of the parabola $k=1-\alpha^2$ which is the envelope of all these $\underline{\mathbf{\Sigma}}$ vectors. Having renamed the half-line $k=2+2\alpha$ by $k=2-2\alpha$ thus highlighting the negative values of α and thus being $k = 2-2\alpha = 2-2(\alpha+1) = 2\alpha$, it is easy to check that the vector $\underline{\mathbf{\Sigma}}$ has components $\equiv (\Sigma_\alpha\ \Sigma_k) \equiv (\Delta\alpha\ \ \Delta k) \equiv (+1\ \ 4\alpha-2)$ and that, while rotating around this point F, it changes its norm according to its application point $(\alpha,\ k) \equiv (\alpha, 4\alpha-2)$ thus having norm

$$\left|\underline{\mathbf{\Sigma}}\right| = \Sigma = \sqrt{[1+(4\alpha-2)^2]} = \sqrt{(16\alpha^2-16\alpha+5)} = \Sigma(\alpha)$$

The next step is straightforward in that, as for the zeroes of Euler's function that is the roots of the equation $\zeta(\alpha) = \sum n^{-\alpha} = 0$, owing to the above-said correspondence between the summation operation $\sum$ and the shift vector operator $\underline{\mathbf{\Sigma}}$ i.e. $\sum \leftrightarrow \underline{\mathbf{\Sigma}}$ it is possible to write for the zeroes of Euler's function

$$\underline{\mathbf{\Sigma}} = \mathbf{0} = \textbf{null vector} \quad \Rightarrow$$
$$\Rightarrow \quad \left|\underline{\mathbf{\Sigma}}\right| = \Sigma = \sqrt{[1+(4\alpha-2)^2]} = \sqrt{(16\alpha^2-16\alpha+5)} = 0$$

and solving this trivial equation leads to the two complex solutions

$$\alpha = +1/2 \pm i/4$$

Again looking at Fig. 7 this means that the condition $\zeta(\alpha) = 0$ is satisfied when the vector $\underline{\mathbf{\Sigma}}$ is horizontal, i.e. $\Sigma_k = \Delta k = 4\alpha - 2 = 0$

The extrusion to the third imaginary axis $i\underline{\mathbf{t}}$ leads to Fig. 8 (having renamed the α axis by σ thus following the standard symbolism with $s = \sigma + it$) where the two half-lines $k=2\pm2\alpha$ have become the two half planes crossing one-each-other along the line ($\sigma=0 \cap k=2$) parallel to the $i\underline{\mathbf{t}}$ axis, thus forming a dihedron.

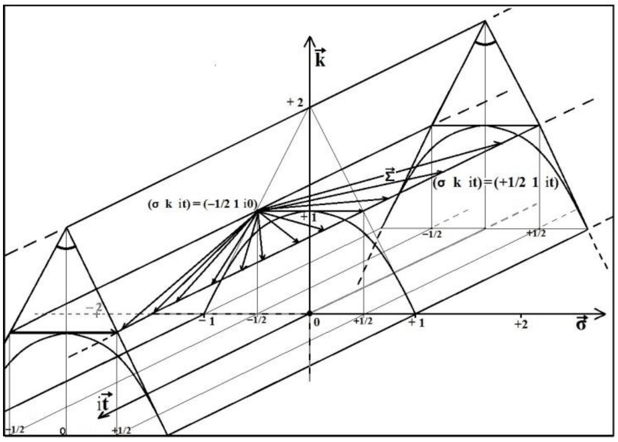

Fig. 8 Extrusion of the whole plane (α k) to the third imaginary axis $i\underline{\mathbf{t}}$ in both positive and negative directions (b) having renamed α by σ

All that means that the generic $\underline{\mathbf{\Sigma}}$ vector in the 3D complex space (σ k it) has components $(\Sigma_\sigma\ \Sigma_k\ \Sigma_{it}) \equiv (\Delta\sigma\ \ \Delta k\ \ i\Delta t) \equiv (1\ \ 4\sigma-2\ \ it)$ and, owing to the fact that we are in the Euclidean space where the metric is simply the unitary diagonal matrix of components δ_{mn} = Kronecker symbol ($\delta_{mn} = 1$ if $m = n$ and $= 0$ otherwise), the norm of this vector is

$$\left|\underline{\mathbf{\Sigma}}\right| = \Sigma = \sqrt{(\Sigma_m \delta_{mn} \Sigma_n)} = \sqrt{[1+(4\sigma-2)^2 + t^2]} =$$
$$= \sqrt{(16\sigma^2 - 16\sigma + 5 + t^2)}$$

In such a way it is possible to write, for the zeroes of Riemann's function,

$$\zeta(s) = \zeta(\sigma + it) = \sum n^{-s} = \sum n^{-(\sigma + it)} = 0$$

that is, again as above, using the correspondence between the summation operation and the shift vector operator

$$\sum \leftrightarrow \underline{\mathbf{\Sigma}} \quad \Rightarrow \quad \underline{\mathbf{\Sigma}} = \mathbf{0} = \textbf{null vector} \quad \Rightarrow$$
$$\Rightarrow \quad \left|\underline{\mathbf{\Sigma}}\right| = \Sigma = \sqrt{(16\sigma^2 - 16\sigma + 5 + t^2)} = 0$$

Solving this equation leads to the complex solutions

$$\sigma = +1/2 \pm i/4 \cdot \sqrt{(1+t^2)} \qquad \forall\ t \in \mathbb{R}$$

that is just Riemann's Hypothesis: *all the non-trivial zeroes of the zeta function have real part equal to +1/2*. In addition all these zeroes are symmetric in respect to the imaginary axis $i\underline{\mathbf{t}}$ as expected.

Going back again to Fig. 8 all the $\underline{\mathbf{\Sigma}}$ vectors lay on the plane k=+1 along the strip $\sigma \in (-1/2, +1/2)$ with their application points on (σ k it)≡(–1/2 1 it) and their end points on (σ k it)≡(+1/2 1 it) being all horizontal that is parallel to the complex plane s that is k=0.

4 Conclusion

The innovative methodology of fitting the numeric progressions $\{n^{\pm\alpha}\}$ as well as $\{\sum n^{\pm\alpha}\}$ and $\{N^{\pm\alpha+1}/(\pm\alpha+1)\}$ by the modified chi-square function in one of its four forms $(\pm 1/\cdot)X_k^{\,2}(\Omega,x/\omega)$ has led to an elementary proof of Riemann's hypothesis (elementary in the sense that it does not use the theory of complex functions), a results that has been awaited since 157 years ago and never attained before now.

It has to be remarked that, in this proof, the real constant $\sigma = +1/2$ arises by itself and automatically in a straightforward way.

As for the future developments, the first one is the study of this shift vector operator also in the light of Hilbert and Pòlya conjecture, while further topics will concern the use of the modified chi-square function with k<2 for the analytical treatment of the finite sequences of prime numbers, with the goal of getting a more refined version of the prime number theorem (PNT), as well as the use of the same function with k>2 for the statistical treatment of the normalized spacing of the non-trivial zeroes of Riemann's zeta function in the frame of random matrices and Gaussian ensembles.

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