Integro-differential splines and quadratic formulae

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Abstract: This work is devoted to the further investigation of splines of the fifth order approximation. Here we present some new formulae which are useful for the approximation of functions with one or two variables. For each grid interval (or elementary rectangular) we construct the approximation separately. Here we construct the basic one-dimensional polynomial splines of the fifth order approximation when the values of the function and the values of its first derivative are known in each point of interpolation. Sometimes it is important that the integrals of the function over the intervals are equal to the integrals of the approximation of the function over the intervals. In that case the approximation has some physical parallel. For this aim we use quadratic formulae here with the sixth order of approximation instead of the value of integral. The one-dimensional case can be extended to multiple dimensions through the use of tensor product spline constructs. Numerical examples are represented.

Key–Words: Polynomial splines, Integro-Differential Splines, Interpolation.

1 Introduction

The idea of the spline interpolation was born in England at the end of the 19th century when British engineers designed the first railroad tracks. The spline interpolation was then considered as a more appropriate alternative to polynomial interpolation. Now there are a variety of different types of splines that are used for solving different mathematical, mechanical, physical and engineering problems.

This method of approximation using polynomial splines is widely used for the interpolation and approximation of discrete data. A lot of research has been devoted to the application of various splines with different properties for approximation and estimation of data. Special attention is given to methods of constructing images [1–10].

As is well known, the one-dimensional case can be extended to multiple dimensions through the use of tensor product spline constructs [11–13].

Suppose that n is a natural number, while a, b are real numbers, $h = (b - a)/n$. Let us build the grid of interpolation nodes $x_j = a + jh$, $j = 0, 1, \dots, n$.

2 Left polynomial splines of one variable

Let the function $u(x)$ be such that $u \in C^5([a, b])$.

Suppose that we know $u(x_j)$, $u'(x_j)$, $j = 0, 1, \dots, n$. The next quadratic formula is well known:

$$\int_{x_{j-1}}^{x_{j+1}} u(x)dx = V_j(u) + O(h^6),$$

where

$$V_j(u) = \frac{(x_{j+1} - x_{j-1})}{30}(7u(x_{j-1}) + 7u(x_{j+1})+$$

$$16u(x_j)) - \frac{(x_{j+1} - x_{j-1})^2}{60}(u'(x_{j+1}) - u'(x_{j-1})).$$

We denote by $\widetilde{u}(x)$ an approximation of the function $u(x)$ on the interval $[x_j, x_{j+1}] \subset [a, b]$:

$$\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x)+$$
$$+u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x)+$$
$$+V_j(u)\,\omega_j^{<0>}(x). \quad (1)$$

The basic splines $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega_j^{<0>}(x)$, we obtain from the system:

$$\widetilde{u}(x) \equiv u(x), \;\; u(x) = x^{i-1}, i = 1, 2, 3, 4, 5. \quad (2)$$

Suppose that $supp\,\omega_{k,\alpha} = [x_{k-1}, x_{k+1}]$, $\alpha = 0, 1$, $supp\,\omega_k^{<0>} = [x_k, x_{k+1}]$. It is easy to see that $\omega_{k,0}, \omega_{k,1}, \omega_k^{<0>} \in C^1(R^1)$. We have for $x = x_j + th$, $t \in [0, 1]$ the next formulae:

$$\omega_{j,0}(x_j + th)=(2t + 1)(t - 1)^2, \quad (3)$$

$$\omega_{j+1,0}(x_j + th)= -(1/8)t^2(15t^2 - 14t - 9), \quad (4)$$

$$\omega_{j,1}(x_j+th)=(1/4)th(5t+4)(t-1)^2, \quad (5)$$

$$\omega_{j+1,1}(x_j+th)=(1/8)ht^2(5t+3)(t-1), \quad (6)$$

$$\omega_j^{<0>}(x_j+th)=(15/16)t^2(t-1)^2/h. \quad (7)$$

Figures 1, 2, 3 show the graphics of the basic functions $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega_j^{<0>}(x)$, when $h=1$. Figure 3 (right) shows the error of approximation of the Runge function $u(x)=1/(1+25x^2)$ with the polynomial splines, $h=0.1$, $x\in[-1,1]$.

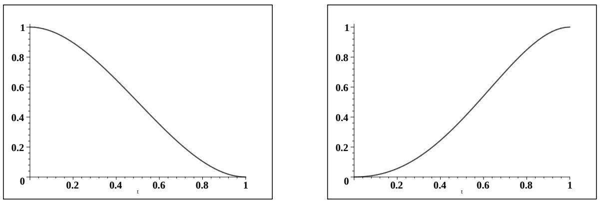

Figure 1: Plots of the basic functions: $\omega_{j,0}(x)$ (left), $\omega_{j+1,0}(x)$ (right)

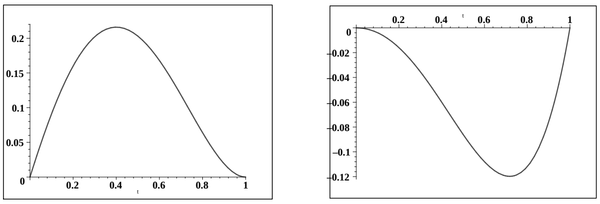

Figure 2: Plots of the basic functions: $\omega_{j,1}(x)$, when $h=1$(left), $\omega_{j+1,1}(x)$, when $h=1$ (right)

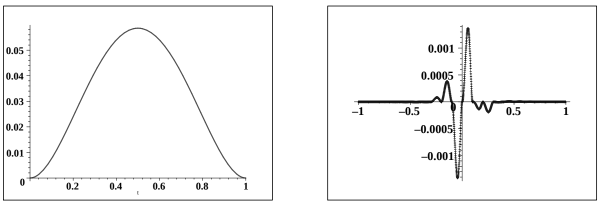

Figure 3: Plots of the basic functions: $\omega_j^{<0>}(x)$, when $h=1$ (left), and the error of approximation of the Runge function with the polynomial splines, $h=0.1$, $x\in[-1,1]$ (right)

Let us take $\tilde{U}(x)$, $x\in[a,b]$, such that $\tilde{U}(x)=u(x)$, $x\in[x_j,x_{j+1}]$. Let $\|u\|_{[a,b]}=\max\limits_{[a,b]}|u(x)|$.

Theorem 1. Let function $u(x)$ be such that $u\in C^5([a,b])$. For approximation $u(x)$, $x\in[x_j,x_{j+1}]$ by (1), (3) – (7) we have:

$$|\tilde{u}(x)-u(x)|_{[x_j,x_{j+1}]}\le K_1h^5\|u^{(5)}\|_{[x_{j-1},x_{j+1}]}, \quad (8)$$

$K_1=0.0225.$

$$|\tilde{u}'(x)-u'(x)|_{[x_j,x_{j+1}]}\le K_2h^4\|u^{(5)}\|_{[x_{j-1},x_{j+1}]}, \quad (9)$$

$K_2=0.0994.$

$$|\tilde{u}(x)-u(x)|_{[a+h,b]}\le K_1h^5\|u^{(5)}\|_{[a,b]}. \quad (10)$$

Proof. Inequality (8) follows from Taylor's theorem and the inequalities:
$|\omega_{j,0}(x)|\le 1$, $|\omega_{j+1,0}(x)|\le 1$,
$|\omega_{j,1}(x)|\le 0.216h$, $|\omega_{j+1,1}(x)|\le 0.1198h$,
$|\omega_j^{<0>}(x)|\le 0.0586/h$.
Inequality (10) follows from (8).

We have the next expressions for derivatives of basic functions:
$\omega'_{j,0}(x_j+th)=6t(t-1)/h$,
$\omega'_{j+1,0}(x_j+th)=-(3/4)t(-7t-3+10t^2)/h$,
${\omega'}_j^{<0>}(x_j+th)=(15/8)t(1+2t^2-3t)/h^2$,
$\omega'_{j,1}(x_j+th)=-(3/2)t-(9/2)t^2+5t^3+1$,
$\omega'_{j+1,1}(x_j+th)=-(3/4)t-(3/4)t^2+(5/2)t^3$.
Inequality (9) follows from Taylor's theorem and the inequalities:
$|\omega'_{j,0}(x)|\le 1.5/h$, $|\omega'_{j+1,0}(x)|\le 1.626/h$,
$|\omega'_{j,1}(x)|\le 1$, $|\omega'_{j+1,1}(x)|\le 1$,
$|{\omega'}_j^{<0>}(x)|\le 0.181/h^2$.

Theorem 2. Let function $u(x)$ be such that $u\in C^5([a,b])$. We have:

$$\int_{x_j}^{x_{j+1}}(\tilde{u}(x)-u(x))\le 0.0081h^6\|u^{(5)}\|_{[x_{j-1},x_{j+1}]}.$$

The proof is obvious and it is similar to what it was in Theorem 1.

3 Compering with the Hermit interpolation

Here we shall compare the approximation that has been constructed above and the Hermite approximation:

$$\widetilde{u}^H(x)=u(x_{j-1})\omega_{j-1,0}(x)+u(x_j)\omega_{j,0}(x)$$
$$+u(x_{j+1})\omega_{j+1,0}(x)+u'(x_j)\omega_{j,1}(x)+$$
$$u'(x_{j+1})\omega_{j+1,1}(x), x\in[x_j,x_{j+1}].$$

The basic splines $\omega_{j-1,0}(x),\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x),\omega_{j+1,1}(x)$ we obtain from the system:

$$\widetilde{u}^H(x)\equiv u(x),\ \ u(x)=x^{i-1}, i=1,2,3,4,5. \quad (11)$$

Suppose that $supp\,\omega_{k,0} = [x_{k-1}, x_{k+2}]$, $supp\,\omega_{k,1} = [x_{k-1}, x_{k+1}]$. It is easy to see that $\omega_{k,0}, \omega_{s,1}, \in C^1(R^1)$, $k = j-1, j, j+1$, $s = j, j+1$.

We have for $x = x_j + th$, $t \in [0,1]$, the next formulae:

$$\omega_{j,0}(x_j+th) = (-1+t)^2(t+1)^2, \qquad (12)$$

$$\omega_{j+1,0}(x_j+th) = -(1/4)t^2(t+1)(5t-7), \qquad (13)$$

$$\omega_{j,1}(x_j+th) = th(t+1)(-1+t)^2, \qquad (14)$$

$$\omega_{j+1,1}(x_j+th) = (1/2)ht^2(-1+t)(t+1), \qquad (15)$$

$$\omega_{j-1,0}(x_j+th) = (1/4)t^2(-1+t)^2. \qquad (16)$$

Table 1 shows the errors $R^I = \max_{x\in[a,b]} |\tilde{u} - u|$, $R^{II} = \max_{x\in[a,b]} |\tilde{u}^H - u|$ when $[a,b] = [-1,1]$, $h = 0.1$. Calculations were done in Maple, Digits=15.

Table 1.

$u(x)$	R^I	R^{II}
x^4	0.0	0.0
$1/((1+25x^2)$	$0.1417e-2$	$0.1531e-2$
$\sin(5x) - \cos(5x)$	$0.2913e-4$	$0.3466e-4$

4 About approximations with two variables

Suppose that n, m are natural numbers, while a, b, c, d are real numbers, $h_x = (b-a)/n$, $h_y = (d-c)/m$. Let us build the grid of interpolation nodes $x_j = a + jh_x$, $j = 0,1,\ldots,n$, $y_k = c + kh_y$, $k = 0,1,\ldots,m$. On every line parallel to axis y, we can construct the approximation in the form:

$$\widetilde{u}(y) = u(y_k)\omega_{k,0}(y) + u(y_{k+1})\omega_{k+1,0}(y) +$$
$$u'(y_k)\omega_{k,1}(y) + u'(y_{k+1})\omega_{k+1,1}(y) +$$
$$+V_k\,\omega_k^{<0>}(y), \quad y \in [y_k, y_{k+1}]. \qquad (17)$$

Now the formulae for $\omega_{k,0}(y)$, $\omega_{k+1,0}(y)$, $\omega_{k,1}(y)$, $\omega_{k+1,1}(y)$, $\omega_k^{<0>}(y)$ are similar to the previous ones.

If $(x,y) \in \Omega_{j,k}$ then we get the next expression using the tensor product:

$$\tilde{u}(x,y) = \sum_{i=0}^{1}\sum_{p=0}^{1} u(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,0}(y) +$$

$$\sum_{i=0}^{1}\sum_{p=0}^{1} u'_y(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,1}(y) +$$

$$\sum_{i=0}^{1} V_{j+i,k}(x)\omega_{j+i,0}(x)\omega_k^{<0>}(y) +$$

$$\sum_{i=0}^{1} V_{j,k+i}\omega_j^{<0>}(x)\omega_{k+i,0}(y) +$$

$$\sum_{i=0}^{1} S_{j,k+i}\omega_j^{<0>}(x)\omega_{k+i,1}(y) +$$

$$W_{j,k}\omega_k^{<0>}(y)\omega_j^{<0>}(x) +$$

$$\sum_{i=0}^{1} u'_x(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,0}(y) +$$

$$+\sum_{i=0}^{1} u''_{xy}(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,1}(y) +$$

$$\sum_{i=0}^{1} P_{j+i,k}\omega_{j+i,1}(x)\omega_k^{<0>}(y), \qquad (18)$$

where

$$V_{j+i,k} = \frac{(y_{k+1}-y_{k-1})}{30}(7u(x_{j+i}, y_{k-1}) +$$
$$7u(x_{j+i}, y_{k+1}) + 16u(x_{j+i}, y_k)) -$$
$$\frac{(y_{k+1}-y_{k-1})^2}{60}(u'_y(x_{j+i}, y_{k+1}) - u'_y(x_{j+i}, y_{k-1})),$$

$$V_{j,k+i} = \frac{(x_{j+1}-x_{j-1})}{30}(7u(x_{j-1}, y_{k+i}) +$$
$$7u(x_{j+1}, y_{k+i}) + 16u(x_j, y_{k+i})) -$$
$$\frac{(x_{j+1}-x_{j-1})^2}{60}(u'_x(x_{j+1}, y_{k+i}) - u'_x(x_{j-1}, y_{k+i})),$$

$$S_{j,k+i} = \frac{(x_{j+1}-x_{j-1})}{30}(7u'_y(x_{j-1}, y_{k+i}) +$$
$$7u'_y(x_{j+1}, y_{k+i}) + 16u_y(x_j, y_{k+i})) -$$
$$\frac{(x_{j+1}-x_{j-1})^2}{60}(u''_{xy}(x_{j+1}, y_{k+i}) - u''_{xy}(x_{j-1}, y_{k+i})),$$

$$P_{j+i,k} = \frac{(y_{k+1}-y_{k-1})}{30}(7u'_x(x_{j+i}, y_{k-1}) +$$
$$7u'_x(x_{j+i}, y_{k+1}) + 16u'_x(x_{j+i}, y_k)) -$$
$$\frac{(y_{k+1}-y_{k-1})^2}{60}(u''_{yx}(x_{j+i}, y_{k+1}) - u''_{yx}(x_{j+i}, y_{k-1})),$$

$$W_{jk} = \frac{(y_{k+1}-y_{k-1})}{30}(7G(x_j, y_{k-1}) +$$
$$7G(x_j, y_{k+1}) + 16G(x_j, y_k)) -$$
$$\frac{(y_{k+1}-y_{k-1})^2}{60}(G'y(x_j, y_{k+1}) - G'y(x_j, y_{k-1})),$$

$$G(x_j, y) = \frac{(x_{j+1}-x_{j-1})}{30}(7u(x_{j-1}, y) +$$

$$7u(x_{j+1},y)+16u(x_j,y))-$$
$$\frac{(x_{j+1}-x_{j-1})^2}{60}(u'_x(x_{j+1},y)-u'_x(x_{j-1},y)).$$

Graphs 4, 5 shows approximations and the errors of approximations $\tilde{u}(x,y)-u(x,y)$ by (18), (3)–(7), (12)–(16) of functions $u_1(x,y) = \sin(3x-3y)\cos(3x-3y)$, $u_2(x,y)=(x-y)^2(x+y)^2$, when $[a,b]=[-1,1]$, $[c,d]=[-1,1]$, $h_x=h_y=h=0.2$. Calculations were done in Maple, Digits=15.

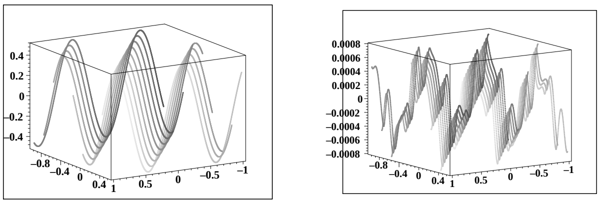

Figure 4: Plots of the functions: $\tilde{u}(x,y)=\sin(3x-3y)\cos(3x-3y)$ (left) and $\tilde{u}(x,y)-u(x,y)$ (right) when $h=0.2$, $[-1,1]\times[-1,1]$

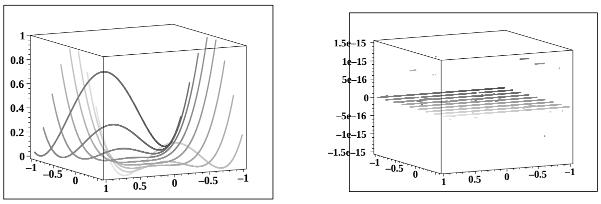

Figure 5: Plots of the functions: $\tilde{u}(x,y)=(x-y)^2(x+y)^2$ (left) and $\tilde{u}(x,y)-u(x,y)$ (right) when $h=0.2$, $[-1,1]\times[-1,1]$

5 Conclusion

Basic spines can be applied for solving various mathematical problems. We can obtain the formulae of our basic splines in the following way. In the interval $[x_{j-1},x_j]$ we obtain basic splines from the system:

$$\widetilde{u}(x)\equiv u(x),\ \ u(x)=x^{i-1}, i=1,2,3,4,5,$$

where

$$\widetilde{u}(x)=u(x_{j-1})\omega_{j-1,0}(x)+u(x_j)\omega_{j,0}(x)+$$
$$u'(x_{j-1})\omega_{j-1,1}(x)+u'(x_j)\omega_{j,1}(x)+V_{j-1}\,\omega_{j-1}^{<0>}(x).$$

If we take the basic splines with the same numbers from $[x_{j-1},x_j]$ and $[x_j,x_{j+1}]$ then we have:

$$\omega_{j,0}(x_j+th)=\begin{cases}-\frac{15}{8}t^4-\frac{23}{4}t^3-\frac{39}{8}t^2+1, t\in[-1,0],\\ 2t^3-3t^2+1, \qquad t\in[0,1],\\ 0, \qquad t\notin[-1,1],\end{cases}$$

$$\omega_{j,1}(x_j+th)=\begin{cases}\frac{5h}{8}t^4+\frac{9h}{4}t^3+\frac{21h}{8}t^2+th,\\ \qquad t\in[-1,0],\\ \frac{5h}{4}t^4-\frac{3h}{2}t^3-\frac{3h}{4}t^2+th, t\in[0,1],\\ 0, \qquad t\notin[-1,1],\end{cases}$$

$$\omega_j^{<0>}(x_j+th)=\begin{cases}\frac{15}{16h}t^2(t-1)^2,\\ \qquad t\in[0,1]\\ 0, \qquad t\notin[0,1].\end{cases}$$

Figure 6 shows the plots of the basic splines $\omega_{j,0}$, $\omega_{j,1}$. The plot of the basic spline $\omega_j^{<0>}$ is shown in Figure 3.

The construction of the nonpolynomial splines with the same properties and their application for the solving of different problems will be regarded in further papers.

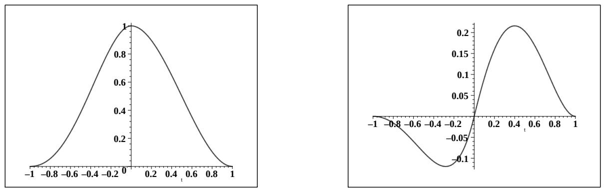

Figure 6: Plots of the basic functions: $\omega_{j,0}(jh+th)$ (left), and $\omega_{j,1}(jh+th)$, when $h=1$ (right)

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