Super-λ_3 and super-κ_3 graphs on girth and diameter

Litao Guo
School of Applied Mathematics
Xiamen University of Technology
Xiamen Fujian 361024
P.R.China
ltguo2012@126.com

Xiaofeng Guo
School of Mathematical Sciences
Xiamen University
Xiamen Fujian 361005
P.R.China
xfguo@xmu.edu.cn

Abstract: Let $G = (V, E)$ be a connected graph. An edge set $F \subset E$ is a 3-restricted edge cut, if $G - F$ is disconnected and every component of $G - F$ has at least three vertices. The 3-restricted edge connectivity $\lambda_3(G)$ of G is the cardinality of a minimum 3-restricted edge cut of G. A graph G is called λ_3-optimal, if $\lambda_3(G) = \xi_3(G)$, where $\xi_3(G)$ is the minimum number of edges between a connected subgraph A with three vertices and $G - A$. A graph G is λ_3-connected, if G contains a 3-restricted edge cut. A λ_3-connected graph G is said to be super-λ_3, if every minimum 3-restricted edge cut isolates a component with exactly three vertices. It is analogous to define $\kappa_3(G)$ and κ_3-connected graph G for the case of vertex. A κ_3-connected graph G is said to be super-κ_3, if $\kappa_3(G) = \xi_3(G)$ and the deletion of a minimum 3-restricted cut isolates a component with exactly three vertices. Let G be a connected graph with girth $g \geq 4$ and minimum degree $\delta \geq 3$. We show that: (1) If diameter $D(G) \leq g - 4$, then G is super-λ_3. (2) If diameter $D(G) \leq g - 5$, then G is super-κ_3. Similar results are also obtained relating the diameter, the girth and the super connectivity of a line graph.

Key–Words: 3-Restricted edge connectivity; Super-λ_3; Super-κ_3

1 Introduction

It is well known that graph theory plays a key role in the analysis and design of reliable or invulnerable networks. A network is often modeled by a graph $G = (V, E)$ with the vertices representing nodes such as processors or stations, and the edges representing links between the nodes. One fundamental consideration in the design of networks is reliability. Connectivity is a parameter to measure the reliability of networks.

In this paper, we only consider simple graphs. Let $G = (V, E)$ be a connected graph. For a vertex $v \in V$, $N(v)$ is the set of all vertices adjacent to v. The degree of a vertex v, denoted by $d(v)$, is the size of $N(v)$. If $u, v \in V$, then $d(u, v)$ denotes the length of a shortest (u, v)-path. For $X, Y \subset V$, $d(X, Y)$ denotes the distance between X and Y; more formally, $d(X, Y) = \min\{d(x, y) :$ for any $x \in X$ and any $y \in Y\}$. If $v \in V, r \geq 0$ is an integer, then let $N_r(v) = \{w \in V : d(w, v) = r\}$, in particular, $N_1(v) = N(v)$. For $X \subset V$, $N_r(X) = \{w \in V : d(w, X) = r\}$ where $d(w, X) = d(\{w\}, X)$, and $N_1(X) = N(X)$. We denote the diameter and girth by D and g, respectively, and write $G - v$ for $G - \{v\}$. A path is called k-path, if its length is k. For $U \subseteq V$, $G[U]$ is the subgraph of G induced by the vertex subset U, and $[U, V - U]$ is the set of edges with one end in U and the other in $V - U$. And $\xi_k(G) = \min\{|[U, V - U]| : U \subset V, |U| = k$ and $G[U]$ is connected$\}$.

Recall that for every graph G we have $\lambda \leq \delta$, where δ is the minimum degree of G. If $\lambda = \delta$, then G is said to be *maximally edge connected* or λ-*optimal*. In the definitions of $\lambda(G)$, no restrictions are imposed on the components of $G - S$, where S is an edge cut. To compensate for this shortcoming, it would seem natural to generalize the notion of the classical connectivity by imposing some conditions or restrictions on the components of $G - S$. Following this idea, k-restricted edge connectivity were proposed in [3,4]. An edge set $F \subset E$ is said to be a *k-restricted edge cut*, if $G - F$ is disconnected and every component of $G - F$ has at least k vertices. The *k-restricted edge connectivity* of G, denoted by $\lambda_k(G)$, is the cardinality of a minimum k-restricted edge cut of G. If $|F| = \lambda_k$, then F is called a λ_k-*cut*. Not all connected graphs have λ_k-cuts ($k \geq 2$), for example $K_{1,n-1}$. A graph G is λ_k-connected, if G contains a k-restricted edge cut. A λ_k-connected graph G is called λ_k-*optimal*, if $\lambda_k(G) = \xi_k(G)$.

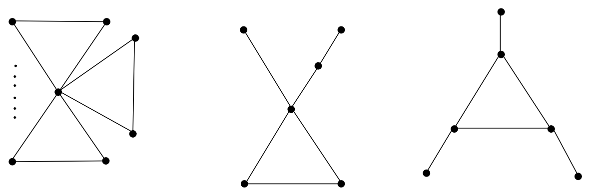
An vertex set X is a *k-restricted cut* of G, if $G - X$ is not connected and every component of $G - X$ has at least k vertices. The *k-restricted connectivity* $\kappa_k(G)$ (in short κ_k) of G, is the cardinality of a minimum k-restricted cut of G. And X is called a κ_k-

cut, if $|X| = \kappa_k$. Not all connected graphs have κ_k-cuts ($k \geq 2$), for example $K_{1,n-1}$. A graph G is κ_k-connected, if a κ_k-cut exists. For $k = 1, 2$ we can see [1, 2, 8]. We will study the case of $k = 3$.

For $X \subset V$, $v \in V \setminus X$ and $u \in N(v)$. Let us introduce the sets $X_u^+(v) = \{z \in N(v) - u : d(z, X) = d(v, X) + 1\}$; $X_u^=(v) = \{z \in N(v) - u : d(z, X) = d(v, X)\}$; $X_u^-(v) = \{z \in N(v) - u : d(z, X) = d(v, X) - 1\}$. Clearly, $X_u^+(v), X_u^=(v)$ and $X_u^-(v)$ form a partition of $N(v) - u$. And $|X_u^+(v)| + |X_u^=(v)| + |X_u^-(v)| = d(v) - 1$. If $d(v) \geq 2, u, w \in N(v)$, then $X_{uw}^+(v) = \{z \in N(v) - \{u, w\} : d(z, X) = d(v, X) + 1\}$; $X_{uw}^=(v) = \{z \in N(v) - \{u, w\} : d(z, X) = d(v, X)\}$; $X_{uw}^-(v) = \{z \in N(v) - \{u, w\} : d(z, X) = d(v, X) - 1\}$. Then $X_{uw}^+(v), X_{uw}^=(v)$ and $X_{uw}^-(v)$ form a partition of $N(v) - \{u, w\}$, and $|X_{uw}^+(v)| + |X_{uw}^=(v)| + |X_{uw}^-(v)| = d(v) - 2$.

Wang et al.[7] obtain the following result for $\lambda_3(G)$.

Theorem 1.1. *Let G be a simple connected graph of order $n \geq 6$. If G is not a subgraph of any of the graphs shown in Fig.1, then both $\lambda_3(G)$ is well defined and $\lambda_3(G) \leq \xi_3(G)$.*



(a) (b) (c)

Fig. 1

From this theorem we can see that if G is a connected graph with girth $g \geq 4$ and $\delta \geq 3$, then G has 3-restricted edge cuts.

We also have the following results for $\lambda_3(G)$ and $\kappa_3(G)$.

Theorem 1.2. (1) [5] *Let G be a λ_3-connected graph with girth $g \geq 4$, minimum degree $\delta \geq 3$ and diameter D. If $D \leq g - 3$, then G is λ_3-optimal.*

(2) [6] *Let G be a connected graph with girth $g \geq 6$, and minimum degree $\delta \geq 3$. Then G is κ_3-connected and $\kappa_3(G) \leq \xi_3(G)$, if $g \geq 7$ or $\delta \geq 4$.*

(3) [6] *Let G be a κ_3-connected graph with girth $g \geq 4$, minimum degree $\delta \geq 3$ and diameter D. If $D \leq g - 4$, then $\kappa_3(G) = \xi_3(G)$.*

In this paper, we investigate super-λ_3 connectivity and super-κ_3 connectivity of graphs with girth $g \geq 4$ and minimum degree $\delta \geq 3$. Some sufficient conditions for the graphs to be super-λ_3 (resp. super-κ_3) are given in Theorem 3.1, which depends on diameters of the graphs and their line graphs.

In Section 2 we shall give some properties of 3-restricted edge cuts and 3-restricted cuts of graphs, in Section 3 we prove the sufficient conditions in Theorem 3.1 for graphs to be super-λ_3 (resp. super-κ_3).

2 Properties of 3-restricted edge cuts and 3-restricted cuts of graphs

If G is a graph with girth $g \geq 4$, then every connected subgraph of G with three vertices is a path xyz of length two. Thus, $\xi_3(G) = \min\{d(x) + d(y) + d(z) - 4 : xyz$ is a path of length two in $G\}$.

Lemma 2.1. *Let G be a connected graph with girth $g \geq 4$, minimum degree $\delta \geq 3$ and $\xi_3(G)$. Let $X \subseteq V$ be a vertex cut with $|X| \leq \xi_3(G)$ and C be any connected component of $G - X$ with $|V(C)| \geq 3$. Then the following assertions hold:*
(1) There exists an edge uv in C such that $d(\{u, v\}, X) \geq \lfloor (g - 4)/2 \rfloor$.
(2) If g is odd and $|V(C)| \geq 4$, then there is a vertex $u \in C$ with $d(u, X) \geq (g - 5)/2$ such that $|N_{(g-5)/2}(u) \cap X| \leq 1$.

Proof. For $g = 4, 5, 6$, both assertions of the lemma hold, since $d(u, X) \geq 1$ for all u in C and $|V(C)| \geq 3$. So suppose that $g \geq 7$ and let $\mu = max\{d(u, X) : u \in V(C)\}$. Note that $\mu \geq 1$. If $\mu \geq \lfloor (g - 2)/2 \rfloor$, then both assertions clearly hold. Thus, we assume that $\mu \leq \lfloor (g - 4)/2 \rfloor$.

(1) If $\mu = 1$, then the result holds. Thus assume that $\mu \geq 2$.

Claim 1. There is an edge uv in C such that $d(\{u, v\}, X) = \mu$.

We argue by contradiction. Suppose that each vertex u in C at $d(u, X) = \mu$ satisfies $d(v, X) = \mu - 1$ for all $v \in N(u)$. As $\delta \geq 3$, take $w, v \in N(u)$, then vuw is a 2-path in C. Thus $d(v, X) = d(w, X) = \mu - 1$. Each vertex in $N(X_u^+(w))$ and $N(X_u^+(v))$ is at distance $\mu - 1$ from X. Moreover, we have $|N_{\mu-1}(X_u^=(w)) \cap X| \geq |X_u^=(w)|$. Otherwise, there are two vertices $x_1, x_2 \in X_u^=(w)$ both at distance $\mu - 1$ from a vertex $x \in N_{\mu-1}(X_u^=(w)) \cap X$. There is a cycle going through $\{x_1, w, x_2, x\}$ of length at most $2\mu \leq 2\lfloor (g - 4)/2 \rfloor \leq g - 4$, contrary to the fact that the length of a shortest cycle in G is equal to g.

Similarly, we have

$$|N_{\mu-1}(N(u)-v-w)\cap X| \geq |N(u)-v-w|,$$
$$|N_{\mu-1}(X_u^{=}(v))\cap X| \geq |X_u^{=}(v)|,$$
$$|N_{\mu-1}(X_u^{=}(w))\cap X| \geq |X_u^{=}(w)|,$$
$$|N_{\mu-1}(w)\cap X| \geq |X_u^{-}(w)|,$$
$$|N_{\mu-1}(v)\cap X| \geq |X_u^{-}(v)|,$$
$$|N_{\mu-1}(N(X_u^{+}(w))-w)\cap X| \geq |X_u^{+}(w)|,$$
$$|N_{\mu-1}(N(X_u^{+}(v))-v)\cap X| \geq |X_u^{+}(v)|.$$

Likewise, the sets $N_{\mu-1}(X_u^{=}(w))\cap X, N_{\mu-1}(N(u)-v-w)\cap X, N_{\mu-1}(X_u^{=}(v))\cap X, N_{\mu-1}(w)\cap X, N_{\mu-1}(v)\cap X, N_{\mu-1}(N(X_u^{+}(w))-w)\cap X$, and $N_{\mu-1}(N(X_u^{+}(v))-v)\cap X$ are pairwise disjoint. Hence we have

$$\begin{aligned}
\xi_3(G) &\geq |X| \\
&\geq |N_{\mu-1}(X_u^{=}(w))\cap X| + |N_{\mu-1}(w)\cap X| \\
&\quad + |N_{\mu-1}(X_u^{=}(v))\cap X| + \\
&\quad |N_{\mu-1}(N(u)-v-w)\cap X| + \\
&\quad |N_{\mu-1}(v)\cap X| + \\
&\quad |N_{\mu-1}(N(X_u^{+}(w))-w)\cap X| + \\
&\quad |N_{\mu-1}(N(X_u^{+}(v))-v)\cap X| \\
&\geq |X_u^{=}(w)| + |X_u^{-}(w)| + |X_u^{=}(v)| + \\
&\quad |N(u)-v-w| + |X_u^{-}(v)| + |X_u^{+}(w)| \\
&\quad + |X_u^{+}(v)| \\
&= d(u)+d(w)+d(v)-4 \geq \xi_3(G).
\end{aligned}$$

Thus, the above inequalities become equalities, yielding

$$\begin{aligned}
X &= (N_{\mu-1}(X_u^{=}(w))\cap X) \cup \\
&\quad (N_{\mu-1}(N(u)-v-w)\cap X) \cup \\
&\quad (N_{\mu-1}(X_u^{=}(v))\cap X) \\
&\quad \cup (N_{\mu-1}(w)\cap X) \cup (N_{\mu-1}(v)\cap X) \\
&\quad \cup (N_{\mu-1}(N(X_u^{+}(w))-w)\cap X) \cup \\
&\quad (N_{\mu-1}(N(X_u^{+}(v))-v)\cap X). \qquad (1)
\end{aligned}$$

And

$$\begin{aligned}
&|N_{\mu-1}(N(u)-v-w)\cap X| = |N(u)-v-w|; \\
&|N_{\mu-1}(N(X_u^{+}(w))-w)\cap X| = \\
&|N(X_u^{+}(w))-w| = |X_u^{+}(w)|; \\
&|N_{\mu-1}(N(X_u^{+}(v))-v)\cap X| = |N(X_u^{+}(v))-v| \\
&= |X_u^{+}(v)|. \qquad (2)
\end{aligned}$$

From (2) it follows that if $|X_u^{+}(w)| > 0$, then every vertex $y \in X_u^{+}(w)$ has degree 2, which contradicts to the fact that $\delta \geq 3$. Then $X_u^{+}(w) = \varnothing$. Similarly, $X_u^{+}(v) = \varnothing$. Furthermore, (2) also implies that each vertex $x \in N(u)-v-w$ has one unique neighbor in X at distance $\mu-1$, that is, $|X_u^{-}(x)| = 1$. Similarly, for the edge ux we obtain that $X_u^{+}(x) = \varnothing$, which implies that $X_u^{=}(x) \neq \varnothing$ because $\delta \geq 3$. Take a vertex $x' \in X_u^{=}(x)$, from (1) we conclude that there is a cycle passing through $\{x', x, u\}$ and the vertex $y \in N_{\mu-1}(x')\cap X$ of length at most $2(\mu-1)+4 \leq g-1$, then there would be a cycle of length less than g, a contradiction.

Claim 2. $\mu \geq \lfloor (g-4)/2 \rfloor$.

By contradiction, suppose that $\mu \leq \lfloor (g-4)/2 \rfloor - 1$. From Claim 1 we know there is an edge uv in C such that $d(\{u,v\}, X) = \mu$. In this case, $X_u^{+}(v) = X_v^{+}(u) = \varnothing$. Then C has a 2-path uvw such that $d(w,X) = \mu$ or $d(w,X) = \mu - 1$.

Firstly, assume that $d(w,X) = \mu$. Thus we have $X_v^{+}(w) = \varnothing$. Arguing as in Claim 1 we have $|N_\mu(X_{uw}^{=}(v))\cap X| \geq |X_{uw}^{=}(v)|$ and $|N_\mu(v)\cap X| \geq |X_{uw}^{-}(v)|$. Furthermore, the sets $N_\mu(X_{uw}^{=}(v))\cap X, N_\mu(v)\cap X, N_\mu(X_v^{=}(u))\cap X, N_\mu(u)\cap X, N_\mu(X_v^{=}(w))\cap X$ and $N_\mu(w)\cap X$ are pairwise disjoint. Therefore we have

$$\begin{aligned}
\xi_3(G) \geq |X| &\geq |N_\mu(X_{uw}^{=}(v))\cap X| + |N_\mu(v)\cap X| \\
&+ |N_\mu(X_v^{=}(u))\cap X| + |N_\mu(u)\cap X| \\
&+ |N_\mu(X_v^{=}(w))\cap X| + |N_\mu(w)\cap X| \\
&\geq |X_{uw}^{=}(v)| + |X_{uw}^{-}(v)| + |X_v^{=}(u)| + \\
&\quad |X_v^{-}(u)| + |X_v^{=}(w)| + |X_v^{-}(w)| \\
&= d(u)+d(w)+d(v)-4 \geq \xi_3(G).
\end{aligned}$$

Thus, the above inequalities become equalities, yielding

$$\begin{aligned}
X &= (N_\mu(X_{uw}^{=}(v))\cap X) \cup (N_\mu(v)\cap X) \cup \\
&\quad (N_\mu(X_v^{=}(u))\cap X) \cup (N_\mu(u)\cap X) \cup \\
&\quad (N_\mu(X_v^{=}(w))\cap X) \cup (N_\mu(w)\cap X) \quad (3)
\end{aligned}$$

and

$$\begin{aligned}
&|N_\mu(X_{uw}^{=}(v))\cap X| = |X_{uw}^{=}(v)|, \\
&|N_\mu(X_v^{=}(u))\cap X| = |X_v^{=}(u)|, \\
&|N_\mu(X_v^{=}(w))\cap X| = |X_v^{=}(w)|. \qquad (4)
\end{aligned}$$

From (4) we know that every vertex $z \in X_{uw}^{=}(v) \cup X_v^{=}(u) \cup X_v^{=}(w)$ has a unique neighbor at distance μ in X. As $\delta \geq 3$, there exists a vertex $z' \in N(z)\cap N_\mu(X)$ and $z' \overline{\in} \{u,v,w\}$, for every $z \in X_{uw}^{=}(v) \cup X_v^{=}(u) \cup X_v^{=}(w)$. From (3) it follows that there is a cycle of length at most $2\mu+5 \leq g-1$, contrary to the fact that the length of a shortest cycle in G is equal to g.

Secondly if $d(w,X) = \mu - 1$, then it is analogous to the case of $d(w,X) = \mu$.

As a consequence of both Claim 1 and Claim 2 we conclude that there exists an edge uv in C such that $d(\{u,v\},X) \geq \lfloor (g-4)/2 \rfloor$.

(2) Suppose now that $\mu = (g-5)/2$ otherwise by item (1) we are done. And we denote $C_X = \{u \in V(C) : d(u,X) = (g-5)/2\}$. By item (1) we can take an edge uv in $G[C_X]$.

Firstly, assume $(N(u)-v) \cap C_X \neq \varnothing$ or $(N(v)-u) \cap C_X \neq \varnothing$, say, $(N(v)-u) \cap C_X \neq \varnothing$. Notice that $X_v^+(u) = X_v^+(w) = X_{uw}^+(v) = \varnothing$ and that the sets $X_v^=(u), X_v^-(u), X_v^=(w), X_v^-(w), X_{uw}^=(v)$ and $X_{uw}^-(v)$ are pairwise disjoint. We will prove it by contradiction.

By contradiction, suppose that any vertex u in C_X satisfies $|N_{(g-5)/2}(u) \cap X| \geq 2$. Then we have $|N_{(g-5)/2}(X_v^=(u)) \cap X| \geq 2|X_v^=(u)|, |N_{(g-5)/2}(X_{uw}^=(v)) \cap X| \geq 2|X_{uw}^=(v)|$, and $|N_{(g-5)/2}(X_v^=(w)) \cap X| \geq 2|X_v^=(w)|$. Since the sets $N_{(g-5)/2}(X_v^=(u)) \cap X, N_{(g-7)/2}(X_v^-(u)) \cap X, N_{(g-5)/2}(X_{uw}^=(v)) \cap X, N_{(g-7)/2}(X_{uw}^-(v)) \cap X, N_{(g-5)/2}(X_v^=(w)) \cap X$ and $N_{(g-7)/2}(X_v^-(w)) \cap X$ are pairwise disjoint, it follows that

$$\begin{aligned}
\xi_3(G) &\geq |X| \\
&\geq |N_{(g-5)/2}(X_v^=(u)) \cap X| + \\
&\quad |N_{(g-7)/2}(X_v^-(u)) \cap X| + \\
&\quad |N_{(g-5)/2}(X_{uw}^=(v)) \cap X| + \\
&\quad |N_{(g-7)/2}(X_{uw}^-(v)) \cap X| + \\
&\quad |N_{(g-5)/2}(X_v^=(w)) \cap X| + \\
&\quad |N_{(g-7)/2}(X_v^-(w)) \cap X| \\
&\geq 2|X_v^=(u)| + |X_v^-(u)| + 2|X_{uw}^=(v)| + \\
&\quad |X_{uw}^-(v)| + 2|X_v^=(w)| + |X_v^-(w)| \\
&\geq \xi_3(G) + |X_v^=(u)| + |X_{uw}^=(v)| + |X_v^=(w)|.
\end{aligned}$$

Then $X_v^=(u) = X_{uw}^=(v) = X_v^=(w) = \varnothing$ and

$$\begin{aligned}
X &= (N_{(g-5)/2}(u) \cap X) \cup (N_{(g-5)/2}(v) \cap X) \cup \\
&\quad (N_{(g-5)/2}(w) \cap X). \qquad (5)
\end{aligned}$$

Furthermore, we can obtain $|N_{(g-5)/2}(u) \cap X| = |X_v^-(u)|, |N_{(g-5)/2}(v) \cap X| = |X_{uw}^-(v)|$ and $|N_{(g-5)/2}(w) \cap X| = |X_v^-(w)|$. This means that $\mu = (g-5)/2 \geq 2$. As $\delta \geq 3$, we have $|N(z) \cap (C_X - u)| \geq d(z) - 2 \geq 1$ for all $z \in X_v^-(u)$ (Otherwise a cycle of length at most $g-2$ would appear). Take a vertex $z \in X_v^-(u)$ and consider a vertex $z' \in N(z) \cap (C_X - u)$. Then from (5) a cycle of length at most $g-1$ would appear, a contradiction.

Secondly, if $(N(u)-v) \cap C_X = \varnothing$ and $(N(v)-u) \cap C_X = \varnothing$, then take a vertex w in $N(v)$ with $d(w,X) = (g-7)/2$. Hence uvw is a 2-path in C, it is analogous to the above case. □

Let $G = (V,E)$ be a λ_3-connected graph. An arbitrary λ_3-cut F can be denoted by $[V(C), V(\overline{C})]$, where C and $\overline{C}$ are the only two components of $G-F$. There are $X \subseteq V(C)$ and $Y \subseteq V(\overline{C})$ such that $X \cup Y$ is the set of the end vertices of $[V(C), V(\overline{C})]$, and so $[V(C), V(\overline{C})] = [X,Y]$.

A λ_3-connected graph G is said to be *super*-λ_3, if G is λ_3-optimal and every minimum 3-restricted edge cut isolates a component with exactly three vertices. A κ_3-connected graph G is said to be *super*-κ_3, if $\kappa_3(G) = \xi_3(G)$ and the deletion of each minimum 3-restricted cut isolates a component with exactly three vertices.

Lemma 2.2. *Let G be a connected graph with girth $g \geq 6$, and minimum degree $\delta \geq 3$. Let $[V(C), V(\overline{C})]$ = $[X,Y]$ be a λ_3-cut. Then the following assertions hold:*
(1) If $V(C) = X$, then G is super-λ_3.
(2) If G is not super-λ_3, then $C - X$ has a component with at least three vertices.

Proof. Since $g \geq 6$ and $\delta \geq 3$, by Theorem 1.1 G is λ_3-connected.

(1) Suppose that $V(C) = X$, then each vertex of C is incident with some edges of $[X,Y]$. If $|V(C)| = 3$, then we are done. So assume that $|V(C)| \geq 4$. Let uvw be a 2-path of C. Because $\delta \geq 3$, we assume that $|X_v^=(u)| \geq 1$. Since girth $g \geq 6$, thus arguing as before, we have

$$\begin{aligned}
\xi_3(G) &\geq \lambda_3(G) = |[X,Y]| \\
&\geq |[u,Y]| + |[v,Y]| + |[w,Y]| \\
&+ |[X_v^=(u),Y]| + |[X_{uw}^=(v),Y]| \\
&+ |[X_v^=(w),Y]| \\
&\geq |[u,Y]| + |[v,Y]| + |[w,Y]| + \\
&\quad |X_v^=(u)| + |X_{uw}^=(v)| + |X_v^=(w)| \\
&\geq 3 + d(u) - 1 + d(v) - 2 + d(w) - 1 \\
&> \xi_3(G),
\end{aligned}$$

which is a contradiction.

(2) By item (1) we have $C - X \neq \varnothing$. Suppose that any component of $C-X$ has at most two vertices. Let $C_1, C_2, \cdots, C_k$ be the components of $C - X$.

Case 1. Each component C_i satisfies $|C_i| = 1$.

Take C_1 from $C_1, C_2, \cdots, C_k$. Let $C_1 = \{v\}$. Then $N(v) \subseteq X$. And $\delta \geq 3$, we pick $u, w \in N(v)$, and thus uvw is a 2-path in C. Arguing as item (1), we have

$$\begin{aligned}\xi_3(G) &\geq \lambda_3(G) = |[X,Y]| \\ &\geq |[N(u)-v,Y]| + |[N(w)-v,Y]| + \\ &\quad |[N(v)-u-w,Y]| \\ &\geq |N(u)-v| + |N(w)-v| + \\ &\quad |N(v)-u-w| \\ &= d(u)+d(v)+d(w)-4 \geq \xi_3(G).\end{aligned}$$

It follows that $|[N(u)-v,Y]| = |N(u)-v|, |[N(v)-u-w,Y]| = |N(v)-u-w|, |[N(w)-v,Y]| = |N(w)-v|$ and $X = (N(u)-v) \cup (N(v)-u-w) \cup (N(w)-v)$. Hence $[\{u,w\},Y] = \varnothing$, which is a contradiction.

Case 2. There is a component C_1 with $|C_1| = 2$.

Assume that $V(C_1) = \{u,v\}$. Then $C_1 = K_2$, and $N(u)-v \subseteq X, N(v)-u \subseteq X$. Take $w \in X \cap (N(v)-u)$. Then uvw is a 2-path in C. As $g \geq 6$, arguing as in (1), we have

$$\begin{aligned}\xi_3(G) &\geq \lambda_3(G) = |[X,Y]| \\ &\geq |[N(u)-v,Y]| + |[N(v)-u-w,Y]| + \\ &\quad |[(N(w)-v)\cap X,Y]| + |[w,Y]| \\ &= d(u)+d(v)+d(w)-4 \geq \xi_3(G).\end{aligned}$$

It follows that $|[N(u)-v,Y]| = |N(u)-v|, |[N(v)-u-w,Y]| = |N(v)-u-w|, |[(N(w)-v)\cap X,Y]| = |(N(w)-v)\cap X|$ and $X = (N(u)-v)\cup(N(v)-u-w)\cup((N(w)-v)\cap X)\cup\{w\}$. Therefore, for any $x \in (N(u)-v)\cup(N(v)-u-w)\cup((N(w)-v)\cap X)$, we have $|[x,Y]| = 1$. Since $g \geq 6$ and $\delta \geq 3$, it follows that $N(x)\cap(X-x) = \varnothing$. So x is adjacent to some C_i's ($2 \leq i \leq k$). If there is a $C_i = \{y\}$ such that $y \in N(x)$, then $N(y) \subseteq X$. As $g \geq 6$ and $\delta \geq 3$, we have $|N(y)\cap(N(u)-v)| \leq 1, |N(y)\cap(N(v)-u)| \leq 1$ and $|N(y)\cap(N(w)\cap X)| \leq 1$.

Without loss of generality, we assume that $|N(y)\cap(N(w)\cap X)| = 1$, then $N(y)\cap(N(v)-u) = \varnothing, \{u,v\} \nsubseteq N(y)$, and we have $|N(y)\cap(N(u)-v)| \geq 2$. There is a cycle with length smaller than g, a contradiction. If $|N(y)\cap(N(w)\cap X)| = 0$, then $|N(y)\cap(N(u)-v)| \geq 2$ or $|N(y)\cap(N(v)-u)| \geq 2$. There is also a cycle of length smaller than g, which is impossible.

If there is a $|C_j| = 2$ which x is adjacent to, then it is analogous to the case of $|C_i| = 1$. We discuss the neighbors of each vertex in C_j, we can obtain the required result. □

Recall that in the line graph $L(G)$ of a graph G, each vertex represents an edge of G, and two vertices in a line graph are adjacent if and only if the corresponding edges of G are adjacent. Let us consider the edges $x_1y_1, x_2y_2 \in E(G)$. The distance between the corresponding vertices of $L(G)$ satisfies

$$d_{L(G)}(x_1y_1, x_2y_2) = d_G(\{x_1,y_1\},\{x_2,y_2\})+1, \qquad (6)$$

which is useful to prove that $D(G)-1 \leq D(L(G)) \leq D(G)+1$.

3 Some sufficient conditions for graphs to be super-λ_3 (resp. super-κ_3)

Now, we will show Theorem 3.1 by contradiction.

Theorem 3.1. *Let G be a connected graph with girth $g \geq 4$ and minimum degree $\delta \geq 3$. The following assertions hold:*

(1) *If $D(G) \leq g-4$, then G is super-λ_3.*

(2) *If $D(G) \leq g-5$, then G is super-κ_3.*

(3) *If the diameter of the line graph $D(L(G)) \leq g-4$, then G is super-λ_3.*

(4) *If the diameter of the line graph $D(L(G)) \leq g-5$, then G is super-κ_3.*

Proof. Since $g \geq 4$, clearly G is different from the graphs in Fig.1. Thus, by Theorem 1.1, G is λ_3-connected. Moreover, if $g \in \{4,5,6\}$, then theorem clearly holds. So we assume that $g \geq 7$. By part (2) of Theorem 1.2, G is κ_3-connected.

(1) From Theorem 1.2 it follows that $\lambda_3 = \xi_3$. Assume that G is not super-λ_3. Let $[V(C), V(\overline{C})] = [X,Y]$ be a λ_3-cut with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. By Lemma 2.2 we know that both $C-X$ and $\overline{C}-Y$ contain a connected component say H and K, respectively, of cardinality at least three vertices. Hence both X and Y are cutsets with $|X|,|Y| \leq \xi_3(G)$. From Lemma 2.1 there exist two vertices $u \in V(H)$ and $\overline{u} \in V(K)$ such that $g-4 \geq D(G) \geq d(u,\overline{u}) \geq d(u,X)+1+d(\overline{u},Y) \geq 2\lfloor(g-4)/2\rfloor+1$, which is a contradiction if g is even.

And for g odd all the inequalities become equalities. This means that $\max\{d(u,X) : u \in V(H)\} = (g-5)/2$ and $\max\{d(\overline{u},Y) : \overline{u} \in V(K)\} = (g-5)/2$. Thus by Lemma 2.1, we can find $u \in V(H)$ with $d(u,X) = (g-5)/2$ such that $N_{(g-5)/2}(u) \cap X = \{x\}$ for some $x \in X$; and we can find $\overline{u} \in V(K)$ with $d(\overline{u},Y) = (g-5)/2$ such that $N_{(g-5)/2}(\overline{u}) \cap Y = \{\overline{x}\}$ for some $\overline{x} \in Y$. As $d(u,\overline{u}) = g-4$, it follows that $x\overline{x} \in [X,Y]$. Clearly we can find a vertex $v \in N(u)$ with $d(v,X) = (g-5)/2$, because otherwise $|N_{(g-5)/2}(u)\cap X| \geq |N(u)| \geq 2$. Since $d(v,\overline{u}) = g-4$ we must have $x \in N_{(g-5)/2}(v)$ or $\overline{x} \in N_{(g-3)/2}(v)$. As a consequence, the path from u to $\overline{x}$ together with the path from v to $\overline{x}$ and the edge uv form a cycle of length at most $g-2$, which is a contradiction.

(2) From Theorem 1.2 it follows that $\kappa_3 = \xi_3$. Assume that G is not super-κ_3. Let X be an any κ_3-cut and consider two connected components $C, \overline{C}$ of $G - X$ with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. From Lemma 2.1 there exist two vertices $u \in V(C)$ and $\overline{u} \in V(\overline{C})$ such that $g - 5 \geq D(G) \geq d(u, \overline{u}) \geq d(u, X) + d(\overline{u}, X) \geq 2\lfloor (g-4)/2 \rfloor$, which is a contradiction if g is even.

And for g odd all the inequalities become equalities. This means that $\max\{d(u, X) : u \in V(C)\} = (g-5)/2$ and $\max\{d(\overline{u}, Y) : \overline{u} \in V(\overline{C})\} = (g-5)/2$. Thus by Lemma 2.1, we can find $u \in V(C)$ with $d(u, X) = (g-5)/2$ such that $N_{(g-5)/2}(u) \cap X = \{x\}$ for some $x \in X$; and we can find $\overline{u} \in V(\overline{C})$ with $d(\overline{u}, Y) = (g-5)/2$ such that $N_{(g-5)/2}(\overline{u}) \cap Y = \{\overline{x}\}$ for some $\overline{x} \in Y$. As $d(u, \overline{u}) = g - 5$, it follows that $x = \overline{x}$. Clearly we can find a vertex $v \in N(u)$ with $d(v, X) = (g-5)/2$. Since $d(v, \overline{u}) = g - 5$ we must have $x \in N_{(g-5)/2}(v)$. As a consequence, the path from u to x together with the path from v to x and the edge uv form a cycle of length at most $g - 4$, which is a contradiction.

(3) Since $D(L(G)) \leq g - 4$, then the diameter $D(G) \leq g - 3$, which means that $\lambda_3 = \xi_3$ by Theorem 1.2. Assume that G is not super-λ_3. Let $[V(C), V(\overline{C})] = [X, Y]$ be a λ_3-cut with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. By Lemma 2.2 we know that both $C - X$ and $\overline{C} - Y$ contain a connected component say H and K, respectively, of cardinality at least three. Hence both X and Y are cutsets with $|X|, |Y| \leq \xi_3(G)$. From Lemma 2.1 there exists an edge uv in $C - X$ and there exist an edge $\overline{u}\,\overline{v}$ in $\overline{C} - Y$ satisfying $d(\{u, v\}, X) \geq \lfloor (g-4)/2 \rfloor$ and $d(\{\overline{u}, \overline{v}\}, Y) \geq \lfloor (g-4)/2 \rfloor$. Then by using (6) we have

$$\begin{aligned} g - 4 \geq D(L(G)) &\geq d_{L(G)}(uv, \overline{u}\,\overline{v}) \\ &= d_G(\{u, v\}, \{\overline{u}, \overline{v}\}) + 1 \\ &\geq d_G(\{u, v\}, X) + 1 + \\ &\quad\; d_G(Y, \{\overline{u}, \overline{v}\}) + 1 \\ &\geq 2\lfloor (g-4)/2 \rfloor + 2, \end{aligned}$$

which is impossible.

(4) Now $D(L(G)) \leq g - 5$. Thus the diameter $D(G) \leq g-4$, which means that $\kappa_3 = \xi_3$ by Theorem 1.2. Assume that G is not super-κ_3. Let X be an any κ_3-cut and consider two connected components $C, \overline{C}$ of $G - X$ with $|V(C)| \geq 4, |V(\overline{C})| \geq 4$. From Lemma 2.1 there exists an edge uv in $C - X$ and there exists an edge $\overline{u}\,\overline{v}$ in $\overline{C} - X$ satisfying $d(\{u, v\}, X) \geq \lfloor (g-4)/2 \rfloor$ and $d(\{\overline{u}, \overline{v}\}, X) \geq \lfloor (g-4)/2 \rfloor$. Then by using (6) we have

$$\begin{aligned} g - 5 \geq D(L(G)) &\geq d_{L(G)}(uv, \overline{u}\,\overline{v}) \\ &= d_G(\{u, v\}, \{\overline{u}, \overline{v}\}) + 1 \\ &\geq d_G(\{u, v\}, X) + d_G(X, \{\overline{u}, \overline{v}\}) \\ &\quad\; +1 \\ &\geq 2\lfloor (g-4)/2 \rfloor + 1, \end{aligned}$$

which is impossible. □

Acknowledgements: The project is supported by NSFC (No.11301440,11301217). We would like to thank the referees for kind help and valuable suggestions.

References:

[1] C. Balbuena, P. Garcia-Vázquez, X. Marcote, Sufficient conditions for λ'-optimality in graphs with girth g, *J. Graph Theroy* 52, 2006, pp. 73-86.

[2] C. Balbuena, Y. Lin, M. Miller, Diameter-sufficient conditions for a graph to be super-restricted connected, *Discrete Appl. Math.* 156, 2008, pp. 2827-2834.

[3] A. Esfahanian, S. Hakimi, On computing a condintional edge connectivity of a graph, *Inform. Process. Lett.* 27, 1988, pp. 195-199.

[4] J. Fábrega, M.A. Foil, Extraconnectivity of graphs with large girth, *Discrete Math.* 127, 1994, pp. 163-170.

[5] L. Guo, W.Yang, X. Guo, On a kind of reliability analysis of networks, *Applied Mathematics and Computation* 218, 2011, pp. 2711-2715.

[6] L. Guo, J.X. Meng, 3-restricted connectivity of graphs with given girth, *Appl. Math. J. Chinese Univ. Series B* 23(3), 2008, pp. 351-358.

[7] Y. Wang, Q. Li, Upper bound of the third edge-connectivity of graphs, *Science in China Ser. A Mathematics* 48(3), 2005, pp. 360ł371.

[8] Z. Zhang, J.J. Yuan, Degree conditions for retricted edge connectivity and isoperimetric-edge-connectivity to be optimal, *Discrete Math.* 307, 2007, pp. 293-298.