Linear Quadratic Regulator Problem for Positive Systems with Polyhedral Cone Constraints

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Abstract: In the paper, the infinite horizon LQR problem of linear discrete time systems with nonnegative state constraints is studied. The state constraints are defined as a polyhedral cone belonging to the nonnegative orthant. It is investigated when the solution of the unconstrained LQR problem coincide with the solution of constrained problem, when there is no solution and when there is solution of constrained problem, but it doesn't coincide with the solution of unconstrained LQR problem.

Key-Words: Positive systems, Invariant sets, Quadratic programming

1 Introduction

In the paper, the infinite horizon LQR problem of linear discrete time systems with state constraints is studied. The state constraints are defined as a polyhedral cone belonging to the nonnegative orthant. The solution of the problem is based on the work of Kalman that identify an analytic expression for optimal control strategy. A number of studies in the literature have focused on systems with state and /or input constraints. Several approaches, methods and algorithms are developed to solve, online or offline, the constrained infinite horizon LQR problem for discrete, continuous, linear and nonlinear systems. Typically in these methods, the constraints sets are convex and contain the origin in their interior [7].

When we consider systems with nonnegative state restriction, known in the literature as positive systems, a question of particular importance is the invariance of the nonnegative orthant of the state space. Typical examples of positive systems are economic models, age-structured populations, sociological processes, etc. An overview of state of the art in positive systems theory is given in [5,2]. Often a stronger restriction is imposed on the system state than nonnegativity, namely the state trajectory must stay in the polyhedral cone belonging to the nonnegative orthant.

The motivation for the study of infinite horizon LQR problem of linear discrete time systems with nonnegative cone constraints on the state is that in this type of constraints sets zero is border point. In all existing methods, the constraints sets are polytopic sets with zero as interior point. So it was interesting to study if and when the solution of the unconstrained problem is a solution to the problem with nonnegative cone constraints on the state.

There are some papers studying LQ-optimal control of positive linear systems [3,6,10,11], but the problem is not solved in general and there are several open questions.

In section 2 we will give some preliminaries concerning polyhedral cone and M-matrices. In section 3 definition of the problem is given and in section 4 we study if and when the solution exists, in which cases the solution of our problem coincide with the solution of the unconstrained problem and give application of the dual mode approach for solving the problem. Some concluding remarks and references are given at the end of the paper.

2 Preliminaries

2.1 Polyhedral cones and *M*-matrices

Polyhedral cones and ***M***-matrices are basic when study positive systems and constraints given as polyhedral cones.

Definition1 [2] A nonempty subset C of $\mathbb{R}^n$ is a polyhedral cone if and only if C is the intersection of a finite number of closed half spaces, each containing the origin on its boundary. A polyhedral cone is closed convex cone.

A polyhedral cone, denoted by $C_{\mathrm{G}} \subseteq \mathbb{R}^n$, can be also considered as a special class of polyhedra, and it can be defined as:

$$C_{\mathrm{G}} = \{x \in \mathbb{R}^n | Gx \geq 0\} \quad (1)$$

Consider the matrix $\boldsymbol{A}$ with nonpositive off-diagonal and nonnegative diagonal entries.

$$\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{n\times n}, a_{ij} \leq 0, i \neq j;\ a_{ij} \geq 0, i = j$$

The matrix $\boldsymbol{A}$ can be expressed in the form:

$$\boldsymbol{A} = s\boldsymbol{I} - \boldsymbol{B}; \boldsymbol{B} \geq 0; s > 0 \quad (2)$$

Definition 2 [1] Any matrix $\boldsymbol{A}$ in the form (2) for which $s > \rho(\boldsymbol{B}), \rho$ −spectral radius of $\boldsymbol{B}$ is called ***M-matrix***

The class of M-matrix have very interesting properties, given in [1]. One of the most applicable is that for every *M*-matrix exists $\boldsymbol{M^{-1}}$ and $\boldsymbol{M^{-1}} \geq \boldsymbol{0}.$

Using the important property of positive inverse, *M*-matrices lead to the construction of polyhedral cones.

If $\boldsymbol{G}$ is *M*-matrix, the polyhedral cone defined by

$$C_G = \{\boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{Gx} \geq 0\} \quad (3)$$

is solid, simplical and interiour of the cone belongs to the nonnegative orthant, i.e. $Int\ C_G \subset Int\ R^n_+$.

Hence when we consider positive linear discrete time systems (PLDS) with cone constraints on the state we can add to the system constraints:

$$0 \leq \boldsymbol{Gx_k}, k = 1,2, \dots; G - M \text{ matrix}$$

Note that in case when the matrix $\boldsymbol{G}$ is equal to identity matrix we have only nonnegativity constraints i.e. the cone

$C_G = \{\boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{Gx} \geq 0\}$ coincide with nonnegative orthant.

2.2. Positive invariant sets

Positive invariant sets play a central role in the theory and applications of dynamical systems. Stability, control and preservation of constraints of dynamical systems can be formulated, somehow in a geometrical way, with the help of positively invariant sets. For a given dynamical system, a set is called a positively invariant set of the system if for any initial state the complete trajectory of the state vector remains in the set [8]. Consider the discrete autonomous system:

$$\boldsymbol{x_{k+1}} = \boldsymbol{Ax_k}, \quad (4)$$

Definition 3 A set $C \subseteq \mathbb{R}^n$ is an positively invariant set for the discrete system (4) if $\boldsymbol{x_k} \in C$ implies $\boldsymbol{x_{k+1}} \in C$, for all $k \in \mathbb{N}$.

Corollary [9] A polyhedral cone C_G given as in (3) is an positive invariant set for the discrete system (4) if and only if there exists a nonnegative matrix $\boldsymbol{H} \in \mathbb{R}^{n\times n}$, such that $\boldsymbol{GA} = \boldsymbol{HG}$ (5)

For a given polyhedral cone and a discrete system, according to Corollary1, to determine whether the set is an invariant set for the system is equivalent to verify the existence of a nonnegative matrix $\boldsymbol{H}$, which is actually a linear optimization problem. Rather than computing $\boldsymbol{H}$ directly, it is more efficient to sequentially solve some small subproblems [9].

Find $\boldsymbol{h_i} \in \mathbb{R}^n$, such that $\boldsymbol{h_i G} = \boldsymbol{G_i A}, \boldsymbol{h_i} \geq \boldsymbol{0}.$

If all of these linear optimization problems are feasible, then their solutions forms such a nonnegative matrix $\boldsymbol{H}$. Otherwise, we can conclude that the set is not an invariant set for the system, and the computation is terminated at the first infeasible subproblem.

2.3 Matrices that leave given polyhedral cone invariant

It is important to know if the closed loop matrix $\boldsymbol{A}$ belongs to the set of all system matrices, $\Delta(C)$, that leave the polyhedral cone C invariant, i.e.

$$\Delta(C) \coloneqq \{\boldsymbol{A} \in \mathbb{R}^{n\times n} | \Delta(C) \in C\}, \quad (6)$$

As it is mentioned in the Corollary 1, a polyhedral cone C_G given as in (3) is an invariant set for a given matrix $\boldsymbol{A}$ if and only if there exists a nonnegative matrix $\boldsymbol{H} \in \mathbb{R}^{m\times m}$, such that $\boldsymbol{GA} = \boldsymbol{HG}$, i.e.

$$\widetilde{\Delta}(C_G) \coloneqq \left\{ \begin{matrix} \boldsymbol{A} \in \mathbb{R}^{n\times n} | \\ \exists\ \boldsymbol{H} \in \mathbb{R}^{n\times n}_+ \text{ such that } \boldsymbol{GA} = \boldsymbol{HG} \end{matrix} \right\} \quad (7)$$

Since the polyhedral cone (3) is finitely generated, it can be represented by its generator matrix $\boldsymbol{N}$, namely:

$$C_N := \{\boldsymbol{x} \in \mathbb{R}^n | \exists\, \boldsymbol{l} \in \mathbb{R}^n_+, \text{such that } \boldsymbol{x} = \boldsymbol{Nl}\}$$

$$\{C_G = C_N\} \Rightarrow \{\boldsymbol{CN} \geq 0\}$$

A representation C_N can be found from a representation C_G by applying known from the literature algorithms.

Analogous to (7) let define the set of all matrices that leave the polyhedral cone C_N invariant as:

$$\bar{\Delta}(C_N) := \left\{ \begin{matrix} \boldsymbol{A} \in \mathbb{R}^{n\times n} | \\ \exists\, \boldsymbol{H} \in \mathbb{R}^{n\times n}_+ \text{ such that } \boldsymbol{AN} = \boldsymbol{NH} \end{matrix} \right\}$$

The following proposition [ ] provides us with necessary and sufficient conditions for a polyhedral cone to be a positively invariant set of a given autonomous system (4).

Proposition1 [9] For the autonomous system with system matrix $\boldsymbol{A}$ and the polyhedral cone, given with two representations C_G and C_N , the following holds

i. $\Delta(C_G) = \tilde{\Delta}(C_G) = \bar{\Delta}(C_N)$

ii. $\boldsymbol{A} \in \Delta(C_G)$ if and only if $\boldsymbol{GAN} \geq 0$

3 Problem formulation

Consider infinite horizon LQR problem for positive linear discrete time system with cone state constraints, denoted with LQR_+^{cone} and described as:

LQR_+^{cone} *PROBLEM*

$$J = \sum_{k=0}^{\infty} (\boldsymbol{x_k^T Q x_k} + \boldsymbol{u_k^T R\, u_k} \Rightarrow min \tag{8}$$

$$\boldsymbol{x_{k+1}} = \boldsymbol{Ax_k} + \boldsymbol{Bu_k} \tag{9}$$

$$\boldsymbol{Gx_k} \geq \boldsymbol{0}$$

$$\boldsymbol{A} \in \mathbb{R}^{n\times n}_+, \boldsymbol{B} \in \mathbb{R}^{m}_+; \boldsymbol{x(0)} = \boldsymbol{x_0}; \boldsymbol{Gx_0} \geq \boldsymbol{0}$$

$$\boldsymbol{0} \leq \boldsymbol{Gx} \subset IntR^n_+$$

The matrix $\boldsymbol{G}$ is such that the polyhedral cone defined by inequality $\boldsymbol{Gx_k} \geq \boldsymbol{0}$, $C_G = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{Gx} \geq 0\}$ belongs to the nonnegative orthant. According to the properties of *M*-matrix, $\boldsymbol{G}$ is *M*-matrix.

For the unconstraint case the following theorem gives the solution of the above problem:

Theorem [12]
If the pair ($\boldsymbol{A}$, $\boldsymbol{B}$) is stabilizable and the pair ($\boldsymbol{M}, \boldsymbol{A}$) ($\boldsymbol{Q} = \boldsymbol{M^T M}$) is detectable then the optimal control (10) which minimizes (8) is determined by

$$\boldsymbol{u} = -\boldsymbol{Kx} \tag{10}$$

$$\boldsymbol{K} = (\boldsymbol{R} + \boldsymbol{B}^T \boldsymbol{PB})^{-1} \boldsymbol{B}^T \boldsymbol{PA}, \tag{11}$$

where $\boldsymbol{P}$ is the unique positive definite solution to the Discrete time Algebraic Riccati Equation (DARE)

$$\boldsymbol{A}^T \boldsymbol{PA} - \boldsymbol{P} - \boldsymbol{A}^T \boldsymbol{PB}(\boldsymbol{R} + \boldsymbol{B}^T \boldsymbol{PB})^{-1} \boldsymbol{B}^T \boldsymbol{PA} + \boldsymbol{Q} = 0$$

Then the optimal discrete-time closed-loop system

$$\boldsymbol{x}_{k+1} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x}_k = \boldsymbol{A_c x_k} \tag{12}$$

is asymptotically stable and the quadratic cost (8) has the minimum value

$$J = \sum_{k=0}^{\infty} (\boldsymbol{x}_k^T \boldsymbol{Qx}_k + \boldsymbol{u}_k^T \boldsymbol{Ru}_k) =$$

$$= \sum_{k=0}^{\infty} \boldsymbol{x}_k^T (\boldsymbol{Q} + \boldsymbol{K^T RK})\boldsymbol{x}_k = \boldsymbol{x}_0^T \boldsymbol{Px_0}$$

The important question is how to use the above results for the positive systems with state constraints defined by a polyhedral cone. The answer to this question depends on the properties of the closed matrix $\boldsymbol{A_c}$, obtained from (12) and the matrix of constraints $\boldsymbol{G}$.

In the next we will investigate how the existence of solution of LQR_+^{cone}, depends on the closed-loop matrix $\boldsymbol{A_c}$ and the constrained matrix $\boldsymbol{G}$. We will remember that matrix $\boldsymbol{A_c}$, obtained from LQR, is stable (maximal eigenvalue of $\boldsymbol{A_c}$ is less than one) and the matrix of constraints $\boldsymbol{G}$ is an *M*-matrix**.**

4 Problem solution

In this section we will investigate if and when the solution of the LQR_+^{cone} problem exists, in which cases the solution of the problem coincide with the solution of the unconstrained problem and application of dual mode approach for solving the LQR_+^{cone} problem will be proposed.

4.1 The solutions of the constrained and unconstrained system coincide for every initial state $\boldsymbol{x_0} \in C_G$

If the matrices $\boldsymbol{A_c}$, $\boldsymbol{G}$ and the generator matrix $\boldsymbol{N}$ of polyhedral cone C_G is such that $\boldsymbol{GAN} \geq 0$ then the matrix $\boldsymbol{A_c}$ makes C_G positively invariant set of a closed loop system (12). This guarantee that the LQR solution of the unconstrained system is the solution of LQR_+^{cone} problem.

Example

Let the matrix $\boldsymbol{A_c}$, which is found from the unconstrained LQR problem and the *M*-matrix, defining polyhedral cone C_G in the nonnegative orthant are:

$$\boldsymbol{A_c} = \begin{bmatrix} 0.4802 & 0.06 \\ 0.2599 & 0.53 \end{bmatrix}; \boldsymbol{G} = \begin{bmatrix} 2 & -0.2 \\ -1 & 3 \end{bmatrix}$$

It is easy to find the generator matrix $\boldsymbol{N}$ of the cone C_G,

$$\boldsymbol{N} = \begin{bmatrix} 1 & 3 \\ 10 & 1 \end{bmatrix}$$

and to check that $\boldsymbol{GAN} \geq 0$.

Hence, according to *Proposition1* the cone C_G is positively invariant set of a closed - loop system (12) and any trajectory beginning from C_G remains in C_G. Then the solution of unconstrained LQR problem and constrained LQR_+^{cone} problem coincide.

The probability this case to be valid is very low, so we will discuss other approaches to solve the problem.

4.2 The solutions of constrained and unconstrained system coincide only for initial states $\boldsymbol{x_0} \in \boldsymbol{\Omega_{x_0}} \in \boldsymbol{C_G}$

Let matrix $\boldsymbol{A_c}$ doesn't make C_G positively invariant set of a closed loop system (12), i.e. the condition for positive invariance of C_G, $\boldsymbol{GAN} \geq \boldsymbol{0}$ is not fulfilled, but there exists set of initial states $\Omega_{x_0} \in C_G$ such that for every $\boldsymbol{x_0} \in \Omega_{x_0} \Rightarrow \boldsymbol{x_k} \in C_G, k = 1,2,....$ Hence the solutions of LQR_+^{cone} and LQR problem coincide for every initial state from Ω_{x_0} .

There are two main questions in this case:

1) How to obtain set Ω, if exists?
2) If the initial set is such that $\boldsymbol{x_0} \notin \Omega_{x_0}$ but $\boldsymbol{x_0} \in C_G$, how to find finite number of controls $\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_{N-1}}$, such that after *N-1* steps system trajectory enter in Ω_{x_0},i.e. $\boldsymbol{x_N} \in \Omega_{x_0}$?

Determination of the set $\boldsymbol{\Omega_{x_0}}$

The set Ω_{x_0} is described as:

$$\Omega_{x_0} := \left\{ \begin{matrix} \boldsymbol{x_0} | \boldsymbol{A_c^k x_0} \in C_G \ i.e. \boldsymbol{G A_c^k x_0} \geq 0, \\ k = 1,2, \dots \end{matrix} \right\}$$

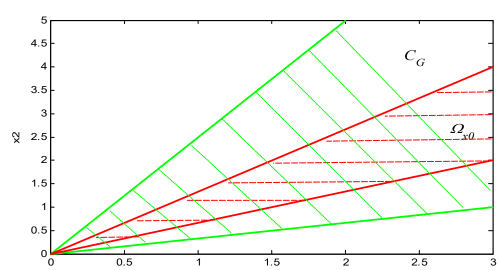
The set Ω_{x_0} is defined with:

$$\left| \begin{matrix} \boldsymbol{G A_c x_0} \geq \boldsymbol{0} \\ \boldsymbol{G A_c^2 x_0} \geq \boldsymbol{0} \\ \dots\dots\dots\dots . \\ \boldsymbol{G A_c^k x_0} \geq \boldsymbol{0} \end{matrix} \right.$$

$$k = 1,2, \dots$$

If the initial state belongs to this set, all states in the future remain inside and the constraints will not be violated. For every $\boldsymbol{x_0} \in \Omega_{x_0}, \boldsymbol{x_k} \in C_G, k = 1,2, \dots$ and system trajectory lie in C_G. The solution exists and the solutions of the unconstrained LQR problem and the constrained LQR_+^{cone} problem coincide.

Both sets are given in the following figure.



4.3 The initial states $\boldsymbol{x_0} \notin \boldsymbol{\Omega_{x_0}} \in \boldsymbol{C_G}$

If the set Ω_{x_0} exists and belongs to C_G , but the initial state $\boldsymbol{x_0} \notin \Omega_{x_0}$, we can apply the well-known from the literature dual mode approach [ ] until the trajectory enters the set Ω_{x_0} and after that we proceed as in 4.2.

4.3.1 Dual mode approach to meet the constraints

In this approach the optimization on the infinite horizon is divided into two pairs.
In order to meet the state constraints the prediction horizon was splitted into two intervals, defining two modes – one from 0 to N-1 and the second from N to infinity.

$$J=\sum_{k=0}^{\infty}(\boldsymbol{x}_k^T\boldsymbol{Q}\boldsymbol{x}_k+\boldsymbol{u}_k^T\boldsymbol{R}\boldsymbol{u}_k)=$$
$$=\sum_{k=0}^{N-1}(\boldsymbol{x}_k^T\boldsymbol{Q}\boldsymbol{x}_k+\boldsymbol{u}_k^T\boldsymbol{R}\boldsymbol{u}_k)+\sum_{k=N}^{\infty}(\boldsymbol{x}_k^T\boldsymbol{Q}\boldsymbol{x}_k+\boldsymbol{u}_k^T\boldsymbol{R}\boldsymbol{u}_k)\Rightarrow min$$

For the second part of the sum we have

$$J1=\sum_{k=N}^{\infty}(\boldsymbol{x}_k^T\boldsymbol{Q}\boldsymbol{x}_k+\boldsymbol{u}_k^T\boldsymbol{R}\boldsymbol{u}_k)=\boldsymbol{x}_L^T\boldsymbol{P}\boldsymbol{x}_L$$

A variety of methods guarantee that the constraints are satisfied on the infinite horizon by checking a finite number of stages $k\geq N$. It is proved that if the unconstrained model is stabilizable, there is an index N such that after N steps, the system trajectory enters the invariant set (the state constraints are satisfied) and after that for k=N, N+1,…, the state will be inside the invariant set (constraints will not be violated). We will use this idea for solving our problem, when the zero is border point of admissible set of states.

4.3.2 Constraints on two modes

We add the constants $\boldsymbol{c_k}\in\Re^{n\times 1}$ as a degree of freedom for cone constraints handling during transient, on the control horizon $k=0,\dots,N-1$. Furthermore, the fixed state feedback $\boldsymbol{K}$ affects the asymptotic behavior.

Hence, the control law that will be used in optimization of future performance, subject to these constraints is:

$$\boldsymbol{u}_k=\{-\boldsymbol{K}\boldsymbol{x}_k+\boldsymbol{c}_k, k=0,\dots,N-1$$
$$\boldsymbol{u}_k=\{-\boldsymbol{K}\boldsymbol{x}_k, k\geq N$$

The constraints satisfaction over MODE1 is ensured by solving the QP problem with constraints

$$\boldsymbol{0}\leq\boldsymbol{G}\boldsymbol{x_k},\qquad k=0,1,2,\dots,N-1$$
$$\boldsymbol{x_{k+1}}=(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x_k}+\boldsymbol{Bc_k};$$
$$\boldsymbol{0}\leq\ \boldsymbol{G}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x_k}+\boldsymbol{GBc_k}$$

Following the approach, described in [4] for constraints satisfaction we define the augmented state vector $\boldsymbol{z_k}$ as:

$$\boldsymbol{z_k}=[\boldsymbol{x_k^T}\ \boldsymbol{c_k^T}\ \boldsymbol{c_{k+1}^T}\ \cdots\ \boldsymbol{c_{k+N-1}^T}]^{\boldsymbol{T}}$$

Let us denote the sequence

$$[\boldsymbol{c_k}\ \boldsymbol{c_{k+1}}\ \cdots\boldsymbol{c_{k+N-1}}]=\overrightarrow{\boldsymbol{c_k}}$$

Then the predicted dynamics will be:

$$\boldsymbol{z_{k+1}}=\widetilde{\boldsymbol{\Phi}}_{\boldsymbol{N}}\boldsymbol{z_k}$$

$$\boldsymbol{u_k}=\ \widetilde{\boldsymbol{K}}\boldsymbol{z_k}$$

$$\widetilde{\boldsymbol{\Phi}}_{\boldsymbol{N}}=\begin{bmatrix}\boldsymbol{A}-\boldsymbol{BK} & \boldsymbol{B} & \boldsymbol{0} & \cdots & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0}\end{bmatrix}$$

$$\widetilde{\boldsymbol{K}}=[-\boldsymbol{K}\quad \boldsymbol{I}\quad \boldsymbol{0}\quad \cdots\quad \boldsymbol{0}]$$

Using the augmented state space model, the constraints on the MODE1 are:

$$\boldsymbol{G}[\boldsymbol{I}\quad \boldsymbol{0}\quad \dots\quad \boldsymbol{0}]\boldsymbol{z_k}\geq\boldsymbol{0}$$

Next, if in MODE 1 we are able to find the sequence $\overrightarrow{\boldsymbol{c_k}}$ so that the constraints are not violated and the end of the state trajectory (at the end of prediction horizon) falls into the invariant set, we continue with MODE 2.

MODE 2
For the constraints satisfaction over the MODE 2, the system is controlled under the feedback law $\boldsymbol{u_k}=-\boldsymbol{K}\boldsymbol{x_k}$ and the constraints can be expressed as:

$$\boldsymbol{G}\boldsymbol{x_k}\geq\boldsymbol{0};\ \ k=N,N+1,\dots$$

4.4 $\boldsymbol{A_c^k}\boldsymbol{x_0}\in\boldsymbol{C_G}$ only for $\boldsymbol{k}=\boldsymbol{1},\boldsymbol{2},\dots\boldsymbol{N}$

If the matrices $\boldsymbol{A_c},\boldsymbol{G}$ and $\boldsymbol{x_0}$ are such that $\boldsymbol{A_c^k}\boldsymbol{x_0}\in\boldsymbol{C_G}$ only for $k=1,2,\dots N$ and $\boldsymbol{A_c^k}\boldsymbol{x_0}\notin C_G$, $k=N+1,N+2,\dots$the solution exists, but it is not equal to the solution of the unconstraint problem.

In this case we define the new set

$$\Omega_{x_0}^N := \begin{Bmatrix} \boldsymbol{x_0} | \boldsymbol{A_c^k x_0} \in \boldsymbol{C_G} \; i.e. \boldsymbol{G A_c^k x_0} \geq \boldsymbol{0}, \\ \boldsymbol{k} = \boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{N} \end{Bmatrix}$$

and proceed as in 4.2 or 4.3 depending on if $\boldsymbol{x_0} \in \Omega_{x_0}^N$ or not. The procedure stop at $\boldsymbol{k} = \boldsymbol{N}$ and after that we have $\boldsymbol{x}_{N+1} = \boldsymbol{x}_{N+2} = \dots$ This case in described in [11] when the cone G_C coincide with nonnegative orthant, i.e. for $G = I$.

4.5. There is no solution to the problem

If the set $\Omega_{x_0}^N$ is empty, i.e. for all initial states $\boldsymbol{x_0} \in C_G$ but $\boldsymbol{A_c x_0} \notin C_G$, $\boldsymbol{A_c^2 x_0} \notin C_G$ ..., there is no solution of the problem.

5 Conclusion

In the paper, the solution of the infinite horizon LQR problem of positive linear discrete time systems with state restrictions, given as polyhedral cone belonging to nonnegative orthant, is studied. The existing of a solution strongly depends on the properties of the closed-loop matrix and matrix of constraints. It is investigated when *(a)* the solution of the unconstrained LQR problem coincide with the solution of LQR_+^{cone} problem, *(b)* there is no solution and *(c)* there is solution of LQR_+^{cone} problem, but it doesn't coincide with the solution of unconstrained problem.

Future research will focus on estimating the necessary and sufficient conditions on the system and the restriction matrices for the solution LQR_+^{cone} problem.

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