Solutions for the Case of Spectrally Separable Kernel Matrices in the Probabilistic Evolution Theory (PREVTH)

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Abstract: This work seeks the possibility of rather simple structures in the applications of the probabilistic evolution theory (PREVTH). We focus on the rather simple forms of the kernel matrix of the system under consideration. Such that for some specific initial vector forms the imaging under the kernel matrix produces an output proportional to the original initial vector. By using this specific kernel matrix forms we have proven that the initial direction is conserved during the evolution. However the magnitude of the solution temporally changes. As we have found these changes may remain in finite domains of the relevant axis while there is also possibilities approaching to infinity.

Key–Words: Probabilistic Evolution Theory, Kernel Matrix, Monocular Matrix, Telescope Matrix, Characteristic Directions.

1 Introduction

Probabilistic Evolution Theory has been proposed and developed in few recent years. It is established to solve the initial value problems of the explicit first order ODEs or ODE sets, beyond that, to investigate certain properties of the solutions. An explicit ODE can be written as follows

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), t\right), \qquad \mathbf{x}(0) = \mathbf{a} \tag{1}$$

where $\mathbf{x}(t)$ stands for the unknown vector which has n temporally varying elements while $\mathbf{a}$ and $\mathbf{f}$ denote given initial value vector and the right hand side descriptive function vector whose functional structure is known respectively. The explicit time (independent variable t) dependence makes the descriptive vector function nonautonomous. This means the structure of the descriptive function changes from time instant to time instant. The absence of this dependence brings certain conservation rules and then $\mathbf{f}$ is called autonomous. In the same way, the coressponding vector ODE is also called autonomous. Without any generality loss we can assume that (1) is autonomous since it is always possible to change nonautonomy to autonomy by adding a new unknown $x_{n+1}(t)$ which is in fact just t and the accompanying initial value a_{n+1} to the unknown vector and initial vector respectively. Hence we can remove explicit t dependence in (1) for our further analysis. Then to proceeed, we assume that $\mathbf{f}\left(\mathbf{x}(t)\right)$ is analitic in the phase space spanned by $\mathbf{x}(t)$ elements. Analyticity enables us to use multivariate Taylor expansion. However it is quite complicated because it necessitates the use of plenty of indices and indexed terms.

Multivariate Taylor series of the descriptive function vector is composed of the products of the powers of the functions $\left(x_j(t) - x_j^{(e)}\right)$, $j = 1, 2, ..., n$ where the superscript (e) recalls the expansion point. This complicates the expressions. To get conciseness it is better to define the following so-called system vector

$$\mathbf{s}(t) \equiv \left[\, \left(x_1(t) - x_1^{(e)}\right) \;\; \dots \;\; \left(x_n(t) - x_n^{(e)}\right) \,\right]^T \tag{2}$$

which urges us to rewrite (1) in terms of $\mathbf{s}(t)$.

$$\dot{\mathbf{s}}(t) = \mathbf{f}\left(\mathbf{s}(t)\right), \qquad \mathbf{s}(0) = \overline{\mathbf{a}} = \mathbf{a} - \mathbf{x}^{(e)} \tag{3}$$

2 Kronecker Products, Powers, and, Series

Now we can give the Kronecker Product of two vectors, say $\mathbf{b}$ and $\mathbf{c}$ whose number of elements need not to be same. The definition is as follows

$$\mathbf{b} \otimes \mathbf{c} \equiv \begin{bmatrix} b_1\mathbf{c} \\ \vdots \\ b_{n_b}\mathbf{c} \end{bmatrix} \tag{4}$$

where the vectors $\mathbf{b}$ and $\mathbf{c}$ are assumed to have n_b and n_c elements. Kronecker product which is a binary operation takes two vectors, which have not necessarily same number of elements, and produces a single vector whose number of element is equal to the product of the operand vector's number of elements. Thus, the output in (4) has $n_b n_c$ elements. Hence this operation generally increases the dimension or the number of elements. The above Kronecker product has been defined between two ordinary linear algebraic vectors. However, there is no such kind of limitation in fact. This product can be defined between one vector and one matrix or one matrix and one vector or one matrix and one matrix.

Kronecker product is not commutative. However, between the original product and its reverted version there is a transformation defined through certain permutation matrices.

The mth Kronecker power of the vector $\mathbf{a}$ is defined as follows

$$\mathbf{a}^{\otimes m} \equiv \mathbf{a} \otimes \mathbf{a} \otimes ... \otimes \mathbf{a} \tag{5}$$

where $\mathbf{a}$ appears m times.

A Kronecker power series is a linear combination of all natural number powers of a vector. The mth power of $\mathbf{a}$ contains mth power of the number of elements in $\mathbf{a}$ as its number of elements. The additive components of Kronecker power series are all same in type and are binary products of the Kronecker powers with the coefficients which are certain matrices structured such that each binary product is in the same type.

3 Kronecker Power Series of Descriptive Function Vector

Now we can write

$$\mathbf{f}\left(\mathbf{s}(t)\right) = \sum_{j=0}^{\infty} \mathbf{F}_j \mathbf{s}(t)^{\otimes j} \tag{6}$$

where $\mathbf{s}(t)$ has n elements each of which is an unknown function. Hence its type $n \times 1$. This makes the number of the elements in $\mathbf{s}(t)^{\otimes j}$ n^j. Hence this Kronecker power's type is $n^j \times 1$. On the other hand, the binary product $\mathbf{F}_j \mathbf{s}(t)^{\otimes j}$ must be of the type $n \times 1$. This means that the type of the matrix $\mathbf{F}_j$ should be $n \times n^j$. In other words, $\mathbf{F}_j$ is a vector of n elements for $j = 0$ while it becomes an $n \times n$ type square matrix whereas for all js greater than or equal to 2 it becomes an horizontal rectanguar matrix whose width increase very rapidly as grows unboundedly while its height remain constant.

Kronecker power series are not unique despite Taylor series' uniqueness. To understand this fact we can focus on the case where $n = 2$ and write the following equality for the Kronecker square of the system vector.

$$\mathbf{s}^{\otimes 2}(t) = \begin{bmatrix} s_1(t)^2 \\ s_1(t)s_2(t) \\ s_2(t)s_1(t) \\ s_2(t)^2 \end{bmatrix} \tag{7}$$

the second and third elements of this vector are same because of the scalars' commutativity. Therefore any four element vector whose second and third elements are equal in magnitude but opposite in sign is orthogonal to the Kronecker square of the system vector. If we define

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \tag{8}$$

then we can construct a one-rank matrix as $\mathbf{b}\mathbf{s}(t)$ where the first factor can be any two element vector. The image of the Kronecker square of the system vector under this one rank matrix vanishes. Hence the addition of this somehow arbitrary matrix to the coefficient matrix $\mathbf{F}_2$ does not change the total Kronecker power series. This means an arbitrariness in $\mathbf{F}_2$.

The above arbitrariness is not peculiar only to the case of $n = 2$. It exists for any n value and for $\mathbf{F}_2$ because the system vector Kronecker square has some identical elements because of the commutativity of the scalars. This arbitrariness does not show up only in $\mathbf{F}_2$. All $\mathbf{F}_j$ coefficients have this kind of arbitrarinesses because of the existence of the identical elements in the jth Kronecker powers of the system vector.

These arbitrarinesses can be removed via certain norm minimizations even though we do not intend to get into the details. If this is done then we can write the following explicit relation

$$\mathbf{F}_j = \frac{1}{j!}\left\{\nabla^{\otimes j}\right\}^T_{\mathbf{s}=\mathbf{0}}, \qquad j = 0, 1, ..., \tag{9}$$

4 Space Extension Concept

(6) contains denumerable infinite number of terms in its general structure and this worsens our analysis in the solution of the corresponding vector ODE. However in many practically encountered cases the situation is not so bad and the descriptive function vector depends of some other functions of unknowns. In other words we cna write

$$\mathbf{f} = \mathbf{f}\left(u_1\left(\mathbf{s}(t)\right), ..., u_m\left(\mathbf{s}(t)\right)\right) \tag{10}$$

where the dependence of $\mathbf{f}$ on us is finitely multinomial by assumption. Now we can convert the relevant vector ODE with the unknown s functions to another ODE with the unknowns, us. To this end we can write the following ODE for the function u_j

$$\dot{u}_j(\mathbf{s}(t)) = f\left(u_1(\mathbf{s}(t)), ..., u_m(\mathbf{s}(t))\right)^T \nabla_{\mathbf{s}} u_j(\mathbf{s}(t)),\quad j = 1, 2, ..., m \tag{11}$$

which urges us to define

$$\widehat{\mathcal{L}} = f\left(u_1(\mathbf{s}(t)), ..., u_m(\mathbf{s}(t))\right)^T \nabla_{\mathbf{s}} \tag{12}$$

and therefore to write

$$\dot{u}_j(\mathbf{s}(t)) = \widehat{\mathcal{L}} u_j(\mathbf{s}(t)), \qquad j = 1, 2, ..., m \tag{13}$$

which implies that the multinomiality not in the original right hand side functions but in these equations can be provided. For this it is sufficient to have a finite set of u functions such that this set is multinomially closed under the operator $\widehat{\mathcal{L}}$ which is also called evolution operator of the system or Lie operator of the system.

We call this procedure to get 13 ODEs “space contraction”for $m < n$ and “space extension $m > n$. In plain speaking the space extension creates new unknowns from the existing ones and increases the phase space of the resulting system when the new case number of function is greater than the old one.

5 Conicality

The multinomiality is not sufficient to get the most simplifying structure in the resulting ODE as long as the degree is greater than 2. The case where the degree of the multinomial is just 2 can be called the conicality where only first three $\mathbf{F}$ matrices exist. Demiralp and Rabitz had proven that the conicality can be achieved by using certain space extensions over certain multinomial functions, if the right hand side function of the existingg ODE is multinomial.

On the other hand, just extending the space by importing a constant to the unknowns, it is possible to get rid of $\mathbf{F}_0$ and to make the matrix $\mathbf{F}_1$ proportional to the correspondant identity matrix. So even conicality is reduced to a very efficient structure which facilitates the further analysis pretty much.

At the end we can write the simplest conical case we obtained through space extensions as follows

$$\mathbf{f}(\mathbf{s}(t)) \equiv \beta \mathbf{I}\mathbf{s}(t)^{\,otimes1} + \mathbf{F}_2 \mathbf{s}(t)^{\,\otimes 2} \tag{14}$$

where β is an unknown scalar whose value can be obtained via certain optimisation procedures.

6 Telescope Matrices and the Solution

Even though we do not intend to give the derivation details the solutiom for the simplest conical case can be written as follows

$$\mathbf{s}(t) = \mathrm{e}^{-\beta t} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{1 - \mathrm{e}^{-\beta t}}{\beta} \right)^j \mathbf{T}_j \mathbf{a}^{\otimes j+1} \tag{15}$$

$$\mathbf{T}_j \equiv \mathbf{M}_1 ... \mathbf{M}_j, \quad \mathbf{T}_0 \equiv \mathbf{I}_n, \quad j = 0, 1, 2, ... \tag{16}$$

$$\mathbf{M}_k \equiv \sum_{\ell=0}^{k-1} \mathbf{I}_n^{\otimes \ell} \otimes \mathbf{F} \otimes \mathbf{I}_n^{\otimes k-1-\ell}, k = 1, 2, ... \tag{17}$$

where $\mathbf{F}$ is a rectangular constant matrix of $n \times n^2$ type. All matrices of the solution are produced from this matrix. Hence we may call it “Kernel Matrix”. The matrix $\mathbf{M}_k$ is produced from the kernel matrix and has the type $n^k \times n^{k+1}$. It maps from n^{k+1} dimensional Cartesian space to n^k dimensional space so somehow gets the images closer. For this reason we call it “Monocular Matrix”. On the other hand, $\mathbf{T}_j$ uses first j monocular matrices in a cascaded way such that it brings the image from n^{j+1} dimensional space to n dimensional space. Even though it realizes somehow a direct transfer it uses the monocular matrices as the intermediate agents.

7 Spectrally Separable Kernels

The kernel matrix $\mathbf{F}$ has n rows and n^2 columns. Its domain is n^2 dimensional while the dimension of the range is n dimensional. We can choose n number of orthonormal vectors and denote bu $\mathbf{u}_1$, ... $\mathbf{u}_n$. There are n^2 linearly independent Kronecker products amongst these vectors. the number of these combined n^2 element vectors is also n^2 and they are mutually orthonormal. Therefore they form an orthonormal basis set spanning the domain kernel matrix. All these urge us to express the kernel matrix as follows

$$\mathbf{F} = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \varphi_{j;k,\ell} \mathbf{u}_j \left(\mathbf{u}_k \otimes \mathbf{u}_\ell \right)^T \tag{18}$$

where $\varphi_{j;k,\ell}$s denote some kernel specific scalars. The case where all φs except the ones denoted by $\varphi_{j;j,j}$s ($j = 1, 2, ...n$) vanish bring high level facilitation.

For these cases we can write

$$\mathbf{F}\left(\mathbf{u}_m \otimes \mathbf{u}_m\right) = \varphi_{m;m,m} \mathbf{u}_m,\quad m = 1, 2, ... \tag{19}$$

$$\mathbf{M}_{j-1}\mathbf{M}_j\mathbf{u}_m^{(\otimes j+1)} = j(j-1)\varphi_{m;m,m}^2\mathbf{u}_m^{(\otimes j-1)},\quad m = 1, 2, ... \tag{20}$$

$$\mathbf{M}_{j-2}\mathbf{M}_{j-1}\mathbf{M}_j\mathbf{u}_m^{(\otimes j+1)} = j(j-1)(j-2)\varphi_{m;m,m}^3\mathbf{u}_m^{(\otimes j-2)},\quad m = 1, 2, ... \tag{21}$$

$$\mathbf{T}_j\mathbf{u}_m^{(\otimes j+1)} = j!\varphi_{m;m,m}^j\mathbf{u}_m, \qquad m = 1, 2, ... \tag{22}$$

$$\mathbf{s}(t) = \frac{\mathrm{e}^{-\beta t}}{1 - \varphi_{j;j,j}\left(\frac{1-\mathrm{e}^{-\beta t}}{\beta}\right)}\mathbf{a} \tag{23}$$

All these analyses urges us to call each of $\mathbf{u}$ vectors “separable eigenvectors”of the kernel matrix. In fact it is better to call each of them “characteristic direction”of the kernel and therefore the system. Each of them is a characteristic direction because the solution conserves the initial direction as long as the initial vector is on this direction. There are n different possibilities to get characteristic directions. Each of these directions defines an axis. Each of these axes spanned by one distinguished $\mathbf{u}$ vector. Since the $\mathbf{u}$ vectors are mutually orthonormal these axes are perpendicular to each other. The system evolves on the relevant axis as time proceeds as long as the initial vector lies on this axis. The temporal variation determines the position of the point on the relevant axis. If the denominator does not vanish than all the motion remains in a finite interval on the relevant axis. This situation corresponds to the convergence of the telescopic Kronecker power series. The agent which controls the convergence is basically relevant φ value. That value is somehow eigenvalue of the kernel matrix. If the φ parameter is sufficiently large to break the convergence down the Kronecker power series diverges. However, like rational functions the temporal magnitude function may represent the true solution. This can be shown by inserting the solution into the ODEs and then check the satisfaction. In the case of divergence the magnitude may oscillate not between two finite values but plus and minus infinities.

8 Asymptotically Characteristic Directional Solutions

We can now consider the cases where the initial vector deviates from one of the $\mathbf{u}$ vectors with a very small amount. That is we can write

$$\mathbf{a} \equiv \mathbf{u}_m + \mathbf{p} \tag{24}$$

where the norm of the perturbation vector $\mathbf{p}$ is very small in comparison with the norm of the vector $\mathbf{u}_m$. p will have the additive components involving other $\mathbf{u}$ but with very small coefficients. The utilization of (24) in the solution formula and the employment of a perturbative scheme to construct the solution may give again analytic or semi analytic expressions for the solution. Here we perturbed only the initial vector. However it is possible also to perturb the kernel matrix. In that case the scheme to evaluate the solution may become more and more complicated. These issues are considered as future works.

9 Conclusion

In this work we have focused on the cases where the kernel matrix in the telescopic representation has additive one rank matrices each of which is composed of only a single vector which is one of the members in a complete orthonormal n-element vectors set. For each of such matrices the image of the vector under the kernel matrix remains proportional to the same vector. This is somehow very similar to the eigenvalue and eigenvectors of square matrices. The directional conservation under the imaging by the kernel matrix does also reflect to the conservation of the direction of the solution for all time instances of the system evolution. There are n number of possibilities for such directional conservations. Each conserved direction can be considered as an axis in the phase space of the system and that axis is spanned by the relevant $\mathbf{u}$ vectors which are mutually orthonormal. These axis are all mutually perpendicular because of this orthonormality. While the direction is conserved during the evolution the magnitude function temporally changes in a finite or semi-infinite or infinite sections of the axis. The convergence and therefore approaching to infinity is controlled by the so-called eigenvalue parameter. The work here can be extended to more complicated cases which are considered as the theme of the future works.

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