# Separate Node Ascending Derivatives Expansion (SNADE) as a Univariate Function Representation

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*Abstract:* This work focuses on the novel approach which is named as "Seperate Node Ascending Derivatives (SNADE)". SNADE is a very recently developed univariate function representation. This method has a similar structure to the Taylor series expansion. Two specific cases related recurrent nodes with reference to certain rules are handled in this paper.

*Key–Words:SNADE, Integral Operator, Taylor Series, Multinode Expansion, Cauchy Contour Integral*

## 1 Introduction

Univariate functions are represented in various different ways and each representation can be appropriately used for approximation via its finite number of ordered terms from the first component, that is, truncations. Among the representations we can mention Taylor series which is at most essential position. The series over orthogonal polynomials or functions are also quite important agents. We do not intend to mention other existing methods. This work is tightly related to Taylor series. Hence the next section is organised to revisit Taylor series. The remaining sections will focus on the many important and even diverse properties of the novel approach SNADE which has infinite number of possibilities in its design [1–6]. The basic issue is the distribution of the nodal points. In this work we consider just two nodel values ($x_1$ and $x_2$) such that ordering of the infinite number of nodes start with say $m_1$ number of $x_1$ followed by say $m_2$ number of $x_2$. Then this ordering structure periodically continues to appear.

## 2 Revisiting Taylor series

Even though the Taylor series is considered for real valued univariate functions because of historical and practicality reason it is in fact defined in a more general way involving the complex variables and complex planes [7–19]. If we consider a complex-valued function $f(z)$ where $z$ stands for the complex valued independent variable. Then the Taylor expansion of the

function $f(z)$ can be given as follows

$$f(z) = \sum_{j=0}^{\infty} f_j \, (z - z_0)^j \tag{1}$$

where $z_0$, denoting a complex value, is called "Expansion Point" since it represents a location (point) in the complex plane of the independent variable $z$. This is an infinite linear combination of the $(z - z_0)^j$ powers with linear combination coefficients $f_j$s. $(z - z_0)^j$ power functions with natural number $j$s are all linearly independent. They can be considered as the elements of a denumerable infinite elements of a set we call "basis set". The linear combination coefficients which are also called "Taylor Series Coefficients" are explicitly defined below

$$f_j \equiv \frac{1}{j!} \frac{d^j f}{dz^j} \, (z_0) \tag{2}$$

which implies that all derivatives of the function at the expansion point of the $z$-complex plane must exist and have unique values. This can rather comfortably accepted in the case of real entities and the exiectence of each order derivative becomes separately important. On the other hand, In the complex plane, the differentiability-once is a sufficient condition for the all derivatives of the univariate function under consideration. This is because the first derivative is defined through a limit when a deviation from the expansion point diminishes to zero without depending on the approaching ditection to the limit point (expansion point). Therefore all approcahing directions must give the same unique values as the derivative of

the function under consideration. The differentiablity is provided as long as the Cauchy-Remain equations over the partial derivatives with respect to real and imaginary parts of the independent variable are satisfied. If this happens then it is possible to prove that all order derivatives with respect to complex variable exists and have unique values. Then the function can be called analytic at the expansion point.

Any point in the complex plane of the independent variable is called "singularity" if the function is not differentiable at that point. Most basic singularity is the pole where the function and its derivatives become plus or minus infinite.

The second type singularity is the branch point which is the resource of the multivaluedness. To get the uniqueness the complex plane is cut and is considered together with one or more than one adjacent complex planes which are called Riemann sheet(s). To get uniqueness it is important to proceed through a path which do not intersects the cut. If done so, then the walking through the path is completed in a rotation with an angle which is multiples of $2\pi$. The multiplicity determines the number of necessary Riemann sheets to get uniqueness. All power functions with rational valued powers have finite number of Riemann sheets while the infinite number of Riemann sheets are encountered in the case of transcendental functions like logarthm.

In the case of essential singularities the Taylor series all coefficients become infinite or identically zero by creating a strong contradiction.

Taylor series expansion is also based on a repetitive use of an identity related to the integral of derivative identity. If done so then a polynomial sum (Taylor polynomial) and a remainder term is obtained. The infinite degree limit of Taylor polynomial defines Taylor series as long as the remainder term, which is responsible for the convergence of the series, tends to vanish as long as the degree of the Taylor polynomial increases unboundedly.

The above analysis has been for determining the formal existence of the Taylor series. However, this formal existence may not mean that they can be used in the function calculations. For the function evaluations, the convergence of the Taylor series gains a lot of importance. For convergence analysis the most helpful tools are found again in complex analysis. To this ends the very well-known and useful definition is the Cauchy contour integral which is defined for a function $f(z)$ as follows

$$f(z) = \frac{1}{2\pi i} \oint_C d\zeta \frac{f(\zeta)}{\zeta - z} \tag{3}$$

where $i$ stands for the imaginary unit number while the symbol $C$ defines a counterclockwise circular con-

tour centered at the point where $\zeta = z$ in $\zeta$ complex plan [20–24]. In this formula, the target function $f(z)$ is assumed not to have any singularity in the disk whose circle is $C$.

Cauchy contour integral formula can be differentiate with respect to the independent variable $z$ without destroying the validity. This procedure enables us to evaluate all derivatives of the target function $f(z)$. The $n$th order derivative contour integral formula can be given as follows

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C d\zeta \frac{f(\zeta)}{\zeta - z}^{n+1} \tag{4}$$

(5) can be used to get a bound for the complex derivatives of the function $f(z)$ as we can write the following inequality without giving the intermediate stages

$$\left| f^{(n)}(z) \right| \leq \frac{n! B_f}{\rho^n}, \qquad n = 0, 1, 2, ... \tag{5}$$

where $\rho$ stands for the radius of the circular contour and $B_f$ characterizes the bound of the function on the contour. The validity of this formula requires the nonsingularity on the contour. All these can be used to prove that the Taylor series of a function $f(z)$ converges to the point which is closest to the expansion point. We do not intend to enter further details of the singularity issues even though they are quite important.

## 3 Recalling SNADE

SNADE can be considered as a new Taylor expansion involving denumerable infinitely many nodes. To formulate the SNADE, identity on the integral of the derivative of a function is used repetitiously but not on the same interval. Each derivative value is calculated at a point that can have different value from the other nodes.

Integral of Derivative Identity for a univariate function $f(x)$ which is analytic in a closed interval $[a, b]$ can be written as follows

$$f(x) = f(x_1) + \int_{x_1}^x d\xi f'(\xi), \quad x, x_1 \in [a, b] \tag{6}$$

If $f(x)$ and $f'(\xi)$ replace with $f'(\xi)$ and $f''(\xi_1)$ respectively, this brings following equality

$$f'(\xi) = f'(x_2) + \int_{x_2}^\xi d\xi_1 f''(\xi_1) \tag{7}$$

SNADE is based on this structure. (6) and (7) can be merged to give

$$
\begin{aligned}
f(x) \;=\;& f(x_1) + f'(x_2)(x - x_1) \\
&+ \int_{x_1}^{x} d\xi_1 \int_{x_2}^{\xi_1} d\xi_2 f''(\xi_2)
\end{aligned} \tag{8}
$$

Taylor series use a single nodal point and all calculations are performed at this point. On the other hand, the function value evaluatedat $x_1$ and the value in the first derivative of function of $x_2$ are used until now to obtain the novel approach, SNADE. The structure which is obtained above can be written in symbolic form as follows

$$
\begin{aligned}
f(x) \;=\;& f(x_1)\mathcal{I}_0 1_f + f'(x_2)\mathcal{I}_1(x_1)1_f \\
&+ f''(x_2)\mathcal{I}_2(x_1, x_2)1_f \\
&+ \mathcal{R}_2(x; x_1, x_2, x_3)
\end{aligned} \tag{9}
$$

where $1_f$ represents the unit constant function in the considered interval. $\mathcal{I}$ stands for an integral operator can be defined by

$$
\begin{aligned}
\mathcal{I}_m(x_1, ..., x_m)g(x) &\equiv \int_{x_1}^{x} d\xi_1 ... \int_{x_m}^{x} d\xi_m g(\xi_m), \\
m &= 1, 2, ..., \quad \mathcal{I}_0 g(x) \equiv g(x)
\end{aligned} \tag{10}
$$

where $g(x)$ is an arbitrary integrable function. m. operator is obtained by m-multiple integration. The remainder term is obtained as follows

$$
\begin{aligned}
\mathcal{R}_m(x; x_1, ..., x_{m+1}) &\equiv \mathcal{I}_{m+1}(x_1, ..., x_{m+1}) \\
&\times f^{(m+1)}(x), m = 0, 1, 2, ...
\end{aligned} \tag{11}
$$

After these definitions the formula given in (9) can be generalized in following form

$$
\begin{aligned}
f(x) \;=\;& \sum_{i=0}^{m} f^{(i)}(x_{i+1})\mathcal{I}_i(x_1, ..., x_i)1_f \\
&+ \mathcal{R}_{m+1}(x; x_1, ..., x_{m+1}), \quad m = 0, 1, ..
\end{aligned} \tag{12}
$$

Since remainder term tends to vanish when m grows, above equation can be written as follows

$$
f(x) = \sum_{i=0}^{\infty} f^{(i)}(x_{i+1})\mathcal{I}_i(x_1, ..., x_i)1_f \tag{13}
$$

and can be called as "Infinite Order SNADE".

## 4   SNADE Basis Functions

The general definition of basis functions can be written as follows

$$
\begin{aligned}
\phi_j(x; x_1, ..., x_j) &\equiv \mathcal{I}_j(x_1, ..., x_j)1_f, \\
j &= 0, 1, ...
\end{aligned} \tag{14}
$$

In the right hand side functions are given through j-multiple integrations. Based on this identity, first three SNADE coefficient functions can be obtained as follows

$$
\begin{aligned}
\phi_0(x) &\equiv 1, \\
\phi_1(x) &\equiv (x - x_1), \\
\phi_2(x) &\equiv (x - x_1)\left(\frac{1}{2}x + \frac{1}{2}x_1 - x_2\right)
\end{aligned} \tag{15}
$$

The basis functions provide the following integral recursion

$$
\begin{aligned}
\phi_i(x; x_1, ..., x_i) = \int_{x_1}^{x} d\xi_1 \phi_{i-1}(\xi_1, x_2, ..., x_i) \\
i = 1, 2, ...
\end{aligned} \tag{16}
$$

The $i$th derivative of $i$th coefficient function with respect to $x$ becomes 1.

$$
\phi_i^{(i)}(x; x_1, ..., x_i) = 1, \quad i = 1, 2, ... \tag{17}
$$

If both sides of (16) is j times differentiated with respect to $x$ then the recursive relation can be obtained as follows

$$
\begin{aligned}
\phi_i^{(j)}(x; x_1, ..., x_i) = \phi_{i-1}^{(j-1)}(x; x_2, ..., x_i), \\
i = 1, 2, 3, ...; \quad j = 1, 2, ..., i
\end{aligned} \tag{18}
$$

which can be iterated to give

$$
\begin{aligned}
\phi_i^{(j)}\left(x_j; \underbrace{x_1, ..., x_i}_{i}\right) &= \phi_{i-1}^{(j-1)}\left(x_j; \underbrace{x_2, ..., x_i}_{i-1}\right) \\
&= \phi_i^{(j-2)}\left(x_j; \underbrace{x_3, ..., x_i}_{i-2}\right) \\
&= \phi_{i-1}^{(1)}\left(x_j; \underbrace{x_j, ..., x_i}_{i-j+1}\right) = 0 \\
i = 1, 2, 3, ...; \quad j &= 1, 2, ..., i-1
\end{aligned} \tag{19}
$$

## 5   A Specific SNADE Case: Infinitely Multiple Node Triplets

In this case we are going to take all the first two nodes same as $x_1$ and the last one same as $x_2$ at each triplet nodes. We are going to present only the following three times indexed basis functions

$$
\mathcal{I}_{3j}(x; x_1, x_2) = \int_{x_1}^{x} d\xi_1 \int_{x_1}^{\xi_1} d\xi_2 \int_{x_2}^{\xi_2} d\xi_3 \mathcal{I}_{3j-3}(\xi_3; x_1, x_2) \tag{20}
$$

Definitions for integration by parts:

$$u = \int_{x_1}^{\xi_1} d\xi_2 \int_{x_2}^{\xi_2} d\xi_3 \mathcal{I}_{3j-3}(\xi_3; x_1, x_2),$$
$$dv = d\xi_1 \tag{21}$$

So the following equations can be written

$$du = d\xi_1 \int_{x_2}^{\xi_1} d\xi_3 \mathcal{I}_{3j-3}(\xi_3; x_1, x_2),$$
$$v = \xi_1 \tag{22}$$

After abovementioned definitions, first integral can be rewritten as follows

$$\left[ \mathcal{I}_{3j}(x; x_1, x_2) = \xi_1 \int_{x_1}^{\xi_1} d\xi_2 \int_{x_2}^{\xi_2} d\xi_3 \mathcal{I}_{3j-3}(\xi_3; x_1, x_2) \right]_{x_1}^{x}$$
$$- \int_{x_1}^{x} d\xi_1 \xi_1 \int_{x_2}^{\xi_1} d\xi_3 \mathcal{I}_{3j-3}(\xi_3; x_1, x_2) \tag{23}$$

If we put limits at first term and associate each integral limits the following equality can be written

$$\mathcal{I}_{3j}(x; x_1, x_2 = x \int_{x_1}^{x} d\xi_1 \int_{x_2}^{\xi_1} d\xi_2 \mathcal{I}_{3j-3}(\xi_2; x_1, x_2)$$
$$- \int_{x_1}^{x} d\xi_1 \xi_1 \int_{x_2}^{\xi_1} d\xi_2 \mathcal{I}_{3j-3}(\xi_2; x_1, x_2)$$
$$= \int_{x_1}^{x} d\xi_1 (x - \xi_1) \int_{x_2}^{\xi_1} d\xi_2 \mathcal{I}_{3j-3}(\xi_2; x_1, x_2) \tag{24}$$

# 6 Another Specific SNADE Case: Infinitely Multiple Node Quartets

In this case we are going to take all the first three nodes same as $x_1$ and the last one same as $x_2$ at each quartet nodes. We are going to present only the following four times indexed basis functions

$$\mathcal{I}_{4j}(x; x_1, x_2) = \int_{x_1}^{x} d\xi_1 \int_{x_1}^{\xi_1} d\xi_2 \int_{x_1}^{\xi_2} d\xi_3 \int_{x_2}^{\xi_3} d\xi_4 \mathcal{I}_{4j-4}(\xi_4; x_1, x_2) \tag{25}$$

Definitions for integration by parts:

$$u = \int_{x_1}^{\xi_1} d\xi_2 \int_{x_1}^{\xi_2} d\xi_3 \int_{x_2}^{\xi_3} d\xi_4 \mathcal{I}_{4j-4}(\xi_4; x_1, x_2),$$
$$dv = d\xi_1 \tag{26}$$

So the following equations can be written

$$du = d\xi_1 \int_{x_1}^{\xi_1} d\xi_2 \int_{x_2}^{\xi_2} d\xi_3 \mathcal{I}_{4j-4}(\xi_3; x_1, x_2),$$
$$v = \xi_1 \tag{27}$$

After abovementioned definitions, first integral can be rewritten as follows

$$\left[ \mathcal{I}_{4j}(x; x_1, x_2) = \xi_1 \int_{x_1}^{\xi_1} d\xi_2 \int_{x_1}^{\xi_2} d\xi_3 \int_{x_2}^{\xi_3} d\xi_4 \mathcal{I}_{4j-4}(\xi_4; x_1, x_2) \right]_{x_1}^{x}$$
$$- \int_{x_1}^{x} d\xi_1 \xi_1 \int_{x_1}^{\xi_1} d\xi_2 \int_{x_2}^{\xi_2} d\xi_3 \mathcal{I}_{4j-4}(\xi_3; x_1, x_2) \tag{28}$$

If we put limits at first term and associate each integral limits the following equality can be written

$$\mathcal{I}_{4j}(x; x_1, x_2) = x \int_{x_1}^{x} d\xi_1 \int_{x_1}^{\xi_1} d\xi_2 \int_{x_2}^{\xi_2} d\xi_3 \mathcal{I}_{4j-4}(\xi_3; x_1, x_2)$$
$$- \int_{x_1}^{x} d\xi_1 \xi_1 \int_{x_1}^{\xi_1} d\xi_2 \int_{x_2}^{\xi_2} d\xi_3 \mathcal{I}_{4j-4}(\xi_4; x_1, x_2)$$
$$= \int_{x_1}^{x} d\xi_1 \int_{x_1}^{\xi_1} d\xi_2 \int_{x_2}^{\xi_2} d\xi_3 (x - \xi_1) \mathcal{I}_{4j-4}(\xi_3; x_1, x_2)$$
$$= \int_{x_1}^{x} d\xi_1 \frac{(x - \xi_1)^2}{2} \int_{x_2} d\xi_2 \mathcal{I}_{4j-4}(\xi_2, x_1, x_2) \tag{29}$$

This can be generalized as

$$\mathcal{I}_{mj}(x, x_1, x_2) = \int_{x_1}^{x} d\xi_1 \frac{(x - \xi_1)^{m-2}}{(m-2)!} \int_{x_2}^{\xi_1} d\xi_2 \mathcal{I}_{mj-m}(\xi_2, x_1, x_2) \tag{30}$$

This formula can more extended to the case where each $m_1$ consecutive $x_1$ is followed by $m_2$ consecutive $x_2$ even though we are not going to give the details.

# 7 Concluding Remarks

What we have shown here the images under certain integral operators of SNADE can be recursively determined each $m_1$-fold plus $m_2$-fold integration can be replaced by just twofold integral which can also be further reduced to one-fold integral with an appropriately defined Kernel. Our further works will focus in the simplification of SNADE in this direction.

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